Stability for Equilibrium Problems: From Variational Inequalities to Dynamical Systems*

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Abstract. We study the connections between solutions of variational inequalities and equilibrium points of a generalized dynamical system. Furthermore, we analyze some stability questions arising in this field.

Key words. Variational inequalities, projected dynamical systems, equilibrium solutions, stability analysis.

1 Mathematical models for equilibrium problems

Equilibrium problems play a central role in the study of complex and competitive systems. There are many classic examples both in Engineering Science (equilibrium problems in a traffic network), and in economics (for example, oligopolistic market equilibrium). Many variational formulations of these problems have been presented in recent years.

Variational inequalities and Minty variational inequalities are two very useful tools in the study of equilibrium solutions and their stability.

Definition 1.1. For a closed convex set $K \subseteq \mathbb{R}^n$ and a vector function $F : K \to \mathbb{R}^n$, a simple variational inequality of Stampacchia-type SVI(F,K) consists in determining a vector $x^* \in K$, such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall\, x \in K,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$; the associated Minty variational inequality MVI(F,K) consists of determining a vector $x^* \in K$, such that

$$\langle F(x), x^* - x \rangle \leq 0, \quad \forall\, x \in K.$$

The Minty Lemma establishes relationships between the solutions to SVI(F,K) and the solutions to MVI(F,K).

Theorem 1.1. (Minty Lemma) Let the operator $F : K \to \mathbb{R}^n$ be given, where $K \subseteq \mathbb{R}^n$ is a closed convex set. Then the following statements hold:

(i) if F is continuous on K, then each solution to MVI(F,K) is a solution to SVI(F,K);

(ii) if F is pseudomonotone on K[1], then each solution to SVI(F,K) is a solution to MVI(F,K).

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[1]That is, $\langle F(y), x - y \rangle \geq 0 \implies \langle F(x), x - y \rangle \geq 0 \quad \forall\, x, y \in K$.

More recently, two dynamical models for equilibrium problems based on projection operators have been proposed: one is known as locally projected dynamical system (Refs. 1-3), and the other as globally projected dynamical system (Refs 4-5). In this paper we study the relationships among the equilibrium points of these dynamical systems and the solutions of an associated variational inequality. Furthermore stability analysis of their solutions is provided.

We now introduce the definition of a locally projected dynamical system with respect to a closed and convex set K and a continuous vector field F defined on an open set containing K.

Given $x \in K$ and $v \in \mathbb{R}^n$, we define the projection of the vector v at x (with respect to K) by

$$\Pi_K(x, v) = \lim_{\delta \to 0} (P_K(x + \delta v) - x)/\delta,$$

where P_K is defined as:

$$P_K(x) = \arg\min_{z \in K} \|x - z\|.$$

Remark 1.1. The above defined projection $\Pi_K(x, v)$ defined above is equivalent to the Euclidean projection of v on the tangent cone to K at x. To see this, let us introduce the normal cone to K at x:

$$N_K(x) = \{y \in \mathbb{R}^n : \ \langle y, x' - x \rangle \leq 0, \quad \forall \ x' \in K\},$$

and the set

$$n(x) = \{\gamma \in \mathbb{R}^n : \ \|\gamma\| = 1, \quad \gamma \in N_K(x)\}.$$

We denote the boundary and interior of K by ∂K and $\operatorname{int} K$, respectively. The following geometric interpretation of Π_K is well known (Ref. 6):

(i) if $x \in \operatorname{int} K$, then $\Pi_K(x, v) = v$;

(ii) if $x \in \partial K$ and $\max_{n \in n(x)} \langle v, n \rangle \leq 0$, then $\Pi_K(x, v) = v$;

(iii) if $x \in \partial K$ and $\max_{n \in n(x)} \langle v, n \rangle > 0$, then $\Pi_K(x, v) = v - \langle v, n^*(x) \rangle n^*(x)$, where

$$n^*(x) = \arg\max_{n \in n(x)} \langle v, n \rangle.$$

Therefore we have $\Pi_K(x, v) = P_{T_K(x)}(v)$, where $T_K(x)$ denotes the tangent cone to K at x.

Definition 1.2. We define the locally projected dynamical system as the following ordinary differential equation LPDS(F,K)

$$\dot{x} = \Pi_K(x, -F(x)),$$

where K is a closed convex set and F is a continuous vector field defined on K.

Remark 1.2. In the definition of LPDS(F,K) we consider the projection on K of the vector field $-F$ because we will characterize the steady points of LPDS(F,K) with the solutions of the variational inequality SVI(F,K), moreover some particular conditions on F will be useful to the stability analysis of LPDS(F,K).

We note that the right-hand side of the previous ordinary differential equation coincides with $-F(x)$ in the interior of K and it can be discontinuous on the boundary of K. Therefore, we need to define what is meant by a solution to an ordinary differential equation with a discontinuous right-hand side.

Definition 1.3. We say that the function $x : [0, +\infty) \to K$ is a solution to the equation LPDS(F,K) if x is absolutely continuous and if

$$\dot{x}(t) = \Pi_K(x(t), -F(x(t))),$$

for all $t \geq 0$ save on a set of Lebesgue measure zero.

We recall the fundamental theorem about LPDS (Ref. 1).

Theorem 1.2. Assume that K is a convex polyhedron. If F is linearly bounded, namely there exists a constant $M > 0$ such that

$$\|F(x)\| \leq M\,(1 + \|x\|), \qquad \forall\, x \in K,$$

and also

$$\langle F(x) - F(y), x - y\rangle \leq M\,\|x - y\|^2, \qquad \forall\, x, y \in K,$$

then

(i) for any $x_0 \in K$, there exists a unique solution $x_0(t)$ to LPDS(F,K), such that $x(0) = x_0$;

(ii) if $x_n \to x_0$ as $n \to +\infty$, then $x_n(t)$ converges to $x_0(t)$ uniformly on every compact set of $[0, +\infty)$.

Remark 1.3. We note that if the vector field F is Lipschitz continuous on K, then assumptions of Theorem 1.2 hold.

We are interested in equilibrium points of LPDS(F,K) defined as follows: the vector $x^* \in K$ is an *equilibrium point* of LPDS(F,K) if

$$\Pi_K(x^*, -F(x^*)) = 0.$$

We observe that x^* is an equilibrium point if, once a solution of the LPDS(F,K) is at x^*, it will remain at x^* for all future times.

In the special case $K = \mathbb{R}^n$, the LPDS(F,K) coincides with the classical autonomous dynamical system DS(F)

$$\dot{x} = -F(x),$$

and its equilibrium points are the solutions to the system of nonlinear equations $F(x) = 0$.

The important relationship announced in Remark 1.2 between equilibrium points of LPDS and solutions of SVI is given in the following known theorem, which we give the proof of for the sake of completeness.

Theorem 1.3. The equilibrium points of the LPDS(F,K) coincide with the solutions of SVI(F,K).

Proof. By Remark 1.2, a vector $x^* \in K$ is an equilibrium point of LPDS(F,K) if and only if $P_{T_K(x^*)}(-F(x^*)) = 0$, that is $-F(x^*) \in N_K(x^*)$, which is a well known characterization of the solutions to SVI(F,K). □

The second dynamical model is the so-called globally projected dynamical system described first in Ref. 4.

Definition 1.4. We define the globally projected dynamical system as the following ordinary differential equation GPDS(F,K,α)

$$\dot{x} = P_K(x - \alpha F(x)) - x,$$

where α is a positive constant.

We note that the right-hand side of GPDS(F,K,α) is continuous on K and it can be different from $-F(x)$ even if x is an interior point to K. Hence the solutions of GPDS(F,K,α) and LPDS(F,K) are different in general.

In the special case $K = \mathbb{R}^n$, the GPDS(F,K,α) coincides with the classical autonomous dynamical system DS(αF)

$$\dot{x} = -\alpha F(x).$$

From ordinary differential equations theory we derive the following result about global existence and uniqueness of solutions to GPDS(F,K,α). Another proof of this result can be found in Ref. 5.

Theorem 1.4. If $F : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and linearly bounded, then for any $x_0 \in \mathbb{R}^n$ there exists a unique solution for GPDS(F,K,α) that is defined for all $t \in \mathbb{R}$.

Proof. Let $T(x) = P_K(x - \alpha F(x)) - x$, then for any $x, y \in \mathbb{R}^n$ one has

$$\begin{aligned} \|T(x) - T(y)\| &\leq \|P_K(x - \alpha F(x)) - P_K(y - \alpha F(y))\| + \|x - y\| \\ &\leq 2\,\|x - y\| + \alpha\,\|F(x) - F(y)\|, \end{aligned}$$

thus T is locally Lipschitz continuous in $\mathbb{R}^n$. Moreover for any $x \in \mathbb{R}^n$

$$\begin{aligned} \|T(x)\| &\leq \|P_K(x - \alpha F(x)) - P_K(x)\| + \|P_K(x) - x\| \\ &\leq \alpha\,\|F(x)\| + \|P_K(x) - P_K(0)\| + \|P_K(0) - x\| \\ &\leq \alpha\,M\,(1 + \|x\|) + 2\,\|x\| + \|P_K(0)\| \\ &= (2 + \alpha\,M)\,\|x\| + \alpha\,M + \|P_K(0)\|. \end{aligned}$$

It follows that T is also linearly bounded; therefore for every $x_0 \in \mathbb{R}^n$ there is a unique solution $x(t)$ of GPDS(F,K,α) defined on $\mathbb{R}$ with $x(0) = x_0$. □

As in LPDS(F,K), a solution to GPDS(F,K,α) starting from a point in K has to remain in K. The proof of this fact is given in Ref. 5 but the correctness of such a proof is, in our opinion, not easily checkable.

Theorem 1.5. If F is locally Lipschitz continuous then for any $x_0 \in K$ the solution $x(t)$ to GPDS(F,K,α), defined in a neighborhood I of 0, is such that $x(t) \in K$ for any $t \in I$.

Proof. First we prove that if $x(t)$ is any solution of GPDS(F,K,α), defined in $[a, b]$ with $x(t) \notin K$ for all $t \in [a, b]$, then the distance $d(x(t), K)$ between $x(t)$ and K is decreasing in t.

Let $t_0 \in [a, b]$ and B be the open ball with center $P_K(x(t_0))$ and radius $\|P_K(x(t_0)) - x(t_0)\|$. Since $x(t_0) \notin K$ then

$$\langle \dot{x}(t_0), P_K(x(t_0)) - x(t_0) \rangle > 0.$$

Thus, there exists $\delta(t_0) > 0$ small enough such that

$$\begin{aligned} &x(t) \notin \overline{B}, \qquad \forall\, t \in (t_0 - \delta(t_0), t_0), \\ &x(t) \in B, \qquad \forall\, t \in (t_0, t_0 + \delta(t_0)), \end{aligned}$$

that is

$$\begin{aligned} &d(x(t), K) > d(x(t_0), K), \qquad \forall\, t \in (t_0 - \delta(t_0), t_0), \\ &d(x(t), K) < d(x(t_0), K), \qquad \forall\, t \in (t_0, t_0 + \delta(t_0)). \end{aligned}$$

We consider $t_1, ..., t_n \in [a, b]$ such that $[a, b] \in \bigcup_{i=1}^{n} (t_i - \delta(t_i), t_i + \delta(t_i))$, hence, using the previous property for each interval $(t_i - \delta(t_i), t_i + \delta(t_i))$, we obtain

$$d(x(s_1), K) > d(x(s_2), K),$$

if $a \le s_1 < s_2 \le b$, namely $d(x(t), K)$ is decreasing in t.

Now let $x(t)$ be the solution of GPDS(F,K,α) with $x(0) = x_0 \in K$. We suppose by contradiction that there are $t', t'' \in I$ such that $x(t') \in K$ and $x(t) \notin K$ for all $t \in (t', t'')$. Then we have $d(x(t'), K) = 0$ and $d(x(t), K)$ is positive and decreasing for all $t \in (t', t'')$, but this is impossible because the distance $d(x(t), K)$ is continuous in t. Therefore $x(t) \in K$ for all $t \in I$. □

The equilibrium points of GPDS(F,K,α) are naturally defined as follows: a vector $x^* \in K$ is an *equilibrium point* of GPDS(F,K,α) if

$$P_K(x^* - \alpha\, F(x^*)) - x^* = 0.$$

It is well known that $x^* \in K$ is a solution to SVI(F,K) if and only if for any $\alpha > 0$ one has

$$x^* = P_K(x^* - \alpha\, F(x^*)).$$

Therefore the equilibrium points of the GPDS(F,K,α) also coincide with the solutions to SVI(F,K), although the solutions to LPDS(F,K) and GPDS(F,K,α), as observed, do not coincide in general.

2 Stability Analysis

In this section we analyze the stability of equilibrium points to locally and globally projected dynamical systems, namely we wish to answer this question: if a solution starts near an equilibrium, will it stay close to it forever? We will assume the property of existence and uniqueness of solutions to locally and globally projected dynamical systems. In the following we will use monotonicity properties of vector field F.

We recall now the most important stability concepts for a classical dynamical system DS(F). We will use $B(x, r)$ to denote the open ball with center x and radius r.

Definition 2.1.

(i) An equilibrium point x^* of DS(F) is *stable* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for every $x_0 \in B(x^*, \delta)$ the solution $x(t)$ of DS(F) with $x(0) = x_0$ is defined and $x(t) \in B(x^*, \varepsilon)$ for all $t > 0$.

(ii) A stable equilibrium point x^* is *asymptotically stable* if there is a $\delta > 0$ such that for every solution $x(t)$ with $x(0) \in B(x^*, \delta)$ one has $\lim_{t\to+\infty} x(t) = x^*$.

(iii) An equilibrium point x^* is *exponentially stable* if there is a $\delta > 0$ and constants $a > 0$ and $C > 0$ such that for every solution $x(t)$ with $x(0) \in B(x^*, \delta)$ one has

$$\|x(t) - x^*\| \le C\, \|x(0) - x^*\|\, exp(-at) \quad \forall\ t \ge 0; \tag{1}$$

x^* is *globally exponentially stable* if (1) holds true for all solutions $x(t)$ of DS(F).

(iv) An equilibrium point x^* is a *monotone attractor* if there exists a $\delta > 0$ such that for every solution $x(t)$ with $x(0) \in B(x^*, \delta)$, $\|x(t) - x^*\|$ is a nonincreasing function of t; x^* is a *global monotone attractor* if $\|x(t) - x^*\|$ is nonincreasing in t for all solutions $x(t)$ of DS(F); x^* is a *strictly monotone attractor* if there exists a $\delta > 0$ such that for every solution $x(t)$ with $x(0) \in B(x^*, \delta)$, $\|x(t) - x^*\|$ is decreasing to zero in t; x^* is a *strictly global monotone attractor* if $\|x(t) - x^*\|$ is decreasing to zero in t for all solutions $x(t)$ of DS(F).

Naturally the previous definitions, with suitable changes, hold even if we consider the equilibrium points of a locally or globally projected dynamical system.

Obviously every monotone attractor is a stable equilibrium point, and there are easy examples of an equilibrium that is stable but neither asymptotically stable nor monotone attractor.

For a dynamical system DS(F), if F is continuously differentiable, a classical method for stability analysis of an equilibrium point x^* is analyzing the eigenvalues of linear part of F at x^*. In particular an equilibrium point x^* of DS(F) is called a *sink* if all eigenvalues of the jacobian matrix $JF(x^*)$ have positive real parts.

We recall the following nonlinear sink theorem.

Theorem 2.1. (Ref. 7) If x^* is a sink of DS(F) then x^* is exponentially stable.

We now present a necessary condition for a monotone attractor of DS(F).

Theorem 2.2. If x^* is a monotone attractor for DS(F), then the Jacobian matrix $JF(x^*)$ is positive semidefinite.

Proof. Suppose that x^* is a monotone attractor with respect to $B(x^*, \delta)$. Let x be an arbitrary point in $B(x^*, \delta)$ and $x(t)$ be the solution to DS(F) passing through x when $t = 0$, then the function $D(t) = \|x(t) - x^*\|^2/2$ is nonincreasing, hence $\dot{D}(t) \leq 0$ for all $t \geq 0$, in particular

$$\dot{D}(0) = \langle -F(x), x - x^* \rangle \leq 0.$$

Thus

$$\langle F(x), x - x^* \rangle \geq 0, \qquad \forall\, x \in B(x^*, \delta).$$

Now we consider $\overline{x} \in B(x^*, \delta)$ and $x = x^* + \alpha\,(\overline{x} - x^*)$ with $\alpha \in (0, 1)$. Since x^* is an equilibrium point and since F is continuously differentiable, we have

$$F_i(x) = \langle \nabla F_i(\xi_i), x - x^* \rangle, \quad \forall\, i = 1, \ldots, n$$

where ξ_i belong to the segment $[x, x^*]$. Hence

$$\begin{aligned} 0 \quad &\leq \quad \langle F(x), x - x^* \rangle = \sum_{i=1}^{n} \langle \nabla F_i(\xi_i), x - x^* \rangle (x_i - x_i^*) = \\ &= \quad \alpha^2 \sum_{i=1}^{n} \langle \nabla F_i(\xi_i), \overline{x} - x^* \rangle (\overline{x}_i - x_i^*), \qquad \forall\, \alpha \in (0, 1); \end{aligned}$$

and thus we have

$$\sum_{i=1}^{n} (\overline{x}_i - x_i^*) \langle \nabla F_i(\xi_i), \overline{x} - x^* \rangle \geq 0, \qquad \forall\, \alpha \in (0, 1).$$

If $\alpha \to 0$, then $x \to x^*$ and also $\xi_i \to x^*$, for all $i = 1, \ldots, n$. Hence

$$\langle \overline{x} - x^*, JF(x^*)(\overline{x} - x^*) \rangle \geq 0,$$

and $JF(x^*)$ is positive semidefinite. □

We remark that monotone attractors cannot be compared to asymptotically stable points. In the trivial example

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= 0, \end{aligned}$$

$x^* = (0,0)$ is a monotone attractor, but it is not asymptotically stable. Moreover if we consider

$$\dot{x}_1 = -2\,x_1 - x_2$$
$$\dot{x}_2 = 2\,x_1,$$

$x^* = (0,0)$ is a sink, thus it is exponentially stable and, in particular, asymptotically stable, because the Jacobian matrix

$$JF(x^*) = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$$

has eigenvalues $1 \pm i$; but

$$\langle x, JF(x^*)x \rangle = x_1(x_2 - 2\,x_1) \not\geq 0 \quad \forall\, x \in \mathbb{R}^2,$$

that is $JF(x^*)$ is not positive semidefinite, and hence, by Theorem 2.2, x^* is not a monotone attractor (see Figure 1).

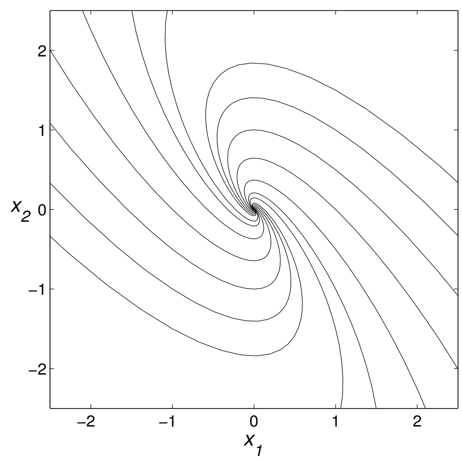

Figure 1: An asymptotically stable equilibrium point which is not a monotone attractor.

We continue by examining the stability of the equilibrium points of LPDS(F,K), which is the main purpose of this paper. In the following theorem we characterize global monotone attractors of LPDS(F,K) with solutions of Minty variational inequality MVI(F,K).

Theorem 2.3. A point $x^* \in K$ is solution to MVI(F,K) if and only if it is a global monotone attractor of LPDS(F,K).

Proof. Since x^* is a solution to MVI(F,K), then by the Minty lemma x^* is also a solution of SVI(F,K) and hence is an equilibrium point of LPDS(F,K). Let $x(t)$ be an arbitrary solution of LPDS(F,K), then the function $D(t) = \|x(t) - x^*\|$ is absolutely continuous and for all $t \geq 0$, save on a set of Lebesgue measure zero, holds

$$\dot{D}(t) = \langle x(t) - x^*, -F(x(t)) \rangle / D(t) - \beta(x(t)) \langle x(t) - x^*, n^*(x(t)) \rangle / D(t),$$

where

$$\beta(x(t)) \geq 0 \quad \text{and} \quad n^*(x(t)) \in N_K(x(t)).$$

Thus,

$$\dot{D}(t) \leq \langle -F(x(t)), x(t) - x^* \rangle / D(t) \leq 0.$$

Therefore if $0 \le t_1 < t_2$ we have

$$D(t_2) - D(t_1) = \int_{t_1}^{t_2} \dot{D}(t)dt \le 0.$$

That is, $D(t)$ is non increasing and hence x^* is a global monotone attractor of LPDS(F,K).

We now assume that $x^* \in K$ is a global monotone attractor. Let T be the smallest affine subspace containing K and let $S = T - x^*$. Let x be an arbitrary point in the relative interior of K and let $x(t)$ be the solution of LPDS(F,K) such that $x(0) = x$. Since in a neighborhood of x in K, LPDS(F,K) coincides with

$$\dot{x} = P_S(-F(x)),$$

then $x(t)$ is continuously differentiable in a neighborhood I of 0 and the function $D(t) = \|x(t) - x^*\|^2/2$ is differentiable and non increasing on I, hence

$$\dot{D}(t) \le 0 \quad \forall\, t \in I,$$

and in particular

$$0 \ge \dot{D}(0) = \langle -F(x), x - x^* \rangle - \langle P_{S^\perp}(-F(x), x - x^* \rangle = \langle F(x), x^* - x \rangle.$$

Since the operator F is continuous then

$$\langle F(x), x^* - x \rangle \le 0, \qquad \forall\, x \in K,$$

that is x^* is a solution of MVI(F,K). □

If we use the monotonicity property of vector field F we can prove directly some stability results for LPDS(F,K) (see also Refs. 2-3).

Theorem 2.4. Suppose that x^* is an equilibrium point to LPDS(F,K). Then

(i) if F is locally pseudomonotone at x^* [2], then x^* is a monotone attractor;

(ii) if F is locally strictly pseudomonotone at x^*, then x^* is a strictly monotone attractor;

(iii) if F is locally strongly monotone at x^*, then x^* is exponentially stable;

(iv) if F is pseudomonotone, strictly pseudomonotone, strongly monotone on K then, respectively, x^* is global monotone attractor, strictly global monotone attractor, globally exponentially stable.

Proof.

(i) If F is pseudomonotone on $N(x^*)$ neighborhood of x^*, then by Theorem 1.3 and Minty lemma, x^* is solution of MVI(F,N(x^*)), and hence x^* is a monotone attractor of LPDS(F,K).

(ii) We now assume that F is strictly pseudomonotone on $N(x^*)$ and we define the continuous function ϕ as follows:

$$\phi(x) = \langle F(x), x - x^* \rangle > 0, \qquad \forall\, x \in N(x^*).$$

Now let us consider an arbitrary solution $x(t)$ of LPDS(F,K) with $x(0) \in N(x^*)$, then $D(t) = \|x(t) - x^*\|$ is absolutely continuous and for all $t \ge 0$, save on a set of Lebesgue measure zero, one has

$$\dot{D}(t) \le \langle -F(x(t)), x(t) - x^* \rangle / D(t) < 0.$$

[2]that is, F is pseudomonotone on a neighborhood of x^*.

Therefore if $0 \leq t_1 < t_2$ we have

$$D(t_2) - D(t_1) = \int_{t_1}^{t_2} \dot{D}(t)dt < 0,$$

that is $D(t)$ is decreasing and there is a $\lim_{t \to +\infty} d(x(t)) = l \geq 0$. We prove $l = 0$. Suppose by contradiction $l > 0$ and consider $\delta = \|x(0) - x^*\|$, then $x(t) \in H = (B(x^*, \delta) \backslash B(x^*, l)) \cap K$ for all $t \geq 0$. Since H is a compact set, we have $\max_{x \in H} \phi(x)/\|x - x^*\| = M < 0$ and so

$$\dot{D}(t) \leq \phi(x(t))/D(t) \leq M < 0.$$

This is impossible and thus x^* is a strictly monotone attractor.

(iii) Suppose that F is strongly monotone on $N(x^*)$ with constant $\eta > 0$. We consider an arbitrary solution $x(t)$ of LPDS(F,K) starting at $x(0) \in N(x^*)$, then for all $t \geq 0$, save on a set of Lebesgue measure zero, one has

$$\dot{D}(t) \leq -\langle F(x(t)) - F(x^*), x(t) - x^* \rangle / \|x(t) - x^*\| \leq -\eta \, \|x(t) - x^*\|,$$

therefore

$$\dot{D}(t) \leq -\eta \, D(t).$$

First we suppose that $D(t) \neq 0$ for all $t > 0$, then for any fixed $\bar{t} > 0$ fixed, $D(t)$ is absolutely continuous on $[0, \bar{t}]$. Since a log function is Lipschitz continuous on $[D(\bar{t}), D(0)]$, then $log(D(t))$ is absolutely continuous on $[0, \bar{t}]$. Therefore we obtain

$$log(D(\bar{t})) - log(D(0)) = \int_0^{\bar{t}} (\dot{D}(t)/D(t))dt \leq -\eta \, \bar{t},$$

and

$$\|x(\bar{t}) - x^*\| \leq \|x(0) - x^*\| exp(-\eta \, \bar{t}),$$

that is x^* is exponentially stable.

If there is some $t_0 > 0$ with $D(t_0) = 0$, then one has

$$\|x(t) - x^*\| = 0, \qquad \forall \; t \geq t_0.$$

Let $C = exp(\eta \, t_0)$. Since $D(t)$ is decreasing on $[0, t_0]$ we have

$$\|x(t) - x^*\| \leq \|x(0) - x^*\| \leq C \, \|x(0) - x^*\| exp(-\eta \, t), \qquad \forall \; t \geq 0,$$

thus x^* is exponentially stable.

(iv) If monotonicity conditions of F hold on K, then global stability results on x^* hold.

□

We now present a stability theorem for LPDS(F,K) analogous to the nonlinear sink theorem; we assume a stronger condition for F, that is the jacobian matrix $JF(x^*)$ is positive definite, but we also obtain a stronger result: x^* is exponentially stable and is a strictly monotone attractor.

Theorem 2.5. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$ vector field and x^* an equilibrium point to LPDS(F,K). If the jacobian matrix $JF(x^*)$ is positive definite, then x^* is a strictly monotone attractor and exponentially stable.

Proof. By continuity there is a neighborhood $N(x^*)$ of x^* in K such that $JF(x)$ is positive definite for all $x \in N(x^*)$. Hence there exists $\eta > 0$ such that

$$\langle y, JF(x)\, y\rangle \geq \eta, \qquad \forall\, x \in N(x^*),\ \forall\, y \text{ s. t. } \|y\| = 1,$$

and thus

$$\langle y, JF(x)\, y\rangle \geq \eta\, \|y\|^2, \qquad \forall\, x \in N(x^*),\ \forall\, y \in \mathbb{R}^n,$$

that is $JF(x)$ is strongly positive definite on $N(x^*)$. Therefore F is strongly monotone on $N(x^*)$ (Ref. 8), and by Theorem 2.4, x^* is a strictly monotone attractor and exponentially stable. □

We go on now to study the stability of the solutions of GPDS(F,K,α). The first crucial remark is that solutions to MVI(F,K) and the global monotone attractor of GPDS(F,K,α) do not coincide, as the following example shows.

Example 2.1. Let $F(x_1, x_2) = (x_2, -x_1) + [(x_1-1)^2 + x_2^2 - 1]^2(1, 1)$ and $K = \mathbb{R}^2_+$. The equilibrium point $x^* = (0, 0)$ solves MVI(F,K) because

$$\langle F(x), x\rangle = [(x_1-1)^2 + x_2^2 - 1]^2(x_1, x_2) \geq 0 \qquad \forall\, x \in \mathbb{R}^2_+.$$

Nevertheless, we consider the semicircle A in K with center $(1, 0)$ and radius 1, then for all $x \in A$ we have $\langle F(x), x\rangle = 0$ and $\|F(x)\| = \|x\|$. Therefore for any fixed $\alpha > 0$, there is $x \in A$ close enough to x^* such that $x - \alpha\, F(x) \notin K$ and hence

$$\langle P_K(x - \alpha\, F(x)) - x, x^* - x\rangle < 0.$$

Then the solution $x(t)$ to GPDS(F,K,α) with $x(0) = x$ goes away from x^* in t within some a neighborhood of 0, thus x^* is not a monotone attractor for GPDS(F,K,α) for any fixed $\alpha > 0$.

As for LPDS(F,K), we prove for GPDS(F,K,α) an analogous result when $JF(x^*)$ is positive definite.

Theorem 2.6. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$ vector field, x^* is an equilibrium point to GPDS(F,K,α). If the jacobian matrix $JF(x^*)$ is positive definite, then there exists $\alpha_0 > 0$ such that x^* is a strictly monotone attractor and exponentially stable for GPDS(F,K,α) for any $\alpha < \alpha_0$.

Proof. By continuity, $JF(x)$ is positive definite for all x in a neighborhood $N(x^*)$ in K of x^*, thus $JF(x)$ is strongly positive definite on $N(x^*)$ and F is strongly monotone on $N(x^*)$ with some constant $\eta > 0$. Moreover, $\|JF(x)\|$ is bounded in $N(x^*)$, hence F is Lipschitz continuous on $N(x^*)$ with constant $L > 0$. Therefore for all $x \in N(x^*)$ the following holds:

$$\begin{aligned}
& \|P_K(x - \alpha\, F(x)) - x^*\|^2 = \\
= \quad & \|P_K(x - \alpha\, F(x)) - P_K(x^* - \alpha\, F(x^*))\|^2 \\
\leq \quad & \|x - x^* - \alpha\,(F(x) - F(x^*))\|^2 \\
= \quad & \|x - x^*\|^2 - 2\alpha\,\langle F(x) - F(x^*), x - x^*\rangle + \alpha^2\,\|F(x) - F(x^*)\|^2 \\
\leq \quad & (1 - 2\,\alpha\,\eta + \alpha^2\, L^2)\|x - x^*\|^2.
\end{aligned}$$

Let $\alpha_0 = 2\eta/L^2$ and $0 < \alpha < \alpha_0$. For any solution $x(t)$ with $x(0) \in N(x^*)$ we denote $D(t) = \|x(t) - x^*\|^2/2$, thus we have

$$\begin{aligned}\dot{D}(t) &= \langle x(t)-x^*, P_K(x(t)-\alpha\,F(x(t)))-x(t)\rangle \\ &= \langle x(t)-x^*, P_K(x(t)-\alpha\,F(x(t)))-x^*\rangle - \|x(t)-x^*\|^2 \\ &\leq \|x(t)-x^*\|\,\|P_K(x(t)-\alpha\,F(x(t)))-x^*\| - \|x(t)-x^*\|^2 \\ &\leq (\sqrt{1-2\,\alpha\,\eta+\alpha^2\,L^2}-1)\,\|x(t)-x^*\|^2 \\ &= -2a\,D(t)\end{aligned}$$

where $a = 1-\sqrt{1-2\,\alpha\,\eta+\alpha^2\,L^2} = 1-\sqrt{1-L^2\alpha(\alpha_0-\alpha)} > 0$. Thus

$$\|x(t)-x^*\| \leq \|x(0)-x^*\| exp(-a\,t), \qquad \forall\ t \geq 0,$$

that is x^* is exponentially stable. Moreover $\dot{D}(t) < 0$ and $\lim_{t\to+\infty} x(t) = x^*$, hence x^* is a strictly monotone attractor as well. □

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