

Hybrid Proximal Methods for Equilibrium Problems*

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Abstract. This paper concerns developing two hybrid proximal point methods (PPMs) for finding a common solution of some optimization-related problems. First we construct an algorithm to solve simultaneously an equilibrium problem and a variational inequality problem, combining the extragradient method for variational inequalities with an approximate PPM for equilibrium problems. Next we develop another algorithm based on an alternate approximate PPM for finding a common solution of two different equilibrium problems. We prove the global convergence of both algorithms under pseudomonotonicity assumptions.

Keywords: equilibrium problems, variational inequalities, hybrid proximal point methods, extragradient methods, variational analysis.

1 Introduction

The proximal point method (PPM), introduced in [18] and further developed in [23], has been used for solving optimization problems and variational inequality problems (VIPs) in the monotone case. There are many interesting papers on this subject; see, e.g., [4, 13, 25] and the references therein. In further developments the PPM has also been applied for solving vector optimization problems (VOPs) as, e.g., in [3] and equilibrium problems (EPs) as in, e.g., [9, 12, 16]. Quite recently several versions of the so-called hybrid approximate proximal method (HAPM) have been introduced and developed in [6] for finding a common solution of VOPs and VIPs. The latter method is based on combining some ideas of the PPM for solving VOPs and the extragradient method to solve VIPs, initiated earlier in [17] and then developed in [14]. Let us finally mention that some variants of the PPM have been recently studied in [8, 22] and the bibliographies therein to solve EPs in combination with fixed point problems.

In this paper we first develop, partly following the approach of [6], a *hybrid proximal algorithm* for finding a common solution of a VIP and an EP by combining the extragradient method for VIPs with an approximate version of the PPM for EPs. Next we construct a hybrid algorithm for finding a common solutions of two different EPs based on an alternate approximate PPM of the HAPM type. We prove that both methods are globally convergent under certain *pseudomonotonicity* assumptions in finite-dimensional spaces.

The general equilibrium problem under consideration in this paper is formulated as follows. Given a closed and convex subset C of \mathbb{R}^n and a bifunction $f: C \times C \rightarrow \mathbb{R}$ with $f(x, x) = 0$ for all $x \in C$, the EP

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consists of finding a point $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0 \text{ for all } y \in C. \quad (\text{EP})$$

Denote by $\mathcal{S}(\text{EP})$ the solution set of (EP) and observe that this model is a common roof for a variety of optimization-related and equilibrium problems including constrained optimization, variational inequalities, Nash equilibria, etc. We particularly refer the reader to [2, 15] and the bibliographies therein for excellent surveys concerning a large spectrum of optimization and equilibrium models that can be written in form (EP).

An important class of optimization-related problems described in the equilibrium form (EP) involves variational inequalities of the following type. Given a mapping $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$, consider the variational inequality problem: find $\bar{x} \in C$ such that

$$\langle F(\bar{x}), y - \bar{x} \rangle \geq 0 \text{ for all } y \in C, \quad (\text{VIP})$$

where $\langle \cdot, \cdot \rangle$ signifies the usual inner product in \mathbb{R}^n . We denote by $\mathcal{S}(\text{VIP})$ the solution set for (VIP) and can easily see that (VIP) is a special case of (EP) with $f(x, y) = \langle F(x), y - x \rangle$.

Recall that a bifunction $f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is pseudomonotone on $C \times C$ if

$$f(x, y) \geq 0 \implies f(y, x) \leq 0 \text{ for all } x, y \in C. \quad (1)$$

A mapping $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is pseudomonotone on C if

$$\langle F(x), y - x \rangle \geq 0 \implies \langle F(y), y - x \rangle \geq 0 \text{ for all } x, y \in C. \quad (2)$$

It is obvious from the definitions that the pseudomonotonicity of F in (2) is equivalent to the pseudomonotonicity of $f(x, y) = \langle F(x), y - x \rangle$ on $C \times C$ in the sense of (1).

The rest of the paper is organized as follows. Section 2 is devoted to developing a new hybrid version of the PPM to find a common solution of the EP and VIP defined above. We give a description of the algorithm and prove its global convergence under appropriate assumptions. Furthermore, we discuss certain modifications of the new algorithm in the case of unknown information and also efficient implementations of some steps.

In Section 3 we develop a hybrid approximate proximal algorithm to solve simultaneously two different equilibrium problems. The main result justifies the global convergence of this algorithm under the pseudomonotonicity and upper semicontinuity assumptions imposed on both equilibrium functions in the problems under consideration.

2 Hybrid PPM for Common Solutions to EP and VIP

In this section we develop a globally convergent hybrid PPM for finding a common solution of an EP and a VIP. The main idea of the algorithm (called below Algorithm 1) is to find first an approximate solution of some Auxiliary Equilibrium Problem, applying a PPM step for EPs, and then to apply a step of the extragradient method for the corresponding VIP.

Algorithm 1

1. **(Initial Step).** Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, $\{\delta_k\}$, and $\{\varepsilon_k\}$ be nonnegative sequences of real numbers such that $\{\alpha_k\}$ is a positive sequence bounded from above, $\{\beta_k\} \subset [\beta', \beta'']$ for some $\beta', \beta'' \in (0, 1)$, $\{\delta_k\} \subset [\delta', \delta'']$ for some $\delta', \delta'' \in (0, 1)$, and $\sum_{k=0}^{\infty} \varepsilon_k < \infty$. Pick an arbitrary vector $x_0 \in C$ and set $k = 0$.

2. **(Stopping Criterion).** If $x_k \in \mathcal{S}(\text{EP}) \cap \mathcal{S}(\text{VIP})$, then stop.

3. **(Proximal Step).** Find $y_k \in C$ such that $\|y_k - \bar{y}_k\| \leq \varepsilon_k$, where \bar{y}_k solves the following *Auxiliary Equilibrium Problem* (AEP):

$$f_k(\bar{y}_k, y) \geq 0 \quad \text{for all } y \in C \quad (3)$$

with $f_k(x, y) := f(x, y) + \alpha_k \langle x - x_k, y - x \rangle$. Set further $z_k := \beta_k x_k + (1 - \beta_k)y_k$.

4. **(Extragradient Step).** Compute

$$\bar{v}_k := P_C(z_k - \gamma_k F(z_k)) \quad \text{and} \quad v_k := P_C(z_k - \gamma_k F(\bar{v}_k)),$$

where P_C denotes the Euclidean projection onto the set C .

5. **(Update of x_k).** Set $x_{k+1} := \delta_k x_k + (1 - \delta_k)v_k$ and then go to Step 2 with $k = k + 1$.

It is clear from the constructions above that Algorithm 1 is well-defined; see Remark 2.2 for more clarification on Step 3. The following theorem justifies the global convergence of this method under appropriate conditions on the initial data of the EP, VIP, and Algorithm 1.

Theorem 2.1. (global convergence of Algorithm 1). *Assume that $\mathcal{S}(\text{EP}) \cap \mathcal{S}(\text{VIP}) \neq \emptyset$, that F is pseudomonotone and Lipschitz continuous with constant $L > 0$ on C , that $\{\gamma_k\} \subset [\gamma', \gamma'']$ for some $\gamma', \gamma'' \in (0, 1/L)$, that f is pseudomonotone on $C \times C$, $f(\cdot, y)$ is upper semicontinuous on C for all $y \in C$, and that the AEP in (3) has a solution for each $k \geq 0$. Then the sequence $\{x_k\}$ generated by Algorithm 1 converges to a common element of $\mathcal{S}(\text{EP})$ and $\mathcal{S}(\text{VIP})$.*

Proof. Let \bar{x} be any element of the intersection $\mathcal{S}(\text{EP}) \cap \mathcal{S}(\text{VIP})$. Then

$$\|x_k - \bar{x}\|^2 = \|x_k - \bar{y}_k + \bar{y}_k - \bar{x}\|^2 = \|x_k - \bar{y}_k\|^2 + \|\bar{y}_k - \bar{x}\|^2 + 2\langle x_k - \bar{y}_k, \bar{y}_k - \bar{x} \rangle.$$

By definition of \bar{y}_k and by the pseudomonotonicity of f , we get

$$\alpha_k \langle x_k - \bar{y}_k, \bar{y}_k - \bar{x} \rangle \geq -f(\bar{y}_k, \bar{x}) \geq 0,$$

which implies the lower estimate

$$\|x_k - \bar{x}\|^2 \geq \|x_k - \bar{y}_k\|^2 + \|\bar{y}_k - \bar{x}\|^2. \quad (4)$$

It follows from (4) and the definition of y_k that

$$\|y_k - \bar{x}\| \leq \|y_k - \bar{y}_k\| + \|\bar{y}_k - \bar{x}\| \leq \varepsilon_k + \|x_k - \bar{x}\|.$$

Taking the latter estimate into account gives us

$$\begin{aligned} \|z_k - \bar{x}\| &= \|\beta_k(x_k - \bar{x}) + (1 - \beta_k)(y_k - \bar{x})\| \\ &\leq \beta_k \|x_k - \bar{x}\| + (1 - \beta_k) \|y_k - \bar{x}\| \\ &\leq \beta_k \|x_k - \bar{x}\| + (1 - \beta_k) [\varepsilon_k + \|x_k - \bar{x}\|] \\ &= \|x_k - \bar{x}\| + (1 - \beta_k) \varepsilon_k \\ &\leq \|x_k - \bar{x}\| + \varepsilon_k. \end{aligned} \quad (5)$$

On the other hand, by the properties of the Euclidean projection onto the convex set C and the *pseudomonotonicity* assumption on F we obtain the relationships:

$$\begin{aligned}
\|v_k - \bar{x}\|^2 &= \|z_k - \gamma_k F(\bar{v}_k) - v_k + v_k - \bar{x}\|^2 - \|z_k - \gamma_k F(\bar{v}_k) - v_k\|^2 \\
&\quad - 2\langle z_k - \gamma_k F(\bar{v}_k) - v_k, v_k - \bar{x} \rangle \\
&\leq \|z_k - \gamma_k F(\bar{v}_k) - \bar{x}\|^2 - \|z_k - \gamma_k F(\bar{v}_k) - v_k\|^2 \\
&= \|z_k - \bar{x}\|^2 - \|z_k - v_k\|^2 + 2\gamma_k \langle F(\bar{v}_k), \bar{x} - v_k \rangle \\
&= \|z_k - \bar{x}\|^2 - \|z_k - v_k\|^2 + 2\gamma_k \langle F(\bar{v}_k), \bar{x} - \bar{v}_k \rangle + 2\gamma_k \langle F(\bar{v}_k), \bar{v}_k - v_k \rangle \\
&\leq \|z_k - \bar{x}\|^2 - \|z_k - v_k\|^2 + 2\gamma_k \langle F(\bar{v}_k), \bar{v}_k - v_k \rangle \\
&= \|z_k - \bar{x}\|^2 - \|z_k - \bar{v}_k\|^2 - \|\bar{v}_k - v_k\|^2 - 2\langle z_k - \bar{v}_k, \bar{v}_k - v_k \rangle \\
&\quad + 2\gamma_k \langle F(\bar{v}_k), \bar{v}_k - v_k \rangle \\
&= \|z_k - \bar{x}\|^2 - \|z_k - \bar{v}_k\|^2 - \|\bar{v}_k - v_k\|^2 + 2\langle z_k - \gamma_k F(\bar{v}_k) - \bar{v}_k, v_k - \bar{v}_k \rangle \\
&= \|z_k - \bar{x}\|^2 - \|z_k - \bar{v}_k\|^2 - \|\bar{v}_k - v_k\|^2 + 2\langle z_k - \gamma_k F(z_k) - \bar{v}_k, v_k - \bar{v}_k \rangle \\
&\quad + 2\gamma_k \langle F(z_k) - F(\bar{v}_k), v_k - \bar{v}_k \rangle \\
&\leq \|z_k - \bar{x}\|^2 - \|z_k - \bar{v}_k\|^2 - \|\bar{v}_k - v_k\|^2 + 2\gamma_k \|F(z_k) - F(\bar{v}_k)\| \cdot \|v_k - \bar{v}_k\| \\
&\leq \|z_k - \bar{x}\|^2 - \|z_k - \bar{v}_k\|^2 - \|\bar{v}_k - v_k\|^2 + \|\bar{v}_k - v_k\|^2 \\
&\quad + \gamma_k^2 \|F(z_k) - F(\bar{v}_k)\|^2 \\
&= \|z_k - \bar{x}\|^2 - \|z_k - \bar{v}_k\|^2 + \gamma_k^2 \|F(z_k) - F(\bar{v}_k)\|^2.
\end{aligned} \tag{6}$$

Since F is *Lipschitz continuous* with constant L , it follows from (5) and (6) that

$$\begin{aligned}
\|v_k - \bar{x}\|^2 &\leq \|z_k - \bar{x}\|^2 - (1 - \gamma_k^2 L^2) \|z_k - \bar{v}_k\|^2 \\
&\leq (\varepsilon_k + \|x_k - \bar{x}\|)^2 - (1 - \gamma_k'^2 L^2) \|z_k - \bar{v}_k\|^2.
\end{aligned} \tag{7}$$

Furthermore, relationships (5) and (7) imply the estimates

$$\begin{aligned}
\|x_{k+1} - \bar{x}\| &= \|\delta_k x_k + (1 - \delta_k) v_k - \bar{x}\| \\
&= \|\delta_k (x_k - \bar{x}) + (1 - \delta_k) (v_k - \bar{x})\| \\
&\leq \delta_k \|x_k - \bar{x}\| + (1 - \delta_k) \|v_k - \bar{x}\| \\
&\leq \delta_k \|x_k - \bar{x}\| + (1 - \delta_k) [\varepsilon_k + \|x_k - \bar{x}\|] \\
&= \|x_k - \bar{x}\| + (1 - \delta_k) \varepsilon_k \\
&\leq \|x_k - \bar{x}\| + \varepsilon_k.
\end{aligned} \tag{8}$$

Employing next Lemma 1.1 from [10, Chapter 3], we get that the sequence $\{x_k\}$ is bounded in \mathbb{R}^n , and thus $\{\|x_k - \bar{x}\|\}$ converges as $k \rightarrow \infty$ with the limit

$$\mu := \lim_{k \rightarrow \infty} \|x_k - \bar{x}\|. \tag{9}$$

It also follows from the estimate in (7) that

$$\limsup_{k \rightarrow \infty} \|v_k - \bar{x}\| \leq \mu.$$

Taking now into account that

$$\lim_{k \rightarrow \infty} \|\delta_k (x_k - \bar{x}) + (1 - \delta_k) (v_k - \bar{x})\| = \lim_{k \rightarrow \infty} \|x_{k+1} - \bar{x}\| = \mu,$$

we obtain from [24] the limiting relationships

$$\lim_{k \rightarrow \infty} \|x_k - v_k\| = 0 \quad \text{and thus} \quad \lim_{k \rightarrow \infty} \|v_k - \bar{x}\| = \mu. \tag{10}$$

It is an immediate consequence of estimate (7) that

$$\|z_k - \bar{v}_k\|^2 \leq \frac{(\varepsilon_k + \|x_k - \bar{x}\|)^2 - \|v_k - \bar{x}\|^2}{1 - \gamma''^2 L^2}.$$

The latter implies by (9) and (10) that

$$\lim_{k \rightarrow \infty} \|z_k - \bar{v}_k\| = 0. \quad (11)$$

Moreover, we have the estimates

$$\|v_k - \bar{v}_k\| \leq \gamma_k \|F(z_k) - F(\bar{v}_k)\| \leq \gamma_k L \|z_k - \bar{v}_k\| \leq \|z_k - \bar{v}_k\|,$$

and thus $\lim_{k \rightarrow \infty} \|v_k - \bar{v}_k\| = 0$. This gives therefore the relationships

$$\lim_{k \rightarrow \infty} \|z_k - v_k\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|z_k - x_k\| = 0, \quad (12)$$

which imply in turn that

$$\lim_{k \rightarrow \infty} \|z_k - \bar{x}\| = \mu. \quad (13)$$

It follows from (4) the estimate $\limsup_{k \rightarrow \infty} \|y_k - \bar{x}\| \leq \mu$ and from (13) the equalities

$$\lim_{k \rightarrow \infty} \|\beta_k(x_k - \bar{x}) + (1 - \beta_k)(y_k - \bar{x})\| = \lim_{k \rightarrow \infty} \|z_k - \bar{x}\| = \mu.$$

Employing [24] again, we get $\lim_{k \rightarrow \infty} \|x_k - y_k\| = 0$, and thus

$$\lim_{k \rightarrow \infty} \|x_k - \bar{y}_k\| = 0. \quad (14)$$

Next let us consider any cluster point \tilde{x} of the sequence $\{x_k\}$, and let $\{x_{k_p}\}$ be a subsequence converging to \tilde{x} . It follows from (14) that the sequence $\{\bar{y}_{k_p}\}$ also converges to \tilde{x} as $p \rightarrow \infty$. By definition of \bar{y}_{k_p} we have furthermore that

$$f(\bar{y}_{k_p}, y) + \alpha_{k_p} \langle \bar{y}_{k_p} - x_{k_p}, y - \bar{y}_{k_p} \rangle \geq 0 \quad \text{for all } y \in C,$$

By passing to the limit as $p \rightarrow \infty$ in the latter inequalities and by using the *upper semicontinuity* of $f(\cdot, y)$, we arrive at the equilibrium condition

$$f(\tilde{x}, y) \geq 0 \quad \text{for all } y \in C,$$

i.e., \tilde{x} solves the EP. On the other hand, it follows from (11) and (12) that the sequences $\{z_{k_p}\}$ and $\{\bar{v}_{k_p}\}$ converge to \tilde{x} . Recall that we have by the construction above the relationships

$$\bar{v}_{k_p} = P_C(z_{k_p} - \gamma_{k_p} F(z_{k_p})) \quad \text{for all } p = 1, 2, \dots,$$

which imply by passing to the limit as $p \rightarrow \infty$ and by taking a subsequence if necessary that

$$\tilde{x} = P_C(\tilde{x} - \tilde{\gamma} F(\tilde{x}))$$

with some $\tilde{\gamma} \in [\gamma', \gamma'']$. Hence \tilde{x} also solves the VIP. It follows finally from (9) that

$$\lim_{k \rightarrow \infty} \|x_k - \tilde{x}\| = 0,$$

which signifies the convergence of the sequence $\{x_k\}$ to an element of $\mathcal{S}(\text{EP}) \cap \mathcal{S}(\text{VIP})$ and thus completes the proof of the theorem. \triangle

The following remarks present some modifications and clarifications of Algorithm 1.

Remark 2.1. (modification of Algorithm 1 with unknown Lipschitz constant). In the statement of Theorem 2.1 the choice of the steps γ_k is based on the exact knowledge of the Lipschitz constant L . We now show that, if such a constant is unknown, the steps γ_k can be computed adaptively; cf. also [20]. If we set parameters $\sigma, \tau \in (0, 1)$ and $\bar{\gamma} > 0$, then at each iteration the Extragradient Step can be replaced by the following procedure:

- 4a. Set $\gamma_k := \bar{\gamma}$ and compute $\bar{v}_k = P_C(z_k - \gamma_k F(z_k))$
- 4b. while $\gamma_k > \sigma \frac{\|z_k - \bar{v}_k\|}{\|F(z_k) - F(\bar{v}_k)\|}$ do
 - Set $\gamma_k := \tau \gamma_k$ and compute $\bar{v}_k = P_C(z_k - \gamma_k F(z_k))$
 - end
- 4c. Compute $v_k = P_C(z_k - \gamma_k F(\bar{v}_k))$.

Since F is Lipschitz continuous with constant L , we have the inequality

$$\sigma \frac{\|z_k - \bar{v}_k\|}{\|F(z_k) - F(\bar{v}_k)\|} \geq \frac{\sigma}{L},$$

and hence the “while” cycle is repeated a finite number of times. At the end of this procedure the step γ_k satisfies the upper estimate

$$\gamma_k \leq \sigma \frac{\|z_k - \bar{v}_k\|}{\|F(z_k) - F(\bar{v}_k)\|}.$$

Thus we obtain from (6) that

$$\|v_k - \bar{x}\|^2 \leq \|z_k - \bar{x}\|^2 - (1 - \sigma^2) \|z_k - \bar{v}_k\|^2.$$

It is easy to check that this procedure gives

$$\gamma_k \in [\min\{\bar{\gamma}, \sigma\tau/L\}, \bar{\gamma}].$$

Following now the proof of Theorem 2.1, we conclude that Algorithm 1, with the above procedure for finding the steps γ_k , converges to an element of the solution set intersection $\mathcal{S}(\text{EP}) \cap \mathcal{S}(\text{VIP})$.

The next remark clarifies the implementation of Proximal Step in Algorithm 1 to solve constructively the Auxiliary Equilibrium Problem (3).

Remark 2.2. (solving the AEP in Algorithm 1). At every iteration of Algorithm 1 we need to solve the AEP in Proximal Step. There is a rather developed existence theory of solutions to equilibrium problems of type (3); see, e.g., [15, 11, 5] and the references therein. However, the constructive implementation of Proximal Step in Algorithm 1 requires the usage of numerical methods converging to an optimal solution of (3) at each iteration of Algorithm 1. For this purpose we can use several descent methods based on gap functions developed in [1, 7, 19, 26]. In order to guarantee the convergence of such methods, it is sufficient to suppose that $f(x, \cdot)$ is convex on C while f is weakly ∇ -monotone on $C \times C$ with constant $\eta > 0$ in the sense that

$$\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x \rangle \geq -\eta \|x - y\|^2 \quad \text{for all } x, y \in C.$$

Indeed, under these assumptions with $\alpha_k \geq \bar{\alpha} > \eta$ for all $k \geq 0$ the bifunction f_k is strongly ∇ -monotone on $C \times C$, which allows us to employ the convergence results of [1, 7, 19].

Furthermore, in order to control the accuracy of an approximate solution to the AEP in Proximal Step we can use the approach described in [16] that is based on gap functions and allows us to compute an error bound of the obtained solution. Such an estimate is applied then to problem (3) provided that f is weakly monotone on $C \times C$ with constant $\eta > 0$, i.e.,

$$f(x, y) + f(y, x) \leq \eta \|x - y\|^2 \text{ for all } x, y \in C.$$

The latter assumption allows us to prove that the bifunction f_k with $\alpha_k \geq \bar{\alpha} > \eta$ in (3) is strongly monotone on $C \times C$; cf. [16] for more details and discussions.

2.1 Numerical examples

In the following we consider some numerical examples to test Algorithm 1. As stopping criterion we used the following error measure:

$$\text{error} := \max\{\|x - \arg \min_{y \in C} [f(x, y) + \|y - x\|^2]\|_\infty, \|x - P_C(x - F(x))\|_\infty\} < 10^{-4}.$$

At each iteration the AEP in Proximal Step was solved applying the descent method based on gap functions developed in [7] and the accuracy of the solution to the AEP was controlled using the approach described in [16]. We implemented Algorithm 1 in MATLAB 7.10 and we used the solver FMINCON from the Optimization Toolbox in order to evaluate the gap function in the descent method. Table 1 reports numerical results on the following three examples. For each starting point (column two) we report the found solution in column three, the number of iterations in column four, and the error of the found solution in column five.

Example 2.1. Let us consider the feasible set $C = [0, 1] \times [0, 1]$. The EP is defined by the bifunction $f(x, y) = (y_1 - y_2)^2 - (x_1 - x_2)^2$ which is monotone and ∇ -monotone on $C \times C$. Indeed, such EP is equivalent to an optimization problem. The solution set of EP is $\mathcal{S}(\text{EP}) = \{x \in C : x_1 = x_2\}$. The VIP is defined by the map $F(x) = (x_2, -x_1)$ which is monotone and Lipschitz continuous with constant 1 on C . The solution set of VIP is $\mathcal{S}(\text{VIP}) = \{x \in C : x_1 = 0\}$. We set the algorithm parameters as follows: $\alpha_k = 1$, $\beta_k = 0.01$, $\gamma_k = 0.5$, $\delta_k = 0.01$ for all $k \geq 0$, and $\varepsilon_k = 1/2^k$. We applied Algorithm 1 starting from 5 different points in the set C . It is shown in Table 1 that from any starting point Algorithm 1 reaches the solution $(0, 0)$ which is the unique element of $\mathcal{S}(\text{EP}) \cap \mathcal{S}(\text{VIP})$.

Example 2.2. Let $C = [0, 1] \times [0, 1]$. The EP is defined by the bifunction $f(x, y) = (x_1 + x_2 - 1)(y_1 - x_1) + (x_1 + x_2 - 1)(y_2 - x_2)$ which is monotone and ∇ -monotone on $C \times C$. Indeed, such EP is equivalent to a variational inequality problem. The solution set of EP is $\mathcal{S}(\text{EP}) = \{x \in C : x_1 + x_2 = 1\}$. The VIP is defined by the map $F(x) = (x_1 - x_2, x_2 - x_1)$ which is monotone and Lipschitz continuous with constant 2 on C . The solution set of VIP is $\mathcal{S}(\text{VIP}) = \{x \in C : x_1 = x_2\}$. We set the algorithm parameters as follows: $\alpha_k = 1$, $\beta_k = 0.01$, $\gamma_k = 0.25$, $\delta_k = 0.01$ for all $k \geq 0$, and $\varepsilon_k = 1/2^k$. We applied Algorithm 1 starting from 5 different points in the set C . It is shown in Table 1 that from any starting point Algorithm 1 reaches the solution $(0.5, 0.5)$ which is the unique element of $\mathcal{S}(\text{EP}) \cap \mathcal{S}(\text{VIP})$.

Example 2.3. Let $C = [0, 1] \times [0, 1]$. The EP is defined by $f(x, y) = (y_1 - x_1)(2y_1 + x_1)$. Such bifunction is pseudomonotone on $C \times C$, it weakly-monotone with constant 1 on $C \times C$ because

$$f(x, y) + f(y, x) = (x_1 - y_1)^2 \leq \|x - y\|^2 \text{ for all } x, y \in C,$$

and is ∇ -monotone on $C \times C$. The solution set of EP is $\mathcal{S}(\text{EP}) = \{x \in C : x_1 = 0\}$. The VIP is defined by the map $F(x) = (-x_2, x_1)$ which is monotone and Lipschitz continuous with constant 1 on C . The solution set of VIP is $\mathcal{S}(\text{VIP}) = \{x \in C : x_2 = 0\}$. We set the algorithm parameters as follows: $\alpha_k = 2$, $\beta_k = 0.01$,

Problem	Starting point	Solution	# Iterations	Error
Example 1	(0.569, 0.469)	(7.67e-16, 7.67e-16)	10	7.67e-16
	(0.012, 0.337)	(1.19e-62, 1.19e-62)	30	1.19e-62
	(0.162, 0.794)	(7.09e-17, 7.09e-17)	9	9.92e-05
	(0.311, 0.529)	(3.11e-63, 3.11e-63)	31	3.11e-63
	(0.263, 0.654)	(2.63e-19, 2.63e-19)	9	2.80e-06
Example 2	(0.757, 0.754)	(5.00e-01, 5.00e-01)	13	8.61e-05
	(0.585, 0.550)	(5.00e-01, 5.00e-01)	21	9.06e-05
	(0.076, 0.054)	(5.00e-01, 5.00e-01)	19	9.87e-05
	(0.569, 0.469)	(5.00e-01, 5.00e-01)	25	8.13e-05
	(0.380, 0.568)	(5.00e-01, 5.00e-01)	27	8.68e-05
Example 3	(0.929, 0.350)	(1.58e-04, 7.66e-22)	11	7.90e-05
	(0.197, 0.251)	(1.08e-04, 1.42e-22)	12	5.40e-05
	(0.616, 0.473)	(1.01e-04, 1.73e-20)	12	5.03e-05
	(0.119, 0.498)	(1.93e-04, 2.36e-16)	11	9.64e-05
	(0.960, 0.340)	(1.60e-04, 6.33e-22)	11	7.98e-05

Table 1: Numerical results for Algorithm 1 tested on Examples 2.1-2.3.

$\gamma_k = 0.5$, $\delta_k = 0.01$ for all $k \geq 0$, and $\varepsilon_k = 1/2^k$. We applied Algorithm 1 starting from 5 different points in the set C . It is shown in Table 1 that from any starting point Algorithm 1 reaches the solution $(0, 0)$ which is the unique element of $\mathcal{S}(\text{EP}) \cap \mathcal{S}(\text{VIP})$.

3 Hybrid PPM for common solutions to two different EPs

In this section we develop a hybrid approximate proximal algorithm to find a common solution of two different EPs defined by bifunctions f and g with the same feasible set C . We denote by $\mathcal{S}(\text{EP}_f)$ and $\mathcal{S}(\text{EP}_g)$ the correspondent sets of solutions.

The idea of the method (called below Algorithm 2) is the following: at each iteration we construct an approximate solution of the corresponding AEP with the bifunction f , then we exploit this solution to define another AEP with the bifunction g , and finally we construct an approximate solution of the latter problem and we use it to update the new iterate of the algorithm.

Algorithm 2

1. **(Initial Step).** Let $\{\alpha_k\}$, $\{\beta_k\}$, $\{\rho_k\}$, $\{\delta_k\}$, $\{\varepsilon_k\}$, and $\{\zeta_k\}$ be nonnegative sequences of real numbers such that $\{\alpha_k\}$ and $\{\rho_k\}$ are positive sequences bounded from above, $\{\beta_k\} \subset [\beta', \beta'']$ for some $\beta', \beta'' \in (0, 1)$, $\{\delta_k\} \subset [\delta', \delta'']$ for some $\delta', \delta'' \in (0, 1)$, $\sum_{k=0}^{\infty} \varepsilon_k < \infty$, and $\sum_{k=0}^{\infty} \zeta_k < \infty$. Let $x_0 \in C$ and set $k = 0$.
2. **(Stopping Criterion).** If $x_k \in \mathcal{S}(\text{EP}_f) \cap \mathcal{S}(\text{EP}_g)$, then stop.
3. **(First Proximal Step).** Find $y_k \in C$ such that $\|y_k - \bar{y}_k\| \leq \varepsilon_k$, where \bar{y}_k solves the following AEP:

$$f_k(\bar{y}_k, y) \geq 0 \text{ for all } y \in C \quad (15)$$

with $f_k(x, y) := f(x, y) + \alpha_k \langle x - x_k, y - x \rangle$. Set $z_k := \beta_k x_k + (1 - \beta_k) y_k$.

4. **(Second Proximal Step).** Find $u_k \in C$ such that $\|u_k - \bar{u}_k\| \leq \zeta_k$, where \bar{u}_k solves the following AEP:

$$g_k(\bar{u}_k, y) \geq 0 \quad \text{for all } y \in C \quad (16)$$

with $g_k(x, y) := g(x, y) + \rho_k \langle x - z_k, y - x \rangle$.

5. **(Update of x_k).** Set $x_{k+1} := \delta_k x_k + (1 - \delta_k)u_k$ and go to Step 2 with $k = k + 1$.

The next theorem establishes the global convergence of Algorithm 2 under appropriate conditions imposed on the initial data of the equilibrium problems of our study.

Theorem 3.1. (global convergence of Algorithm 2). *Assume that $\mathcal{S}(EP_f) \cap \mathcal{S}(EP_g) \neq \emptyset$, that f is pseudomonotone on $C \times C$ and $f(\cdot, y)$ is upper semicontinuous on C for all $y \in C$, that g is pseudomonotone on $C \times C$ and $g(\cdot, y)$ is upper semicontinuous on C for all $y \in C$, and that the AEPs in (15) and (16) have solutions for each $k \geq 0$. Then the sequence $\{x_k\}$ generated by Algorithm 2 converges to a common element of $\mathcal{S}(EP_f)$ and $\mathcal{S}(EP_g)$.*

Proof. Let $\bar{x} \in \mathcal{S}(EP_f) \cap \mathcal{S}(EP_g)$. Then $\|y_k - \bar{x}\| \leq \|x_k - \bar{x}\| + \varepsilon_k$ and

$$\|z_k - \bar{x}\| \leq \|x_k - \bar{x}\| + \varepsilon_k \quad (17)$$

similarly to the proof of Theorem 2.1. On the other hand, we have

$$\|z_k - \bar{x}\|^2 = \|z_k - \bar{u}_k + \bar{u}_k - \bar{x}\|^2 = \|z_k - \bar{u}_k\|^2 + \|\bar{u}_k - \bar{x}\|^2 + 2\langle z_k - \bar{u}_k, \bar{u}_k - \bar{x} \rangle.$$

The latter implies, by the definition of \bar{u}_k and by the *pseudomonotonicity* of g , that

$$\rho_k \langle z_k - \bar{u}_k, \bar{u}_k - \bar{x} \rangle \geq -g(\bar{u}_k, \bar{x}) \geq 0,$$

which yields in turn the estimate

$$\|z_k - \bar{x}\|^2 \geq \|z_k - \bar{u}_k\|^2 + \|\bar{u}_k - \bar{x}\|^2. \quad (18)$$

It follows from (18) and the definition of u_k that

$$\|u_k - \bar{x}\| \leq \|u_k - \bar{u}_k\| + \|\bar{u}_k - \bar{x}\| \leq \zeta_k + \|z_k - \bar{x}\|. \quad (19)$$

Therefore from (17) and (19) we derive the relationships

$$\begin{aligned} \|x_{k+1} - \bar{x}\| &= \|\delta_k(x_k - \bar{x}) + (1 - \delta_k)(u_k - \bar{x})\| \\ &\leq \delta_k \|x_k - \bar{x}\| + (1 - \delta_k) \|u_k - \bar{x}\| \\ &\leq \delta_k \|x_k - \bar{x}\| + (1 - \delta_k) [\zeta_k + \|z_k - \bar{x}\|] \\ &\leq \delta_k \|x_k - \bar{x}\| + (1 - \delta_k) [\zeta_k + \varepsilon_k + \|x_k - \bar{x}\|] \\ &\leq \|x_k - \bar{x}\| + \zeta_k + \varepsilon_k. \end{aligned} \quad (20)$$

Employing Lemma 1.1 from [10, Chapter 3] allows us to conclude that the sequence $\{x_k\}$ is bounded and thus the one of $\{\|x_k - \bar{x}\|\}$ converges as $k \rightarrow \infty$ with the limit

$$\mu := \lim_{k \rightarrow \infty} \|x_k - \bar{x}\|. \quad (21)$$

Observe also that estimates (17) and (19) yield $\limsup_{k \rightarrow \infty} \|u_k - \bar{x}\| \leq \mu$. We get furthermore

$$\lim_{k \rightarrow \infty} \|x_k - u_k\| = 0 \quad (22)$$

from [24] and the obvious equalities

$$\lim_{k \rightarrow \infty} \|\delta_k(x_k - \bar{x}) + (1 - \delta_k)(u_k - \bar{x})\| = \lim_{k \rightarrow \infty} \|x_{k+1} - \bar{x}\| = \mu$$

and thus arrive at the limiting relationships

$$\lim_{k \rightarrow \infty} \|u_k - \bar{x}\| = \mu, \quad (23)$$

$$\lim_{k \rightarrow \infty} \|x_k - \bar{u}_k\| = 0. \quad (24)$$

Applying further (17) and (18) gives us the estimates

$$\begin{aligned} \|u_k - \bar{x}\|^2 &= \|u_k - \bar{u}_k + \bar{u}_k - \bar{x}\|^2 \\ &= \|u_k - \bar{u}_k\|^2 + \|\bar{u}_k - \bar{x}\|^2 + 2\langle u_k - \bar{u}_k, \bar{u}_k - \bar{x} \rangle \\ &\leq \zeta_k^2 + \|\bar{u}_k - \bar{x}\|^2 + 2\|u_k - \bar{u}_k\| \cdot \|\bar{u}_k - \bar{x}\| \\ &\leq \zeta_k^2 + \|\bar{u}_k - \bar{x}\|^2 + 2\zeta_k \|\bar{u}_k - \bar{x}\| \\ &\leq \zeta_k^2 + 2\zeta_k \|\bar{u}_k - \bar{x}\| + \|z_k - \bar{x}\|^2 - \|z_k - \bar{u}_k\|^2 \\ &\leq \zeta_k^2 + 2\zeta_k \|\bar{u}_k - \bar{x}\| + [\|x_k - \bar{x}\| + \varepsilon_k]^2 - \|z_k - \bar{u}_k\|^2, \end{aligned}$$

which imply in turn the following one:

$$\|z_k - \bar{u}_k\|^2 \leq \zeta_k^2 + 2\zeta_k \|\bar{u}_k - \bar{x}\| + [\|x_k - \bar{x}\| + \varepsilon_k]^2 - \|u_k - \bar{x}\|^2. \quad (25)$$

Passing to the limit in (25) as $k \rightarrow \infty$ and using (21) and (23) ensure that

$$\lim_{k \rightarrow \infty} \|z_k - \bar{u}_k\| = 0. \quad (26)$$

Moreover, from the above we have the estimate

$$\|x_k - z_k\| \leq \|x_k - u_k\| + \|u_k - \bar{u}_k\| + \|\bar{u}_k - z_k\|,$$

and hence $\lim_{k \rightarrow \infty} \|x_k - z_k\| = 0$ due to (22) and (26). Thus

$$\lim_{k \rightarrow \infty} \|z_k - \bar{x}\| = \mu. \quad (27)$$

Taking now into account that $\limsup_{k \rightarrow \infty} \|y_k - \bar{x}\| \leq \mu$ and that

$$\lim_{k \rightarrow \infty} \|\beta_k(x_k - \bar{x}) + (1 - \beta_k)(y_k - \bar{x})\| = \lim_{k \rightarrow \infty} \|z_k - \bar{x}\| = \mu$$

by (27), we deduce from [24] that $\lim_{k \rightarrow \infty} \|x_k - y_k\| = 0$ and therefore

$$\lim_{k \rightarrow \infty} \|x_k - \bar{y}_k\| = 0. \quad (28)$$

Consider next a cluster point \tilde{x} of $\{x_k\}$ (which always exists due to the finite dimensionality of the space in question), and let $\{x_{k_p}\}$ be a subsequence converging to \tilde{x} . It follows from (28) that $\{\bar{y}_{k_p}\}$ also converges to \tilde{x} . By the definition of \bar{y}_{k_p} we have

$$f(\bar{y}_{k_p}, y) + \alpha_{k_p} \langle \bar{y}_{k_p} - x_{k_p}, y - \bar{y}_{k_p} \rangle \geq 0 \quad \text{for all } y \in C$$

Passing to the limit as $p \rightarrow \infty$ in the latter inequalities and employing the assumed *upper semicontinuity* of $f(\cdot, y)$ give us

$$f(\tilde{x}, y) \geq 0 \quad \text{for all } y \in C,$$

which means that $\tilde{x} \in \mathcal{S}(\text{EP}_f)$. On the other hand, we get from the definition of \bar{u}_{k_p} that

$$g(\bar{u}_{k_p}, y) + \gamma_{k_p} \langle \bar{u}_{k_p} - z_{k_p}, y - \bar{u}_{k_p} \rangle \geq 0 \text{ for all } y \in C. \quad (29)$$

Since the sequence $\{\bar{u}_{k_p}\}$ converges to \tilde{x} by (24), it follows from (29) by passing to the limit as $p \rightarrow \infty$ and using (26) together with the assumed *upper semicontinuity* of $g(\cdot, y)$ that $\tilde{x} \in \mathcal{S}(\text{EP}_g)$. Finally, we get from (21) that

$$\lim_{k \rightarrow \infty} \|x_k - \tilde{x}\| = 0,$$

which justifies the convergence of the sequence $\{x_k\}$ to an element of $\mathcal{S}(\text{EP}_f) \cap \mathcal{S}(\text{EP}_g)$ and thus completes the proof of the theorem. \triangle

Note that the discussions of Remark 2.2 on solving Auxiliary Equilibrium Problems equally apply to the AEPs given in (15) and (16) of Algorithm 2.

3.1 Numerical examples

In this subsection we report some numerical examples to test Algorithm 2. We chose the feasible set $C = [0, 1] \times [0, 1]$ in all the examples. As stopping criterion we used an error measure similar to that of the previous section:

$$\text{error} := \max\{\|x - y_f(x)\|_\infty, \|x - y_g(x)\|_\infty\} < 10^{-4},$$

where $y_f(x) = \arg \min_{y \in C} [f(x, y) + \|y - x\|^2]$ and $y_g(x) = \arg \min_{y \in C} [g(x, y) + \|y - x\|^2]$. At each iteration both the AEPs in Proximal Steps were solved applying the descent method developed in [7] and the accuracy of the solution to the AEP was controlled using the solution bound introduced in [16]. We implemented Algorithm 2 in MATLAB 7.10 and we used the solver FMINCON from the Optimization Toolbox in order to evaluate the gap function in the descent method. Table 2 reports numerical results on the following four examples.

Example 3.1. We consider the same problems of Example 2.1. The EP_f is defined by $f(x, y) = (y_1 - y_2)^2 - (x_1 - x_2)^2$ which is monotone and ∇ -monotone on $C \times C$ and its solution set is $\mathcal{S}(\text{EP}_f) = \{x \in C : x_1 = x_2\}$. The EP_g is defined by $g(x, y) = x_2(y_1 - x_1) - x_1(y_2 - x_2)$ which is monotone and ∇ -monotone on $C \times C$ and its solution set is $\mathcal{S}(\text{EP}_g) = \{x \in C : x_1 = 0\}$. We set the algorithm parameters as follows: $\alpha_k = \rho_k = 1$, $\beta_k = \delta_k = 0.01$ for all $k \geq 0$, and $\varepsilon_k = 1/2^k$. We applied Algorithm 2 starting from 5 different points in the set C . It is shown in Table 2 that from any starting point Algorithm 2 reaches the solution $(0, 0)$ which is the unique element of $\mathcal{S}(\text{EP}_f) \cap \mathcal{S}(\text{EP}_g)$.

Example 3.2. We consider the same problems of Example 2.2. The EP_f is defined by the bifunction $f(x, y) = (x_1 + x_2 - 1)(y_1 - x_1) + (x_1 + x_2 - 1)(y_2 - x_2)$ which is monotone and ∇ -monotone on $C \times C$, its solution set is $\mathcal{S}(\text{EP}_f) = \{x \in C : x_1 + x_2 = 1\}$. The EP_g is defined by $g(x, y) = (x_1 - x_2)(y_1 - x_1) + (x_2 - x_1)(y_2 - x_2)$ which is monotone and ∇ -monotone on $C \times C$, its solution set is $\mathcal{S}(\text{EP}_g) = \{x \in C : x_1 = x_2\}$. We set the algorithm parameters as follows: $\alpha_k = \rho_k = 1$, $\beta_k = \delta_k = 0.01$ for all $k \geq 0$, and $\varepsilon_k = 1/2^k$. We applied Algorithm 2 starting from 5 different points in the set C and Table 2 shows that from any starting point Algorithm 2 reaches the solution $(0.5, 0.5)$ which is the unique element of $\mathcal{S}(\text{EP}_f) \cap \mathcal{S}(\text{EP}_g)$.

Example 3.3. We consider the same problems of Example 2.3. The EP_f is defined by $f(x, y) = (y_1 - x_1)(2y_1 + x_1)$ which is pseudomonotone, weakly-monotone with constant 1, and ∇ -monotone on $C \times C$. Its solution set is $\mathcal{S}(\text{EP}_f) = \{x \in C : x_1 = 0\}$. The EP_g is given by $g(x, y) = -x_2(y_1 - x_1) + x_1(y_2 - x_2)$ which is monotone and ∇ -monotone on $C \times C$. Its solution set is $\mathcal{S}(\text{EP}_g) = \{x \in C : x_2 = 0\}$. We set the algorithm parameters as follows: $\alpha_k = 2$, $\beta_k = 0.01$, $\rho_k = 1$, $\delta_k = 0.01$ for all $k \geq 0$, and $\varepsilon_k = 1/2^k$. We

applied Algorithm 2 starting from 5 different points in the set C and Table 2 shows that from any starting point Algorithm 1 reaches the solution $(0, 0)$ which is the unique element of $\mathcal{S}(\text{EP}_f) \cap \mathcal{S}(\text{EP}_g)$.

Example 3.4. Here we consider two general EPs. The EP_f the same as in Example 3.3. The EP_g is given by $g(x, y) = e^{x^2}(y_2^2 - x_2^2)$ which is monotone and ∇ -monotone on $C \times C$. Its solution set is $\mathcal{S}(\text{EP}_g) = \{x \in C : x_2 = 0\}$. We set the algorithm parameters as follows: $\alpha_k = 2$, $\beta_k = 0.01$, $\rho_k = 1$, $\delta_k = 0.01$ for all $k \geq 0$, and $\varepsilon_k = 1/2^k$. We applied Algorithm 2 starting from 5 different points in the set C and Table 2 shows that from any starting point Algorithm 1 reaches the solution $(0, 0)$ which is the unique element of $\mathcal{S}(\text{EP}_f) \cap \mathcal{S}(\text{EP}_g)$.

Problem	Starting point	Solution	# Iterations	Error
Example 4	(0.084 , 0.400)	(8.05e-32 , 2.64e-04)	19	8.79e-05
	(0.260 , 0.800)	(6.72e-23 , 2.07e-04)	20	6.90e-05
	(0.431 , 0.911)	(1.68e-23 , 2.07e-04)	20	6.91e-05
	(0.182 , 0.264)	(1.68e-25 , 2.57e-04)	19	8.59e-05
	(0.146 , 0.136)	(1.68e-23 , 2.08e-04)	19	6.95e-05
Example 5	(0.780 , 0.390)	(5.00e-01 , 5.00e-01)	8	3.37e-05
	(0.242 , 0.404)	(5.00e-01 , 5.00e-01)	9	7.02e-05
	(0.547 , 0.296)	(5.00e-01 , 5.00e-01)	9	3.10e-05
	(0.235 , 0.353)	(5.00e-01 , 5.00e-01)	10	8.16e-05
	(0.575 , 0.060)	(5.00e-01 , 5.00e-01)	8	7.23e-05
Example 6	(0.644 , 0.379)	(1.82e-04 , 2.93e-06)	13	9.09e-05
	(0.812 , 0.533)	(1.06e-04 , 2.16e-06)	14	5.30e-05
	(0.351 , 0.939)	(1.23e-04 , 6.65e-06)	13	6.14e-05
	(0.226 , 0.171)	(1.33e-04 , 1.17e-06)	14	6.64e-05
	(0.622 , 0.587)	(1.84e-04 , 2.29e-06)	13	9.22e-05
Example 7	(0.086 , 0.262)	(1.72e-04 , 1.31e-04)	12	8.61e-05
	(0.801 , 0.029)	(1.13e-04 , 1.33e-04)	12	6.63e-05
	(0.929 , 0.730)	(1.31e-04 , 8.80e-05)	12	6.53e-05
	(0.489 , 0.579)	(1.67e-04 , 8.08e-05)	12	8.34e-05
	(0.237 , 0.459)	(8.10e-05 , 7.02e-05)	13	4.05e-05

Table 2: Numerical results for Algorithm 2 tested on Examples 3.1-3.4.

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