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Proper Measures of Connectedness

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Abstract The concept of connectedness has been widely used in financial applications, in particular for systemic risk detection. Despite its popularity, at the state of the art, a rigorous definition of connectedness is still missing. In this paper we propose a general definition of connectedness introducing the notion of Proper Measures of Connectedness (PMCs). Based on the classical concept of mean introduced by Chisini, we define a family of PMCs and prove some useful properties. Further, we investigate whether the most popular measures of connectedness available in the literature are consistent with the proposed theoretical framework. We also compare different measures in terms of forecasting performances on real financial data. The empirical evidence shows the forecasting superiority of the PMCs compared to the measures that do not satisfy the theoretical properties. Moreover, the empirical results support the evidence that the PMCs can be useful to detect in advance financial bubbles, crises, and, in general, for systemic risk detection.

Keywords Connectedness · Systemic risk · Market risk · Financial crisis

JEL classification C60 · E30 · G01 · G10 · G17

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1 Introduction

In the last decade a wide stream of research focused on the notion of connectedness and its applications. Despite the growing attention on the topic, at the state of the art, a formal definition of connectedness is not yet available and the concept remains vague and elusive (see Diebold and Yilmaz 2014). The concept of connectedness together with a proposal for its measurement has been first introduced by Diebold and Yilmaz (2009). Then, the connectedness of financial systems has been investigated in many subsequent papers, see among the others Andersen et al. (2010), Barunkin et al. (2016), Diebold and Yilmaz (2014), Demirer et al. (2018), Zhang and Broadstock (2018). The idea behind connectedness is to measure the degree of the inter-relations and the inter-dependencies between the components of a whole system concentrating this information in a single number or index. For this reason, the connectedness is defined on a multidimensional domain representing, in general, the elements of a set, for instance, the agents of a system, the firms quoted in a given market.

A high connectedness reflects a strong interdependence between the elements of the considered set. In economic and financial applications, the relation between connectedness and systemic risk is intuitive. In financial markets, where the elements can be the returns of the assets, a high connectedness means that there is at least a subset of returns that behave in a similar way. Obviously, this constitutes a limit to the effective diversification opportunities.

Different models and approaches to predict financial crises and, in general, to detect systemic risk have been proposed in the economic literature starting from the late 1970s, when currency crises increased the interest both in specific indicators (Bilson 1979) and in the theoretical models (Krugman 1979) able to explain and predict such crises. Kaminsky (1999) identifies a single indicator following variable selection criterium; moreover, Frankel and Saravelos (2010), Rose and Spiegel (2009), Sarlin and Holopainen (2017) and Sarlin and Mezei (2016) apply multivariate models for estimating the probability of a future event from a set of several potential warning indicators.

In this paper we focus on the concept of connectedness and its possible application to systemic risk detection. Various measures of connectedness have been proposed and studied for market risk (return connectedness and market diversification), see among others Belsey et al. (1980), credit risk (default connectedness), see for example Merton (2014), and in general systemic risk (for system wide connectedness), see Billio et al. (2012), Figini et al. (2018), Acemoglu et al. (2015), Acharya et al. (2012). In particular, financial interconnectedness is well-known as a potential source of systemic risk. Volatility spillovers across and within markets are significant when the interdependence is high (see Diebold and Yilmaz 2012). Furthermore, market returns show high positive correlation when their volatility increases such that periods of high volatility are associated with market bubbles or crashes (see Wu 2001). Market volatility, especially in association with a high degree of connectedness, is the main cause of contagion during severe financial crises. Markets had become so intricately connected to each other, that it is impossible to effectively shield against risks across global financial markets, though a source of competitive gain had proved itself also as a mechanism for propagating risks.

Considering the previous discussion, it is natural to use a measure of connectedness as an early warning indicator of financial crises and bubbles. On one hand, a high degree of connectedness of an economic system is an indicator of the potential severity of a crisis, when occurring. On the other hand, it is interesting to understand whether an increase in connectedness leads to a higher probability of a financial crisis. One appealing feature of this approach is that the definition of an economic model is not needed and macroeconomic variables are not taken into account as direct explanatory variables.

As pointed out by Diebold and Yilmaz (2014), in the literature on connectedness the use of correlation-based measures is widespread, despite their intrinsic limitations on the pairwise analysis and the linearity assumptions. The concept of connectedness may be seen as a generalization of the concept of correlation. For example, while the connectedness of the assets returns on a given market directly reports to the classical portfolio diversification problem, default connectedness in credit risk reflects a contagion problem that can not be measured through the standard correlation-based approaches. Indeed, the idea behind connectedness is much broader and this strengthens the opinion that it would be too restrictive to consider it as a simple matter of linear correlation between the variables of a system.

In this paper we propose a general theoretical framework to identify the measures of connectedness: we define a Proper Measures of Connectedness (PMC) as a real-valued function defined on the set of full-rank matrices and satisfying a given set of minimal properties. Among the PMCs we introduce a specific class of measures based on the notion of power mean, a special case of the classical concept of mean introduced by Chisini (1929), and investigate the theoretical properties. In particular, we show that the proposed family of PMCs is a generalization of the condition number of a matrix (see Golub and Van Loan 1989) and its minimum is attained at special orthogonal matrices. Furthermore, we compare different measures of connectedness in a real data out-of-sample exercise aimed at assessing the forecasting performances. In the empirical experiment, the PMCs show an effective capacity for discriminating high volatility negative returns periods with respect to normal volatility periods, providing a useful instrument to predict future market distresses.

The paper is organized as follows: in Section 2 we discuss some literature on systemic risk; in Section 3 we provide the definition of the PMCs commenting on the required minimal properties; in Section 4 we recall the notions of Chisini and power means and, based on them, we discuss general classes of PMCs along with their properties; in Section 5 we consider some widely used measures of connectedness proposed in the literature and investigate whether they belong to the PMC class; finally, in Section 6, using real financial data, we set up an empirical application and compare the forecasting performances of the considered measures of connectedness. We gather comments and conclusions in Section 7, which ends the paper.

2 Systemic risk and its measurement

A typical characteristic of connectedness stands in its good capability of measuring the propagation strength of a signal within a set of elements. This feature proves to be fundamental for systemic risk analysis. In fact, although it does not exist a unique and globally accepted definition of systemic risk, it can be generally defined as the risk of catastrophic events affecting the vast majority of the elements of a system (companies, banks, governments, economic sectors, ...). For example, systemic risk can be defined as "any set of circumstances that threatens the stability of or public confidence in the financial system" (see Billio et al. 2012), while the European Central Bank generally refers to a financial instability "so widespread that it impairs the functioning of a financial system to the point where economic growth and welfare suffer materially." The plurality of definitions and their vagueness give rise to the proposal of a multitude of measures, each of them focusing on some specific aspect of systemic risk.

In the recent literature, various other different approaches have been proposed for systemic risk detection in finance: for qualitative models see Gaytán and Johnson (2002), for network theory based models see Elsinger et al. (2006), for artificial intelligence based models see Chin-Shien et al. (2006), for machine learning based approaches see Manasse and Roubini (2009), for the scenario analysis framework see Aikman et al. (2009), for principal components analysis approaches see Kritzman et al. (2011) and Zheng et al. (2012), for market based measures and the use of marginal expected shortfall see Pederzoli and Torricelli (2017) and Acharya et al. (2017), for pivotal systemic risk measures see Stolbov and Shchepeleva (2018).

For a comprehensive review of the measures proposed in the literature we refer to the survey Bisias et al. (2012). In this paper, the measures of systemic risk are divided into different categories on the base of the types of inputs required, analysis performed, and outputs produced.

In particular, the survey considers the measures of connectedness proposed in Billio et al. (2012) that we prove to belong to the class of PMCs (see Section 5), providing the natural link between our proposal and the existing measures. Systemic risk emerged as a relevant issue principally as a consequence of the global financial crisis of 2007–2009. The subject has been treated from different points of view, stressing various aspects of the phenomenon. A typical notion of systemic risk analysis is that of systemically important institution. A systemically important institution is an element of a system with strong effects on a large number of elements, in case of negative events (large price drops, financial distress, default). The Financial Stability Board publishes the list of the global systemically important financial institutions, after a monitoring activity of the main banks and financial institutions. In the academic literature, among the technique to detect such institutions, we recall CoVaR analysis (see Adrian and Brunnermeier 2009; 2011) and network theory (see Caccioli et al. 2018; for a complete review). In this field it is possible to assign each element a measure of systemic danger, and thus they are suited for monitoring and regulation of financial institutions.

In this paper we refer to another line of research. In fact, we are interested to an overall measure of the systemic fragility of a whole system (see Bisias et al. 2012; and the references therein). This way, the single elements are not individually analyzed and the focus is placed on the comprehensive effects. In our approach the systemic risk is considered as endogenous, in the sense that it generates within the system and does not comes from an external source of risk. Thus, the definition and the application of the measures of connectedness appear to be valuable tools to assess the fragility of the system. This is an important aspect, for instance when considering the policy makers' attention to systemic risk, as the notion of connectedness can add relevant information to the risk analysis. In financial applications, the connectedness is a measure of how strong the effects of systematic bad news can be on financial returns. A highly connected financial system is exposed to large fluctuations, because the possible diversification effect is weak and all the returns display common pattern. In such financial markets, the risk is hardly diversifiable and the possible losses can be severe. In this situation negative feedback effects can worsen the initial shock, producing a risk amplification. In this sense, it is also possible to consider the connectedness as a candidate predictor for market risk increase. This is the motivation of the empirical analysis we present in Section 6, where various measures of connectedness are compared on the basis of their predicting power, with respect to an increase of the foregoing market risk. Moreover, we consider connectedness measures that can be computed directly on the same variables which defines the elements of the system. In other words, we do not insert in the analysis exogenous variables, such as macroeconomic variables.

3 Proper Measures of Connectedness

In this section we present the theoretical framework to introduce the Proper Measures of Connectedness: we provide a general definition of a measure of connectedness as a real-valued function satisfying a given set of minimal properties.

Let $m \geq n \geq 2$, let $\mathrm{Mat}_{m\times n}$ be the set of $m \times n$ real matrices and $\mathcal{M}_{m\times n}$ be the subset of $\mathrm{Mat}_{m\times n}$ containing all the full-rank matrices, i.e. $\mathrm{rank}(A) = n,\ \forall A \in \mathcal{M}_{m\times n}$. Throughout the paper, we will interpret the columns A^j, with $j = 1, \ldots, n$, of A as the m realizations of a random variable representing the j^{th} element of the system. We indicate with A^t the transposed of any matrix A, with $\langle \cdot, \cdot \rangle$ the scalar product and with $\| \cdot \|$ the Euclidean norm.

Definition 1 (Proper Measure of Connectedness) A real-valued function $C : \mathcal{M}_{m\times n} \to \mathbb{R}$ is a *Proper Measure of Connectedness (PMC)* if it satisfies the following minimal properties 1, 2, 3 and 4.

Property 1 $C(A) \geq 0$, for any $A \in \mathcal{M}_{m\times n}$.

A PMC is a non-negative function of a given input matrix.

Property 2 $C(A)$ is invariant for any permutation of the columns of $A \in \mathcal{M}_{m\times n}$.

This property states that the connectedness is required to be independent from the order the constituents of a system are considered. For example, if we consider a financial market, its connectedness is independent from the order the assets are considered in the calculation.

Property 3 $C(A) > C(B)$ if and only if $C(\alpha A) > C(\alpha B)$, for any $A, B \in \mathcal{M}_{m\times n}$ and $\alpha > 0$.

This property requires that a positive rescaling in the data does not significantly impact the structure of connectedness of the corresponding system; in other words, if one system results to be more connected compared to another one, it remains so, after any rescaling of the data.

Remark 1 If C is a homogeneous function, that is if there exists an integer $r \in \mathbb{R}$ such that $C(\alpha A) = \alpha^r C(A)$ for any $A \in \mathcal{M}_{m\times n}$ and $\alpha > 0$, then C satisfies Property 3.

It is worthwhile to give an interpretation of Remark 1 in the special case $r = 0$: the PMCs contain the functions whose value is independent from any positive rescaling of the input data, i.e. a scale invariant measures.

Property 4 Let ρ be the linear correlation coefficient and $a_1, a_2, a_3 \in \mathcal{M}_{m\times 1}$, with $\|a_2\| = \|a_3\|$ and $\langle \mathbf{1}, a_2\rangle = \langle \mathbf{1}, a_3\rangle = 0$, where $\mathbf{1}$ is the unit vector. Let $A_{1j} = (a_1|a_j) \in \mathcal{M}_{m\times 2}$, with $j \in \{2, 3\}$. If $|\rho(a_1, a_2)| \geq |\rho(a_1, a_3)|$ then $C(A_{12}) \geq C(A_{13})$.

This last property reflects the relation between correlation and connectedness: a higher correlation results in a higher connectedness. On the opposite, nothing is specified when the connectedness is high: a high level of connectedness could mean something more than a simple high correlation in the data. Property 4 requires a PMC to incorporate at least the information provided by the correlation and, if possible, some extra information. The technical conditions $\|a_2\| = \|a_3\|$ and $\langle \mathbf{1}, a_2\rangle = \langle \mathbf{1}, a_3\rangle = 0$ are required to avoid potential numerical distortions related to a wide difference in the norm of a_2 and a_3. Moreover, they imply that the additional columns contain data with zero mean and equal variance, focusing the comparison on the correlation.

4 A class of PMCs

In this section, based on the classical concept of mean introduced by Chisini (1929), we define a family of measures respecting the requirements for a PMC.

Definition 2 (Chisini mean) Let I be a real interval, $x_1, \ldots, x_n \in I$ and $f : I^n \to \mathbb{R}$. If there exists a value $\overline{x} \in \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) = f(\overline{x}, \ldots, \overline{x})$$

then $\overline{x} = \overline{x}(x_1, \ldots, x_n)$ is called the *Chisini mean of* $x_1, \ldots, x_n$ *with respect to* f (or more simply the *Chisini mean* of $x_1, \ldots, x_n$).

As pointed out by De Finetti (1931) (see in particular page 19), despite the fact that Chisini has been the first mathematician to shed light on the deep meaning of the concept of mean, Chisini's definition is so general that it does not even satisfy the property to be an internal function, which is required in many practical situations. Therefore, very useful properties, like *idempotency*, *inbetweenness*, *monotonicity* and *homogeneity*, do not generically hold and need to be imposed upon request, as in our case. We state Definition 3 by referring to the state-of-the-art (see Grabisch et al. 2011).

Definition 3 Let I be a real interval and let us consider a Chisini mean $\overline{x}(x_1, \ldots, x_n)$ defined on $(x_1, \ldots, x_n) \in I^n$. Then the Chisini mean is:

- *idempotent*: if $x_1 = \ldots = x_n = x \in I$ implies $\overline{x}(x, \ldots, x) = x$, for any $x \in I$;
- *internal*: if $\min\{x_1, \ldots, x_n\} \leq \overline{x}(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}$ for any $x_1, \ldots, x_n \in I$;
- *nondecreasing*: if $\overline{x}(x_1, \ldots, x_n) \leq \overline{x}(y_1, \ldots, y_n)$ for any tuples $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in I^n$ such that $x_i \leq y_i$, $i = 1, \ldots, n$;
- *strictly increasing*: if $\overline{x}(x_1, \ldots, x_n) < \overline{x}(y_1, \ldots, y_n)$ for any $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) \in I^n$ such that $x_i \leq y_i$, $i = 1, \ldots, n$;
- *homogeneous (of degree 1)*: if $\overline{x}(\alpha x_1, \ldots, \alpha x_n) = \alpha \overline{x}(x_1, \ldots, x_n)$ for any $x_1, \ldots, x_n \in I$ and for any $\alpha > 0$.

Starting from an idempotent, internal, nondecreasing, homogeneous Chisini mean, we introduce a general class of PMCs (see Definition 1).

Definition 4 (Chisini measure) Let $A \in \mathcal{M}_{m\times n}$ and let $\sigma_1(A) \geq \ldots \geq \sigma_n(A) > 0 \in I \subset \mathbb{R}$ be the *singular values* of A (for instance, see Golub and Van Loan 1989) listed, as usual, in nonincreasing order. For each $k \in \{1, \ldots, n\}$ the *Chisini measure* $m_k(A)$ is defined as

$$m_k(A) = \frac{\sigma_1(A)}{\overline{\sigma}(\sigma_{n-k+1}(A), \ldots, \sigma_n(A))},$$

where $\overline{\sigma}$ is an idempotent, internal, nondecreasing, homogeneous Chisini mean.

In the following proposition we prove that m_k is a PMC.

Proposition 1 *The Chisini measure* m_k *is a PMC, for each* $k \in \{1, \ldots, n\}$.

Proof Let $k \in \{1, \dots, n\}$. From Definition 4 each m_k is a real-valued function defined on the full-rank matrices $\mathcal{M}_{m\times n}$.

Let $A \in \mathcal{M}_{m\times n}$ and $\sigma_1(A) \geq \dots \geq \sigma_n(A) > 0$ be its singular values. Since the considered Chisini mean is internal, then $\overline{\sigma}(\sigma_{n-k+1}(A), \dots, \sigma_n(A)) \geq \min\{\sigma_{n-k+1}(A), \dots, \sigma_n(A)\} > 0$, and so

$$m_k(A) = \frac{\sigma_1(A)}{\overline{\sigma}(\sigma_{n-k+1}(A), \dots, \sigma_n(A))} > 0$$

therefore Property 1 is satisfied.

To prove Property 2 it is enough to observe that the singular values of a matrix do not change under rows or columns permutation.

To prove Property 3, we recall that $\sigma_j(\alpha A) = \alpha\sigma_j(A)$ for each $\alpha > 0$ and for each $j \in \{1, \dots, n\}$ (it immediately follows from the singular value decomposition, see Golub and Van Loan (1989)). Further, since the considered Chisini mean is homogeneous of degree 1, it is easy to show that m_k is homogeneous of degree 0. In fact

$$m_k(\alpha A) = \frac{\sigma_1(\alpha A)}{\overline{\sigma}(\sigma_{n-k+1}(\alpha A), \dots, \sigma_n(\alpha A))} = \frac{\alpha\sigma_1(A)}{\alpha\overline{\sigma}(\sigma_{n-k+1}(A), \dots, \sigma_n(A))} = m_k(A)$$

for each $\alpha > 0$ and for each $k \in \{1, \dots, n\}$. So Property 3 is proved.

To prove Property 4, we note that the assumptions $|\rho(a_1, a_2)| \geq |\rho(a_1, a_3)|$, $\|a_2\| = \|a_3\|$ and $\langle \mathbf{1}, a_2\rangle = \langle \mathbf{1}, a_3\rangle = 0$ are equivalent to $\langle a_1, a_2\rangle^2 \geq \langle a_1, a_3\rangle^2$. Since $n = 2$, then $k \in \{1, 2\}$. If $k = 1$, for the idempotency property of the Chisini mean $\overline{\sigma}(\sigma_2(A_{1j})) = \sigma_2(A_{1j})$, for any $j \in \{2, 3\}$, and so:

$$m_1(A_{1j}) = \frac{\sigma_1(A_{1j})}{\sigma_2(A_{1j})}, \quad j \in \{2, 3\}. \tag{1}$$

If $k = 2$, using the homogeneity property of the Chisini mean, we have:

$$m_2(A_{1j}) = \frac{\sigma_1(A_{1j})}{\overline{\sigma}\left(\sigma_1(A_{1j}), \sigma_2(A_{1j})\right)} = \frac{1}{\overline{\sigma}\left(1, \frac{\sigma_2(A_{1j})}{\sigma_1(A_{1j})}\right)}, \quad j \in \{2, 3\}. \tag{2}$$

Since the singular values are invariant to matrix transpose and recalling a result of Torrente and Uberti (2018) (see Proposition 6, formula (14)) it follows that

$$\frac{\sigma_1(A_{1j})}{\sigma_2(A_{1j})} = \sqrt{1 + 2\frac{1}{\sqrt{h(a_1^t, a_j^t)} - 1}} \tag{3}$$

where $h(a_1^t, a_j^t)$ is defined by

$$h(a_1^t, a_j^t) = \frac{(\|a_1\|^2 + \|a_j\|^2)^2}{(\|a_1\|^2 - \|a_j\|^2)^2 + 4\langle a_1, a_j\rangle^2}.$$

Since $\|a_2\| = \|a_3\|$ and $\langle a_1, a_2\rangle^2 \geq \langle a_1, a_3\rangle^2$ then $h(a_1^t, a_2^t) \leq h(a_1^t, a_3^t)$, and consequently, using formula (3), it follows

$$\frac{\sigma_1(A_{12})}{\sigma_2(A_{12})} \geq \frac{\sigma_1(A_{13})}{\sigma_2(A_{13})}. \tag{4}$$

Using expression (1) and inequality (4), we have:

$$m_1(A_{12}) = \frac{\sigma_1(A_{12})}{\sigma_2(A_{12})} \geq \frac{\sigma_1(A_{13})}{\sigma_2(A_{13})} = m_1(A_{13}).$$

Analogously, using expression (2), inequality (4) and the nondecreasing property of the Chisini mean, we have:

$$m_2(A_{12}) = \frac{1}{\overline{\sigma}\left(1, \frac{\sigma_2(A_{12})}{\sigma_1(A_{12})}\right)} \geq \frac{1}{\overline{\sigma}\left(1, \frac{\sigma_2(A_{13})}{\sigma_1(A_{13})}\right)} = m_2(A_{13}).$$

Therefore, we conclude that m_k is a PMC, for each $k \in \{1, \ldots, n\}$.

The proposed class of PMCs $\{m_k\}_{k=1,\ldots,n}$ verifies a bunch of interesting properties in addition to the minimal properties required for a PMC, presented in Proposition 2, whose proof requires the following technical lemma.

Lemma 1 *Let m_k, $k \in \{1, \ldots, n\}$, be the Chisini measure defined over $\mathcal{M}_{m\times n}$ and let $A \in \mathcal{M}_{m\times n}$. Assume that the Chisini mean that defines m_k is strictly increasing. If there exists $i \in \{1, \ldots, n\}$ such that $m_i(A) = 1$, then $\sigma_1(A) = \ldots = \sigma_n(A) = \|A\|$ and $m_k(A) = 1$ for each $k \in \{1, \ldots, n\}$.*

Proof Let $m_i(A) = 1$ for some $i \in \{1, \ldots, n\}$; using the internal property of the Chisini mean we have:

$$1 = m_i(A) = \frac{\sigma_1(A)}{\overline{\sigma}(\sigma_{n-i+1}(A), \ldots, \sigma_n(A))} \geq \frac{\sigma_1(A)}{\sigma_{n-i+1}(A)} \geq 1, \tag{5}$$

from which it follows that $\sigma_{n-i+1}(A) = \sigma_1(A)$, and therefore

$$\sigma_j(A) = \sigma_1(A), \qquad j = 1, \ldots, n-i+1. \tag{6}$$

If $i = 1$, the proof is complete. If $i > 1$, since in particular $\sigma_1(A) = \sigma_{n-i+1}(A)$, we rewrite (5) as follows:

$$\overline{\sigma}(\sigma_{n-i+1}(A), \ldots, \sigma_n(A)) = \sigma_{n-i+1}(A) = \max\{\sigma_{n-i+1}(A), \ldots, \sigma_n(A)\},$$

from which, since by assumption the Chisini mean is strongly increasing, we have

$$\sigma_j(A) = \sigma_{n-i+1}(A), \qquad j = n-i+1, \ldots, n. \tag{7}$$

Consequently, from (6) and (7) and the equality $\sigma_1(A) = \|A\|$, it follows $\sigma_1(A) = \ldots = \sigma_n(A) = \|A\|$; therefore

$$m_k(A) = \frac{\sigma_1(A)}{\overline{\sigma}(\sigma_{n-k+1}(A), \ldots, \sigma_n(A))} = \frac{\sigma_1(A)}{\overline{\sigma}(\sigma_1(A), \ldots, \sigma_1(A))} = \frac{\sigma_1(A)}{\sigma_1(A)} = 1,$$

for each $k \in \{1, \ldots, n\}$, so the lemma is proved.

Proposition 2 (Properties of the Chisini measures m_k) *Let m_k, $k \in \{1,\dots,n\}$, be the Chisini measure defined over $\mathcal{M}_{m\times n}$, let $A \in \mathcal{M}_{m\times n}$ and let $\mathcal{K}(A) = \frac{\sigma_1(A)}{\sigma_n(A)}$ be the condition number of A (see Golub and Van Loan 1989). Then:*

1. $m_1(A) = \frac{\sigma_1(A)}{\sigma_n(A)} = \mathcal{K}(A)$;
2. $1 \le m_k(A) \le \mathcal{K}(A)$, *for each* $k \in \{1,\dots,n\}$.

Furthermore, if the Chisini mean identifying m_k is strongly increasing, the following properties hold:

3. $m_k(A) = 1$, *for each* $k = 1,\dots,n$, *if and only if* $\frac{1}{\|A\|}A$ *is unitary;*
4. *let $m > n$, let $b \in \mathcal{M}_{m\times 1}$, let A^j denote the j-th column of A, $j = 1,\dots,n$, and let $(A^j \mid b) \in \mathcal{M}_{m\times 2}$. If $m_1(A^j \mid b) = 1$, for each $j \in \{1,\dots,n\}$, then the matrix $(A \mid b) \in \mathcal{M}_{m\times(n+1)}$ and*

$$m_k(A \mid b) = m_k(A), \quad \forall k \in \{1,\dots,\overline{k}\},$$

where $\overline{k} = \max\{j \in \{1,\dots,n\} \mid \sigma_{n-j+1}(A) < \|b\|\}$.

Proof 1: It is enough to exploit the idempotency property of the Chisini mean which yields $\overline{\sigma}(\sigma_n(A)) = \sigma_n(A)$.

2: It easily follows using the internal property of the Chisini mean

$$m_k(A) \ge \frac{\sigma_1(A)}{\max\{\sigma_{n-k+1}(A),\dots,\sigma_n(A)\}} = \frac{\sigma_1(A)}{\sigma_{n-k+1}(A)} \ge 1,$$
$$m_k(A) \le \frac{\sigma_1(A)}{\min\{\sigma_{n-k+1}(A),\dots,\sigma_n(A)\}} = \frac{\sigma_1(A)}{\sigma_n(A)} = \mathcal{K}(A).$$

3: If there exists $k \in \{1,\dots,n\}$ such that $m_k(A) = 1$ then, applying Lemma 1, it follows that $\sigma_1(A) = \ldots = \sigma_n(A) = \|A\|$. Therefore the singular value decomposition of $\frac{1}{\|A\|}A$ is

$$\frac{1}{\|A\|}A = U \begin{pmatrix} I_n \\ \mathbf{0}_{(m-n)\times n} \end{pmatrix} V^t,$$

with $U \in \mathrm{Mat}_{m\times m}(\mathbb{R})$ and $V \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be orthonormal matrices. Consequently, $\frac{1}{\|A\|^2}A^tA = I_n$, that is, $\frac{1}{\|A\|}A$ is unitary. Vice versa, by definition of unitary matrix, it follows that $A^tA = AA^t = \|A\|^2 I_n$. Therefore $\sigma_1(A) = \ldots = \sigma_n(A) = \|A\|$ and, by an easy computation, using the idempotency property of the Chisini mean, it follows that $m_k(A) = 1$ for each $k \in \{1,\dots,n\}$.

4: Let $j \in \{1,\dots,n\}$; using the assumption $m_1(A^j \mid b) = 1$ and applying Lemma 1 and item 3, it follows that $\frac{1}{\|(A^j \mid b)\|}(A^j \mid b)$ is orthonormal, therefore, in particular, b is orthogonal to each A^j. Consequently, $b \notin \mathrm{span}(A^1,\dots,A^n)$, so the matrix $(A \mid b) \in \mathcal{M}_{m\times(n+1)}$. Using again Lemma 1 it follows that $\sigma_1(A^j|b) = \sigma_2(A^j|b) = \|(A^j|b)\|$ and from the orthonormality of $\frac{1}{\|(A^j \mid b)\|}(A^j|b)$ we conclude that $\|(A^j|b)\| = \|A^j\| = \|b\|$, for each $j \in \{1,\dots,n\}$. In particular, the squared Frobenius norm of $(A \mid b)$ is $\|(A \mid b)\|_F^2 = (n+1)\|b\|^2$. We consider the characteristic polynomial of the matrix $(A \mid b)^t(A \mid b)$:

$$p_\lambda((A \mid b)^t(A \mid b)) = \det((A \mid b)^t(A \mid b) - \lambda I_{n+1}) =$$
$$= \det \begin{pmatrix} A^tA - \lambda I_n & \mathbf{0} \\ \mathbf{0} & b^tb - \lambda \end{pmatrix} = (\|b\|^2 - \lambda) \det(A^tA - \lambda I_n),$$

from which it easily follows that the singular values of the matrix $(A \mid b)$ are the (not ordered) values $\|b\|, \sigma_1(A), \ldots, \sigma_n(A)$. We prove that $\sigma_n(A) \leq \|b\| \leq \sigma_1(A)$. In fact, by contradiction, if $\|b\| > \sigma_1(A)$, then we have

$$(n+1)\|b\|^2 = \|(A \mid b)\|_F^2 = \sigma_1^2(A) + \ldots + \sigma_n^2(A) + \|b\|^2 < (n+1)\|b\|^2.$$

Analogously, if $\|b\| < \sigma_n(A)$, then we have

$$(n+1)\|b\|^2 = \|(A \mid b)\|_F^2 = \sigma_1^2(A) + \ldots + \sigma_n^2(A) + \|b\|^2 > (n+1)\|b\|^2.$$

So, in particular, we have $\sigma_1(A|b) = \sigma_1(A)$ and $\sigma_{(n+1)-k+1}(A|b) = \sigma_{n-k+1}(A)$, for each $k \in \{1, \ldots, \overline{k}\}$. Therefore, by computing $m_k(A \mid b)$, for each $k \in \{1, \ldots, \overline{k}\}$, we obtain:

$$m_k(A \mid b) = \frac{\sigma_1(A \mid b)}{\overline{\sigma}(\sigma_{(n+1)-k+1}(A \mid b), \ldots, \sigma_{n+1}(A \mid b))} =$$
$$= \frac{\sigma_1(A)}{\overline{\sigma}(\sigma_{n-k+1}(A), \ldots, \sigma_n(A))} = m_k(A),$$

so the proposition is proved.

Proposition 2 provides interesting properties of the measures $\{m_k\}_{k=1,\ldots,n}$ highlighting the relations between such class of PMCs and well-known linear algebra concepts. Item 1 shows that m_1 coincides with $\mathcal{K}$, so that the measure m_k, whose values $m_k(A)$ belong to the interval $[1, \mathcal{K}(A)]$ (see item 2), is a generalization of the condition number of a matrix. Further, items 3 and 4 relate the minimum value of m_k to special geometrical properties of the input matrix.

4.1 The Power Mean Measure

In this section we explicitly propose a family of PMCs by simply specifying a class of means chosen among the Chisini means. A well-studied class is represented by the *quasi-arithmetic means (QAM)* (for instance, see Grabisch et al. 2011), yielding the general form of each associative and monotone Chisini mean (*Nagumo-Kolmogoroff Theorem*, see Kolmogoroff 1930). We construct the proposed PMCs using the class of *power means*, which basically coincides with the homogeneous QAM (see Grabisch et al. 2011; Theorem 3).

Definition 5 (Power mean) Let $p \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ and $x_1, \ldots, x_n$ be n real positive numbers. The *power mean* with exponent p of $x_1, \ldots, x_n$, denoted by $M_p(x_1, \ldots, x_n)$, is defined as follows:

- if $p > 0$:

$$M_p(x_1, \dots, x_n) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}};$$

- if $p = 0$:

$$M_0(x_1, \dots, x_n) = \lim_{p \to 0} M_p(x_1, \dots, x_n) = \sqrt[n]{x_1 \cdot \dots \cdot x_n};$$

- if $p = +\infty$:

$$M_{+\infty}(x_1, \dots, x_n) = \lim_{p \to +\infty} M_p(x_1, \dots, x_n) = \max\{x_1 \dots x_n\}.$$

We list some special cases:

- if $p = 0$ then $M_0(x_1, \dots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$ is the *geometric mean*;
- if $p = 1$ then $M_1(x_1, \dots, x_n) = \frac{\sum_{i=1}^{n} x_i}{n}$ is the *arithmetic mean*;
- if $p = 2$ then $M_2(x_1, \dots, x_n) = \left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right)^{\frac{1}{2}}$ is the *quadratic mean* or *root mean square*;
- if $p = +\infty$ then $M_{+\infty}(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$ is the *maximum*.

The power mean just defined allows us to introduce the *Power Mean Measure* μ_k^p.

Definition 6 (Power Mean Measure) Let $A \in \mathcal{M}_{m \times n}$, let $\sigma_1(A), \dots, \sigma_n(A)$ be the singular values of A and let $p \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$. For each $k \in \{1, \dots, n\}$ the *Power Mean Measure* $\mu_k^p(A)$ is defined by:

$$\mu_k^p(A) = \frac{\sigma_1(A)}{M_p(\sigma_{n-k+1}(A), \dots, \sigma_n(A))},$$

where M_p is the p-power mean.

Proposition 3 *The Power Mean Measure μ_k^p, $p \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, is a PMC for each $k \in \{1, \dots, n\}$.*

Proof It is enough to observe that the power mean $M_p, p \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is an idempotent, internal, nondecreasing, homogeneous Chisini mean (see Definition 3) and to apply Proposition 1.

It is immediate to observe that μ_k^p also satisfies additional properties. In particular, if $p \neq +\infty$, μ_k^p satisfies Lemma 1 and Proposition 2; whereas, in the case $p = +\infty$, since the power mean $M_{+\infty}$ is not strongly increasing, μ_k^p only satisfies Proposition 2, items 1 and 2. Further, for $p \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ other properties of μ_k^p are outlined in the following proposition; in order to prove it, we recall that power means are *decomposable* (see Grabisch et al. 2011), i.e. for the integers $0 \leq k \leq n$

$$M_p(x_1, \dots, x_n) = M_p(\underbrace{M_p(x_1, \dots, x_k), \dots, M_p(x_1, \dots, x_k)}_{k \text{ times}}, x_{k+1}, \dots, x_n).$$

Proposition 4 *Let μ_k^p, $p \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, be the Power Mean Measure defined on $\mathcal{M}_{m\times n}$, with $k \in \{1, \ldots, n\}$. Then the following properties hold:*

1. $\mu_{k+1}^p(A) \leq \mu_k^p(A)$, *for each* $A \in \mathcal{M}_{m\times n}$ *and for each* $k \in \{1, \ldots, n\}$*;*
2. *if* $p < q$ *then* $\mu_k^p(A) \geq \mu_k^q(A)$ *for each* $A \in \mathcal{M}_{m\times n}$ *and for each* $k \in \{1, \ldots, n\}$*; further,* $\mu_k^p(A) = \mu_k^q(A)$ *if and only if* $\frac{1}{\|A\|}A$ *is orthonormal.*

Proof 1: Let $\overline{M} = M_p(\sigma_{n-k+1}(A), \ldots, \sigma_n(A))$. Thanks to the decomposability of the power means

$$M_p(\sigma_{n-k}(A), \sigma_{n-k+1}(A), \ldots, \sigma_n(A)) = M_p(\sigma_{n-k}(A), \underbrace{\overline{M}, \ldots, \overline{M}}_{k \text{ times}}).$$

Since the power means are internal and the singular values are listed in decreasing order then $\sigma_{n-k}(A) \geq \overline{M}$; moreover, power means are nondecreasing and idempotent, so

$$M_p(\sigma_{n-k}(A), \underbrace{\overline{M}, \ldots, \overline{M}}_{k \text{ times}}) \geq M_p(\underbrace{\overline{M}, \ldots, \overline{M}}_{k+1 \text{ times}}) = \overline{M} = M_p(\sigma_{n-k+1}(A), \ldots, \sigma_n(A)).$$

Consequently,

$$\mu_{k+1}^p(A) = \frac{\sigma_1(A)}{M_p(\sigma_{n-k}(A), \ldots, \sigma_n(A))} \leq \frac{\sigma_1(A)}{M_p(\sigma_{n-k+1}(A), \ldots, \sigma_n(A))} = \mu_k^p(A).$$

2: since $p < q$, by applying the power mean inequality, we have:

$$M_p(\sigma_{n-k+1}(A), \ldots, \sigma_n(A)) \leq M_q(\sigma_{n-k+1}(A), \ldots, \sigma_n(A)),$$

and therefore

$$\mu_k^p(A) = \frac{\sigma_1(A)}{M_p(\sigma_{n-k+1}(A), \ldots, \sigma_n(A))} \geq \frac{\sigma_1(A)}{M_q(\sigma_{n-k+1}(A), \ldots, \sigma_n(A))} = \mu_k^q(A),$$

for each $A \in \mathcal{M}_{m\times n}$ and for each $k \in \{1, \ldots, n\}$. Further, using again the power mean inequality, which states that two means with different powers p and q are equal if and only if all the elements of the list are the same, we have that $\mu_k^p(A) = \mu_k^q(A)$ for each $k \in \{1, \ldots, n\}$ if and only if $\sigma_1(A) = \ldots = \sigma_n(A)$, which is equivalent to the condition $\frac{1}{\|A\|}A$ is orthonormal.

5 Some measures of connectedness proposed in the literature

A bunch of indexes has been proposed in the literature in order to measure the connectedness of the financial markets and possibly to prevent systemic risk and financial crises. In this section, we review some of the most used connectedness indicators and investigate whether these measures verify the minimal properties required by the PMCs definition. We focus our attention on the following most commonly used measures: the *Total Connectedness* (see Diebold and Yilmaz 2009), the *Cumulative Risk Fraction* (see Billio et al. 2012), the *Market Rank Indicator* (see Figini et al. 2018), the *Average Correlation*, the *Variance Inflation Factor* (see Belsey et al. 1980), the *Mahalanobis distance* (see Mahalanobis 1936).

5.1 The Total Connectedness

The Total Connectedness (TC) has been first introduced in (see Diebold and Yilmaz 2009). The measure of connectedness is evaluated on the base of an average of the entries of the so called Connectedness Table, a matrix where the entries depends on a “variance decomposition matrix” and the estimation of a Vector autoregression (VAR) model. For more details on the calculation of TC see (see Diebold and Yilmaz 2009). Considering that the value of TC depends on the estimation of an econometric model, it is not immediate to prove if the TC belongs to the class of PMCs. In our opinion, this is behind the scope of the present paper. Nevertheless, given the importance of the TC in the framework of connectedness measures, we decided to include it in our empirical experiment as a benchmark for the PMCs.

5.2 The Cumulative Risk Fraction

The notion of Cumulative Risk Fraction (see Billio et al. 2012) is defined as the portion of the variability of the returns explained by the first principal components.

Definition 7 (Cumulative Risk Fraction) The *Cumulative Risk Fraction* h_k, $k = 1, \dots, n$, is a real-valued function of the matrices $A \in \mathcal{M}_{m\times n}$ defined as

$$h_k(A) = \frac{\sum_{j=1}^{k} \sigma_j^2(A)}{\sum_{j=1}^{n} \sigma_j^2(A)}.$$

Proposition 5 *The Cumulative Risk Fraction h_k is a PMC, for each $k \in \{1, \dots, n\}$.*

Proof Let $k \in \{1, \dots, n\}$. From Definition 7 each h_k is a real-valued function defined on the full-rank matrices $\mathcal{M}_{m\times n}$.

Let $A \in \mathcal{M}_{m\times n}$ and $\sigma_1(A) \geq \dots \geq \sigma_n(A) > 0$ be its singular values. Obviously, we have

$$0 < h_k(A) = \frac{\sum_{j=1}^{k} \sigma_j^2(A)}{\sum_{j=1}^{n} \sigma_j^2(A)} \leq 1,$$

therefore Property 1 is satisfied.

To prove Property 2 it is enough to observe that the singular values of a matrix do not change under rows or columns permutation.

To prove Property 3, it is enough to note that h_k is a positive homogeneous function of degree 0 (see Remark 1).

Now we prove Property 4. Recall that the assumptions $|\rho(a_1, a_2)| \geq |\rho(a_1, a_3)|$, $\|a_2\| = \|a_3\|$ and $\langle \mathbf{1}, a_2\rangle = \langle \mathbf{1}, a_3\rangle = 0$ are equivalent to $\langle a_1, a_2\rangle^2 \geq \langle a_1, a_3\rangle^2$. Since $n = 2$, then $k \in \{1, 2\}$. If $k = 1$, then:

$$h_1(A_{1j}) = \frac{\sigma_1^2(A_{1j})}{\sigma_1^2(A_{1j}) + \sigma_2^2(A_{1j})} = \frac{1}{1 + \left(\frac{\sigma_2(A_{1j})}{\sigma_1(A_{1j})}\right)^2}, \quad j \in \{2, 3\}. \tag{8}$$

Arguing in the same way as in the proof of Proposition 1 (see formula (3) and (4)), we obtain

$$\frac{\sigma_1(A_{12})}{\sigma_2(A_{12})} \geq \frac{\sigma_1(A_{13})}{\sigma_2(A_{13})}$$

from which, using expression (8) it follows $h_1(A_{12}) \geq h_1(A_{13})$.

Finally, if $k = 2$, then:

$$h_2(A_{1j}) = \frac{\sigma_1^2(A_{1j}) + \sigma_2^2(A_{1j})}{\sigma_1^2(A_{1j}) + \sigma_2^2(A_{1j})} = 1, \quad j \in \{2, 3\},$$

so the proof is concluded.

5.3 The Market Rank Indicator

The notion of Market Rank Indicator (see Figini et al. 2018) is defined as a ratio of some of the eigenvalues of a given matrix and it is useful to detect the collinearity between the columns of the input matrix.

Definition 8 (Market rank indicator) (see Figini et al. 2018) The *market rank indicator* s_k, $k = 1, \ldots, n$, is a real-valued function of the matrices $A \in \mathcal{M}_{m\times n}$ defined as

$$s_k(A) = \frac{\sigma_1(A)}{\left(\prod_{j=1}^{k} \sigma_{n-j+1}(A)\right)^{\frac{1}{k}}}.$$

We point out that s_k is a special case of the Power Mean Measure μ_k^p when $k = 0$, i.e. $s_k = \mu_k^0$ for each $k \in \{1, \ldots, n\}$. Therefore, from Proposition 3, it immediately follows that s_k is a PMC. In particular, from Proposition 4, item 2, it follows that

$$s_k(A) \geq \mu_k^p(A), \quad \forall p > 0 \text{ or } p = +\infty.$$

5.4 The Average Correlation

We recall the definition of *Average Correlation* commonly used to measure the internal reliability of the set of variables in a matrix.

Definition 9 (Average correlation) The *average correlation* c_{ave} is a real-valued function of the matrices $A \in \mathcal{M}_{m\times n}$ defined as

$$c_{ave}(A) = \frac{2}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} |\rho(A^i, A^j)|, \tag{9}$$

where $A^1, \ldots, A^n$ denote the columns of A and $\rho(A^i, A^j)$ is the *Pearson correlation coefficient* between A^i and A^j.

Proposition 6 *The average correlation c_{ave} is a PMC.*

Proof From Definition 9 the average correlation c_{ave} is a real-valued function defined over the full-rank matrices $\mathcal{M}_{m\times n}$.

Property 1 immediately follows from the definition.

To prove Property 2 it is enough to observe that any permutation of the columns of A does not change (9).

To prove Property 3, we show that c_{ave} is a positive homogeneous function of degree 0 (see Remark 1). Recalling that the correlation coefficient ρ is invariant under (even different) scale changes in the two arguments, it easily follows that, for each $\alpha > 0$:

$$
\begin{aligned}
c_{ave}(\alpha A) &= \frac{2}{n(n-1)} \sum_{i,j=1,i<j}^{n} |\rho(\alpha A^i, \alpha A^j)| = \\
&= \frac{2}{n(n-1)} \sum_{i,j=1,i<j}^{n} |\rho(A^i, A^j)| = c_{ave}(A).
\end{aligned}
$$

Regarding Property 4, since $n = 2$, the average correlation is simply

$$c_{ave}(A_{1j}) = |\rho(a_1, a_j)|, \qquad j \in \{2, 3\},$$

therefore the statement immediately follows.

5.5 The Variance Inflation Factor

The *Variance Inflation Factors (VIFs)* are measures commonly used by econometricians to detect the presence of collinearity in a multiple linear model, see for instance Belsey et al. (1980). From the VIFs of the data matrix, it is possible to obtain a measure of connectedness as follows.

Definition 10 (Maximum Variance Inflation Factor - M-VIF) Let $A \in \mathcal{M}_{m\times n}$ and $A^1 \ldots, A^n$ be the columns of A. The *Variance Inflation Factors* of A are defined by

$$\mathrm{VIF}_j(A) = \frac{1}{1 - R_j^2}, j = 1, 2, \ldots, n,$$

where R_j^2 is the *coefficient of determination* of the linear regression of A^j with respect to $\{A^i \mid i = 1, \ldots, n, i \neq j\}$. The *Maximum Variance Inflation Factor* M-VIF(A) is defined as

$$\text{M-VIF}(A) = \max\{\mathrm{VIF}_1(A), \ldots, \mathrm{VIF}_n(A)\}.$$

Proposition 7 *The Maximum Variance Inflation Factor M-VIF is a PMC.*

Proof From Definition 10, M-VIF is a real-valued function defined over the full-rank matrices $\mathcal{M}_{m\times n}$.

To prove Property 1 we recall that for each $j \in \{1,\ldots,n\}$ the coefficient of determination $R_j^2 \in [0,1]$, therefore $\mathrm{VIF}_j(A) \geq 1$, and so M-VIF$(A) \geq 1 > 0$.

To prove Property 2 it is enough to observe that any permutation of the columns of A induces the same permutation in the set of values $\{\mathrm{VIF}_j(A)\,|\,j \in \{1,\ldots,n\}\}$, but this does not affect their maximum value, and so M-VIF does not change.

To prove Property 3, it is enough to show that the function VIF(A) is homogeneous of degree 0 (see Remark 1). Recalling that, for each j, the coefficient of determination R_j^2 is invariant under any scale change of the problem, it follows that the same holds for $\mathrm{VIF}_j(A)$, and this allows us to conclude that M-VIF$(\alpha A) =$ M-VIF(A) for each $\alpha > 0$.

Finally, we consider Property 4. First of all, we recall that with $n = 2$ the two VIFs of a data matrix are equal. Therefore, we need only to show that $\mathrm{VIF}_1(A_{12}) \geq \mathrm{VIF}_1(A_{13})$ or, equivalently, that $R_1^2(A_{12}) \geq R_1^2(A_{13})$. Thanks to the definition of the coefficient of determination we obtain, for $j \in \{2,3\}$,

$$R_1^2(A_{1j}) = \frac{\beta_{1,j}^2\|a_j\|^2}{\|a_1\|^2},$$

where $\beta_{1,j} = \frac{\mathrm{cov}(a_1,a_j)}{\mathrm{Var}(a_j)}$ is the slope of the regression line, and therefore

$$R_1^2(A_{1j}) = \rho(a_1,a_j)^2\frac{\mathrm{Var}(a_1)}{\mathrm{Var}(a_j)}\frac{\|a_j\|^2}{\|a_1\|^2}.$$

Since the assumptions $\|a_2\| = \|a_3\|$ and $\langle \mathbf{1}, a_2\rangle = \langle \mathbf{1}, a_3\rangle = 0$ lead to $\mathrm{Var}(a_2) = \mathrm{Var}(a_3)$, using the hypothesis $|\rho(a_1,a_2)| \geq |\rho(a_1,a_3)|$ we get

$$R_1^2(A_{12}) \geq R_1^2(A_{13}),$$

so that M-VIF$_1(A_{12}) \geq$ M-VIF$_1(A_{13})$.

5.6 The Mahalanobis distance

The *Mahalanobis distance* is a measure of a distance between a point and a distribution, see Mahalanobis (1936).

Definition 11 (Mahalanobis distance) The *Mahalanobis distance* d_M is a real-valued function of the matrices $A \in \mathcal{M}_{m\times n}$ defined by:

$$d_M(A) = \sqrt{A_m S_A^{-1} A_m^t},$$

where $S_A \in \mathcal{M}_{n\times n}$ is the covariance matrix of the column vectors $A^1,\ldots,A^n$ of A and $A_m \in \mathrm{Mat}_{1\times n}(\mathbb{R})$ is the mth row vector of A.

The Mahalanobis distance is not a PMC: indeed, it satisfies Property 1, 2 and 3, as proved in Proposition 8, but it does not satisfy Property 4, as shown in Example 1.

Proposition 8 *The Mahalanobis distance d_M satisfies Property 1, 2, 3 of PMCs.*

Proof Property 1 follows from the definition of d_M.

To prove Property 2, let $\Pi \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a permutation matrix and $A\Pi \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. Let $S_{A\Pi}$ be the covariance matrix of the column vectors of $A\Pi$ and $(A\Pi)_m \in \mathrm{Mat}_{1\times \mathrm{n}}(\mathbb{R})$ be the mth row of $A\Pi$. It is immediate to verify that $S_{A\Pi} = \Pi^t S_A \Pi$ and $(A\Pi)_m = A_m\Pi$. Therefore

$$\begin{aligned} d_M(A\Pi) &= \sqrt{(A\Pi)_m S_{A\Pi}^{-1} (A\Pi)_m^t} = \sqrt{A_m\Pi(\Pi^t S_A \Pi)^{-1}(A_m\Pi)^t} = \\ &= \sqrt{A_m\Pi\, \Pi^{-1} S_A^{-1} (\Pi^t)^{-1}\Pi^t A_m^t} = \sqrt{A_m S_A^{-1} A_m^t} = d_M(A), \end{aligned}$$

thus Property 2 is proved.

Regarding Property 3, it is enough to show that d_M is a positive homogeneous function of degree 0 (see Remark 1). To this end, we consider the matrix αA, with $\alpha > 0$. The covariance matrix of the columns of αA is $S_{\alpha A} = \alpha^2 S_A$, and the m-th row of αA is $(\alpha A)_m = \alpha A_m$; therefore

$$d_M(\alpha A) = \sqrt{(\alpha A)_m S_{\alpha A}^{-1} (\alpha A)_m^t} = \sqrt{\alpha A_m (\alpha^2 S_A)^{-1} \alpha A_m^t} = d_M(A).$$

Example 1 We consider $a_1, a_2, a_3 \in \mathrm{Mat}_{4\times 1}(\mathbb{R})$, $A_{12} = (a_1 \mid a_2)$ and $A_{13} = (a_1 \mid a_3)$ in $\mathcal{M}_{4\times 2}$. Note that the assumptions $\|a_2\| = \|a_3\|$ and $\langle \mathbf{1}, a_2\rangle = \langle \mathbf{1}, a_3\rangle = 0$ of Property 4 are satisfied.

In the following two cases we compare the ordering of $|\rho(a_1, a_2)|$ and $|\rho(a_1, a_3)|$ with the ordering of the Mahalanobis distance $d_M(A_{12})$ and $d_M(A_{13})$.

First case:

$$a_1 = \begin{pmatrix} -0.7609 \\ 0.6421 \\ 0.0880 \\ 0.0309 \end{pmatrix} \qquad a_2 = \begin{pmatrix} -0.6485 \\ 0.6718 \\ 0.2411 \\ -0.2644 \end{pmatrix} \qquad a_3 = \begin{pmatrix} 0.6213 \\ 0.2844 \\ -0.7007 \\ -0.2050 \end{pmatrix}$$

$$\begin{aligned} |\rho(a_1, a_2)| &= 0.9379 > 0.3581 = |\rho(a_1, a_3)| \\ d_M(A_{12}) &= 1.4659 > 0.3638 = d_M(A_{13}). \end{aligned}$$

Second case:

$$a_1 = \begin{pmatrix} 0.0693 \\ 0.7904 \\ -0.4511 \\ -0.4086 \end{pmatrix} \qquad a_2 = \begin{pmatrix} -0.8558 \\ 0.2244 \\ 0.4102 \\ 0.2213 \end{pmatrix} \qquad a_3 = \begin{pmatrix} -0.1810 \\ 0.1147 \\ -0.6567 \\ 0.7230 \end{pmatrix}$$

$$|\rho(a_1, a_2)| = 0.1574 > 0.0790 = |\rho(a_1, a_3)|$$
$$d_M(A_{12}) = 0.7593 < 1.4910 = d_M(A_{13}).$$

Since, under the assumption $|\rho(a_1, a_2)| > |\rho(a_1, a_3)|$, both cases $d_M(A_{12}) < d_M(A_{13})$ or $d_M(A_{12}) < d_M(A_{13})$ may occur, we conclude that the Mahalanobis distance does not satisfy Property 4.

6 Empirical application

In this section various measures of connectedness are considered and compared according to their capabilities to forecast the risk of a market. To this aim, following Figini et al. (2018), we build up a financial application in which, for each considered measure of connectedness, the capability to detect possible financial tensions in the system is related to an increase in the probability of overall financial losses. We measure the market risk of losses by means of the Value at Risk at 5% level ($\mathrm{VaR}_{5\%}$) of a market index. We consider three financial indexes: the S&P (SPX), Eurostoxx (EUROSTOXX50) and DAX (DAX30), together with their components. For the SPX we use its sector sub-indexes[1] instead of its components. For all datasets, the observations are daily from July 9, 2005 to June 5, 2019.[2] We consider the measures of connectedness of Section 4 and Section 5, namely:

- The Total Connectedness (TC) (see Section 5.1).
- The Cumulative Risk Fraction (CRF) (see Section 5.2).
- The Power Mean Measure μ_k^p (see Section 4.1) with various parameter settings:
  (i) μ_1^p, i.e. the condition number $\mathcal{K}$ (note that the value of p is not relevant);
  (ii) μ_3^0, i.e. the market rank indicator s_3 (see Section 5.3);
  (iii) μ_3^2.
  Note that, in cases (ii) and (iii), the choice $k = 3$ turns out to be a good trade off between the number of dimensions and the information about the numerical rank of the data matrix.
- The average correlation (AC) (see Section 5.4).
- The Maximum Variance Inflation Factor (M-VIF) (see Section 5.5).
- The Mahalanobis distance (M) (see Section 5.6).

We remark that all the above measures except TC and M are proper measures of connectedness.

In order to investigate the behavior of the distribution of each index's return conditioned on the value of the above measures of connectedness we proceed as follows. Using the components or the sector sub-indexes of each market, whose cardinality is denoted by n, we compute the value of each measure of connectedness on a sliding window of length w_e days, and relate the value of the measure to the return on 20 days (i.e. one month) ahead.

[1] The sectors obtained from Bloomberg are the 10 level 1 sector aggregations.
[2] Remark that the sample period covers the global financial crisis of 2008.

Although the computation of each measure of connectedness is feasible for sliding windows of length $w_e \geq n$, we opt for a longer estimation window of length $w_e = 3n$, that is, we set $w_e = 30$ for the SPX, $w_e = 141$ for the EUROSTOXX50 and $w_e = 81$ for the DAX30. This choice is related to the computation of TC that, being based on a VAR model, performs better using longer estimation windows. Indeed, we observe that this settings could favour TC, as higher values of w_e may turn the other measures to be less reactive and, consequently, may worsen their performances. However, a detailed analysis of the optimal settings is out of the scope of this application.

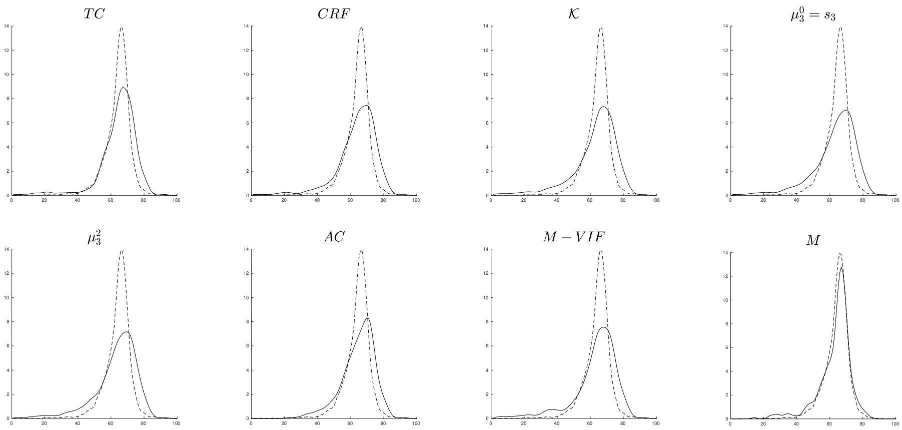

Fig. 1 SPX. Comparison between unconditional (dashed) and conditional (solid) densities of the 20-days ahead return of the index return. The conditional densities are conditioned to values of the connectedness indicators belonging to their largest 10% quantile.

In Figures 1, 2 and 3 we compare the unconditional distributions of the 20-days ahead return of the the financial indexes SPX, EUROSTOXX50 and DAX30 respectively, with the corresponding conditional distributions, conditioned to the values of each considered measure of connectedness belonging to their largest 10% quantile. We observe that, in order to be useful as an early warning signal, the conditional distribution should display a fatter left tail than the unconditional one; moreover, larger differences between the left tails of the two distributions lead to better forecasting performances. In our application we note that, as a general tendency, the conditional distributions present fatter tails, mainly the left ones, if compared to the unconditional ones, and this could be interpreted assessing that high values of the connectedness indicators can anticipate an increase of the market risk. Further, it is evident that the considered measures of connectedness show different performances: in particular, the Power Mean Measures and, in general, the proper measures of connectedness display overall good performances. On the other side, the worst results are given by the Mahalanobis measure, which is not a proper measure of connectedness.

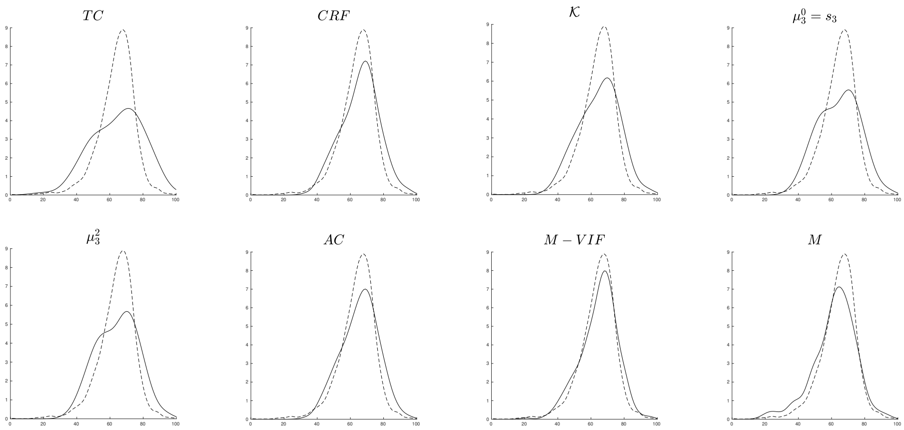

Fig. 2 EUROSTOXX50. Comparison between unconditional (dashed) and conditional (solid) densities of the 20-days ahead return of the index return. The conditional densities are conditioned to values of the connectedness indicators belonging to their largest 10% quantile.

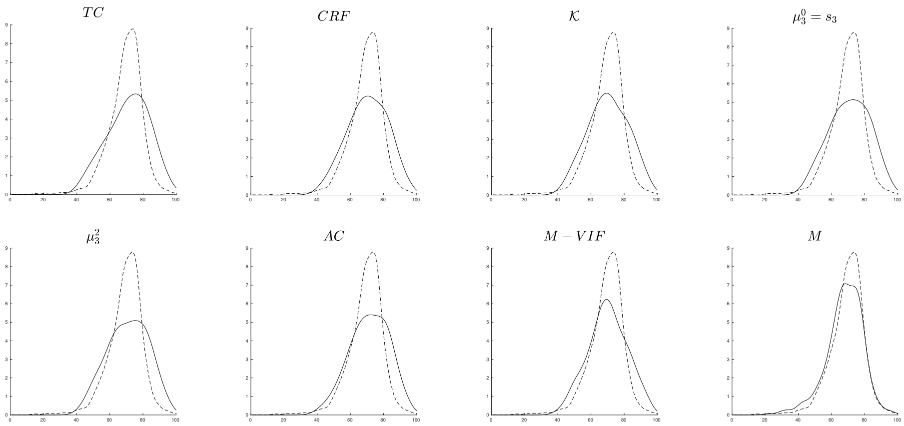

Fig. 3 DAX30. Comparison between unconditional (dashed) and conditional (solid) densities of the 20-days ahead return of the index return. The conditional densities are conditioned to values of the connectedness indicators belonging to their largest 10% quantile.

In the following, using the above setting, we propose an alternative way to estimate the forecasting capability of the various measures of connectedness. We compare the Value at Risk at 5% level of the 20-days ahead returns of each financial index, SPX, EUROSTOXX50 and DAX30, conditioned to low and high values of the measures of connectedness computed on estimation windows on length $w_e = 30$ for the SPX, $w_e = 141$ for the EUROSTOXX50 and $w_e = 81$ for the DAX30. We consider two different quantile levels, namely 25% and 10%, and gather the corresponding results in Table 1, subtables (a) and (b), respectively. More in details, each subtable of Table 1 reports the Value at Risk at 5% level of the return of the considered financial market, conditioned to the value of the measures of connectedness listed in the column labeled with "Measures". The column labeled with q_l contains the $\mathrm{VaR}_{5\%}$ conditioned to the measure's values belonging to its lowest quantile; analogously, the column labeled with q_h contains the $\mathrm{VaR}_{5\%}$ conditioned to the measure's value belonging to its highest quantile; finally, the column labeled with Δ contains the difference between the two previous values. Note that a large value of Δ indicates that the measure can well discriminate between periods with low and large VaR. Therefore, this difference may be used as a performance measure of the various connectedness indicators. From Table 1 we observe that in all but two cases the values of Δ are strictly positive. Further, in most cases, proper measures of connectedness yield the best results. In fact, except for the case of the EUROSTOXX50 and 0.10 quantiles, proper measures outperform non proper ones. We also remark that the class of Power Mean Measures shows consistently good results, whereas the bad performances of CRF and AC corresponding to the DAX30 in the case of 0.10 quantiles may highlight the fact that these kind of measures strongly rely on the linear correlation structure of the returns.

(a) Quantile level 25%

	SPX			EUROSTOXX50			DAX30		
Measures	q_l	q_h	Δ	q_l	q_h	Δ	q_l	q_h	Δ
TC	0.0559	0.0891	0.0332	0.0623	0.1112	0.0489	0.0816	0.1031	0.0216
CRF	0.0483	0.1037	0.0554	0.0722	0.0984	0.0262	0.0833	0.1027	0.0194
$\mathcal{K}$	0.0469	0.1046	0.0577	0.0648	0.1240	0.0592	0.0776	0.1176	0.0400
$\mu_3^0 = s_3$	0.0478	0.1210	0.0732	0.0637	0.1106	0.0468	0.0762	0.1048	0.0286
μ_3^3	0.0474	0.1236	0.0762	0.0637	0.1106	0.0468	0.0794	0.1031	0.0238
AC	0.0496	0.1018	0.0522	0.0637	0.0969	0.0332	0.0855	0.0973	0.0118
M-VIF	0.0518	0.1192	0.0675	0.0683	0.1240	0.0558	0.0715	0.1008	0.0294
M	0.0696	0.0917	0.0221	0.0857	0.1169	0.0312	0.0840	0.1014	0.0174

(b) Quantile level 10%

	SPX			EUROSTOXX50			DAX30		
Measures	q_l	q_h	Δ	q_l	q_h	Δ	q_l	q_h	Δ
TC	0.0487	0.1067	0.0581	0.0661	0.1249	0.0588	0.0835	0.1061	0.0226
CRF	0.0468	0.1293	0.0825	0.0718	0.0936	0.0218	0.1019	0.1004	-0.0015
$\mathcal{K}$	0.0483	0.1634	0.1151	0.0707	0.1042	0.0335	0.0735	0.0994	0.0258
$\mu_3^0 = s_3$	0.0528	0.154	0.1012	0.0677	0.1032	0.0354	0.0735	0.0994	0.0258
μ_3^3	0.0528	0.1517	0.0988	0.0677	0.1032	0.0354	0.0735	0.0994	0.0258
AC	0.0487	0.1021	0.0534	0.0677	0.0927	0.0251	0.1035	0.0922	-0.0113
M-VIF	0.0579	0.1668	0.1089	0.0659	0.1013	0.0354	0.0720	0.1004	0.0284
M	0.0785	0.1018	0.0233	0.0791	0.1351	0.0561	0.0893	0.1128	0.0235

Table 1 Value at Risk at 5% level of the return of the three indexes, conditioned to the value of the listed measures of connectedness for quantile levels 25% (table (a)) and 10% (table (b)). In columns q_l the $\mathrm{VaR}_{5\%}$ is conditioned to the measure's values belonging to its lowest quantile; in columns q_h the $\mathrm{VaR}_{5\%}$ is conditioned to the measure's values belonging to its lowest quantile; columns Δ report the difference between the two previous values.

From this empirical application, we can conclude that proper measures of connectedness are able to detect some tensions on the financial markets related to an increase in the co-movement or the similarity of the returns of the component of a market. This can be related to an increase in the riskiness of the market, resulting in a larger probability of market losses.

7 Conclusions

In this paper we provide a rigorous definition of the concept of connectedness, identifying the class of Proper Measures of Connectedness (PMCs) through a set of minimal required properties. These properties are justified from a theoretical point of view. We propose a general class of PMCs, proving they satisfy the minimal required properties. We also proved some interesting properties of the proposed class of measures. The interpretation of the concept of connectedness is determined by the meaning of the required minimal properties and its relations to some well known concepts of linear algebra; for example, we show that the class of the so called Power Mean Measure is a generalization of the condition number of a matrix. We also investigate if the most popular measure of connectedness are included in the proposed general theoretical framework. The empirical exercise on real financial data highlights the following results. First, the PMCs seem to be a useful instrument to forecast systemic risk and financial bubbles/crashes. Second, the measures belonging to the class of the PMCs show a clear forecasting superiority with respect to the measures that do not satisfy the minimal properties.

Declarations

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Conflict of interest The authors declare that they have no conflict of interest.
Availability of data and material Data are obtained from the Bloomberg database, provided by the University of Pavia.
Code availability The codes written by the authors are not published.
Author's contributions All authors contributed equally to this paper.

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