Nash Equilibria, Variational Inequalities and Dynamical Systems*

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Abstract. In this paper we introduce some relationships between Nash equilibria, variational equilibria and dynamical equilibria for noncooperative games.

Key words. Nash Equilibria, Variational and Quasi-Variational Inequalities, Dynamical Systems, Performance and Gap Functions.

1 Introduction

In this paper we investigate some basic relations among the three equilibrium concepts for three different mathematical models which are referred in the title. The first one is Nash-type equilibrium solution for game model, the second one is connected with variational formulation of the problem and the third one arises from dynamical approach to the problem. Some material that we expose has already appeared in the literature; here we deepen some basic relations and we give comments on the topics. Because this field is largely open, we don't achieve definitive results, but, on the contrary, we try to convince the reader that some progress in one field may help to solve problems for the others. In section 4 we study also the relationships between performance functions for Nash equilibria and merit functions for variational inequalities. Recall, in fact, that to optimize performance functions gives Nash equilibria, whereas to optimize merit functions gives variational equilibria.

2 Optimization

Here we expose two basic known lemmas that are essential for further results. Let $f : X \to \mathbb{R}$ be a Gateaux differentiable function, $X \subseteq E$ a closed convex subset of the topological vector space E.

Lemma 2.1. (Ref. 1) Let $x^* \in \arg\min(f, X)$, i.e.

$$x^* \in X \quad \text{and} \quad f(x^*) \leq f(x) \quad \forall\, x \in X,$$

then

$$\langle f'(x^*), x - x^* \rangle \geq 0 \quad \forall\, x \in X. \tag{1}$$

If f is pseudoconvex then (1) is also sufficient for optimality.

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In other words (1) constitutes the N.O.C. (necessary optimality condition) for a constrained minimum x^*, for each Gateaux differentiable function. If we replace f' with an operator $A : E \to E^*$, then (1) is called SVI (Stampacchia type variational inequality)

Lemma 2.2. If there is $x^* \in X$ such that

$$\langle f'(x), x - x^* \rangle \geq 0 \quad \forall\, x \in X, \tag{2}$$

then $x^* \in \arg\min(f, X)$. If f is quasiconvex, then (2) is also necessary for optimality.

Proof. Condition (2) is a well known sufficient optimality condition (S.O.C.) for each Gateaux differentiable function on a convex set X (Ref. 1). When f is quasiconvex and $x^* \in \arg\min(f, X)$, then

$$\langle f'(x), x - x^* \rangle = -\partial f(x)/\partial(x^* - x) = \lim_{t \to 0^+} [f(x) - f(x + t(x^* - x))]/t \geq 0,$$

because

$$f(x + t(x^* - x)) \leq f(x) \quad \forall\, x \in X,\ \forall\, t \in [0, 1],$$

being f quasiconvex, thus (2) is a necessary condition. □

If we replace f' with an operator $A : E \to E^*$, then (2) is called MVI (Minty type variational inequality).

Lemma 2.3. Let f be convex and Gateaux differentiable on X, then the following statements are equivalent:

$$\begin{aligned} &x^* \in \arg\min(f, X); \\ &\langle f'(x^*), x - x^* \rangle \geq 0 \ \ \forall\, x \in X \quad (SVI); \\ &\langle f'(x), x - x^* \rangle \geq 0 \ \ \forall\, x \in X \quad (MVI). \end{aligned}$$

Furthermore the following inequalities hold true

$$\langle f'(x), x - y \rangle \geq f(x) - f(y) \geq \langle f'(y), x - y \rangle \quad \forall\, x, y \in X. \tag{3}$$

3 Noncooperative Games

In what follows we always consider noncooperative games in normal form, with a finite set $I = \{1, \ldots, n\}$ of players (Refs 2-3). Every player $i \in I$ has a set Σ_i of moves (or actions, pure strategies), which is supposed a closed convex subset of an Hilbert space H_i. The ambient space $H = \prod_{i \in I} H_i$ is endowed with the customary inner product $\langle , \rangle$ inherited from its factor spaces H_i. The constrained set Σ_i are often specified by individual (technological) constraints. Every player $i \in I$ has a payoff function $J_i : \prod_{i \in I} \Sigma_i \to \mathbb{R}$. When all these are specified we have a game in normal (strategic) form.

When the sets of pure strategies Σ_i are not closed and convex, then it is better to consider a whole plane of moves for each player.

Definition 3.1. We call mixed strategies for player $i \in I$, the set

$$S_i = \{p \colon p \text{ is a probability measure on } \Sigma_i\}.$$

In this way we assure that S_i are convex subset of H_i, moreover payoff functions are naturally extended to the space $\prod_{i \in I} S_i$ by

$$\tilde{J}_i(p_i, p_{-i}) = \int_{\prod_{i \in I} S_i} J_i(x_i, x_{-i})\, dp_i\, dp_{-i},$$

to obtain a new game in normal form on the space of mixed strategies $\prod_{i\in I} S_i$. Furthermore we may consider the possibility to have a set $K \subseteq \prod_{i\in I} \Sigma_i$ (or $K \subseteq \prod_{i\in I} S_i$) often specified by a social restrictions imposed on the set of all players. It will be assumed closed and convex.

The aim of each player (unless we specify the contrary) will be

$$\min_{\{x_i:\ (x_i,x_{-i})\in K\}} J_i(x_i, x_{-i}),$$

that is to minimize his own utility with respect to the only variable x_i under his control, given the pattern of actions of his adversaries $x_{-i} = (x_j)_{j\in I\setminus\{i\}}$.

When a game in normal form is given, with a social constraint, and played without any agreement or cooperation among players, one of the most frequently used concept of equilibrium is that of Nash.

Definition 3.2. We say that $x^* \in K$ is a *Nash equilibrium* of the game (J, K) if for all $i \in I$

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*) \qquad \forall\ (x_i, x_{-i}^*) \in K.$$

We remark that in the case $K = \prod_{i\in I} \Sigma_i$ we simply have for all $i \in I$:

$$J_i(x_i^*, x_{-i}^*) = \min_{x_i\in\Sigma_i} J_i(x_i, x_{-i}^*).$$

It is possible and easy to state necessary and sufficient conditions, for Nash equilibria of a game in normal form, in terms of the optimization problems that it generates. We omit to do this and expose new conditions in a more compact form. We need some concepts and notations that we now explain.

Definition 3.3. We call the i-th section of $K \subseteq \prod_{i\in I} H_i$ at x^* the set

$$S_i(x^*) = \{w_i \in E_i :\ (x_i^* + w_i, x_{-i}^*) \in K\},$$

i.e. it is the translation of the usual section to the origin.

We call sections of K at x^* the set

$$S(x^*) = \bigcup_{i\in I} S_i(x^*).$$

We call internal cone or inner cone to K at x^*, and we denote it $I_K(x^*)$, the smallest pointed, closed, convex cone which contains sections $S(x^*)$. With $T_K(x^*)$, we will denote the tangent cone to K at x^*.

To produce a compact form of necessary and sufficient conditions we introduce an operator that we call "gradient" of a game in normal form which has the formal expression of a gradient but it is not.

Definition 3.4. We call "gradient" of a game in normal form, with Gateaux differentiable costs w.r.t. the actions of players, at a point $x \in K$, the vector

$$\nabla J(x) = (\partial J_1(x)/\partial x_1, \ldots, \partial J_n(x)/\partial x_n)\,.$$

Theorem 3.1. (Ref. 4) Let (J, K) be a noncooperative game and x^* a Nash equilibrium, then

$$\langle \nabla J(x^*), x - x^* \rangle \geq 0 \quad \forall\ x \in x^* + I_K(x^*) \cap K. \tag{4}$$

When $J_i(\cdot, x_{-i})$ is pseudoconvex for all $i \in I$, then (4) is also sufficient.

The condition (4) is a necessary optimality condition in the form of a Stampacchia type quasi variational inequality for a noncooperative game in normal form. We remark that the quasi-variational inequality (4) is equivalent to the following equation

$$P_{I_K(x^*)}(-\nabla J(x^*)) = 0,$$

where $P_{I_K(x^*)}$ denotes the orthogonal projection onto the internal cone $I_K(x^*)$ of K at $x^* \in K$.

Theorem 3.2. Let x^* satisfy

$$\langle \nabla J(x), x - x^* \rangle \geq 0 \quad \forall\, x \in x^* + S(x^*), \tag{5}$$

then x^* is a Nash equilibrium. When $J_i(\cdot, x_{-i})$ is quasiconvex for all $i \in I$, then the sufficient condition (5) is also necessary.

Proof. It is a direct consequence of Lemma 2.2. □

The condition (5) is a first order sufficient condition in the form of a Minty type quasi variational inequality for the Nash equilibrium of a noncooperative game in normal form.

Theorem 3.3. In addition to the usual K closed convex and J_i Gateaux differentiable with respect to x_i for all $i \in I$, assume that, for all $i \in I$, $J_i(\cdot, x_{-i})$ is pseudoconvex, then (4) and (5) are equivalent and any solution of both of them is a Nash equilibrium, i.e. (4) and (5) are both necessary and sufficient for x^* to be a Nash equilibrium.

Remark 3.1. We have written the necessary and sufficient conditions for a Nash equilibrium in the form of quasi variational inequalities, but in a simple and important case in which social restriction are not considered, i.e. when $K = \prod_{i \in I} \Sigma_i$, inequalities (4) and (5) take the more usual form of a variational inequality and they are equivalent when ∇J is a pseudomonotone operator, i.e.

$$\langle \nabla J(x^*), x - x^* \rangle \geq 0 \quad \forall\, x \in K,$$
$$\langle \nabla J(x), x - x^* \rangle \geq 0 \quad \forall\, x \in K.$$

A short summary of the results is shown in the Figure 1.

If we consider solutions of (4) in which the domain is the feasible set K, we obtain an other important type of equilibria of the game, the so-called variational equilibria.

Definition 3.5. A point $x^* \in K$ is said a variational equilibrium of the game (J, K) if

$$\langle \nabla J(x^*), x - x^* \rangle \geq 0 \quad \forall\, x \in K. \tag{6}$$

We now expose a uniqueness result for Nash and variational equilibria under strict monotonicity assumption.

Theorem 3.4. Let (J, K) be a game in normal form with the usual conditions and ∇J be a strictly monotone operator, then we have

(i) (J, K) has at most one variational equilibrium.

(ii) (J, K) has at most one Nash equilibrium in the relative interior of K.

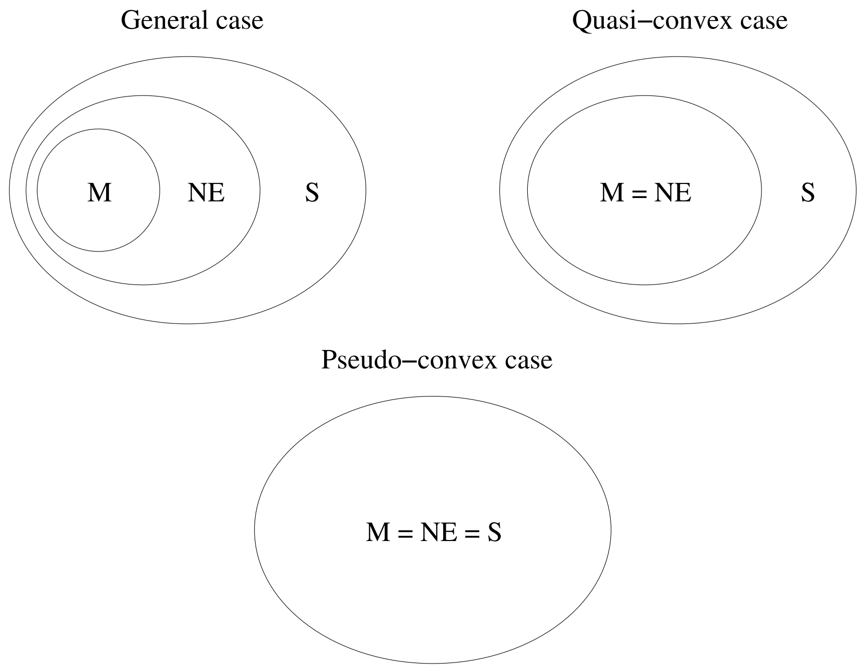

Figure 1: M = set of solutions to Minty type quasi variational inequality (5); NE = set of Nash equilibria of the game; S = set of solutions to Stampacchia type quasi variational inequality (4).

(iii) If $K = \prod_{i \in I} \Sigma_i$ then (J, K) has at most one Nash equilibrium.

Proof.

(i) Suppose, by contradiction, the game has two variational equilibria x^1 and x^2, with $x^1 \neq x^2$. Then we have

$$\langle \nabla J(x^1), x^2 - x^1 \rangle \geq 0,$$
$$\langle \nabla J(x^2), x^1 - x^2 \rangle \geq 0.$$

By strict monotonicity assumption we have

$$0 \leq \langle \nabla J(x^1) - \nabla J(x^2), x^2 - x^1 \rangle < 0,$$

which is absurd.

(ii) It follows from (i) because each Nash equilibrium in the relative interior of K is a variational equilibrium.

(iii) It follows from (i) because, in this case, each Nash equilibrium is a variational equilibrium.

□

When the assumption of pseudo-convexity on functions J_i holds true, then a variational equilibrium is a sufficient but, in general, not necessary condition for a Nash equilibrium. The difference between Nash equilibria and variational equilibria is explained, as we will see in what follows, by the inclusion of the internal cone $I_K(x^*)$ in the tangent cone $T_K(x^*)$ of K at x^*. If x^* is a Nash equilibrium then no deviation from x^* in the internal cone will be accepted by players; whereas if x^* is a variational equilibrium, then no deviations from x^* in the tangent cone will be accepted by players. Thus variational equilibria have greater social stability than Nash equilibria. Obviously, in the particular case in which there is no social constraint, that is $K = \prod_{i \in I} \Sigma_i$, then Nash equilibria and variational equilibria coincide (the internal cone $I_K(x)$ coincides with $T_K(x)$ for all $x \in K$). It may happen that there are infinitely many Nash equilibria but only one of them is a variational one, as the following example shows.

Example 3.1. (Ref. 5) (To locate a post office) This is a two-person noncooperative game: player i select the coordinate $x_i \in \mathbb{R}$ subject to the private restriction $\Sigma_i = \{x_i \leq 0\}$, and the social constraint is $x_1 + x_2 \leq -1$. The aim of player i is to minimize the distance between (x_1, x_2) and his favourite goal $P_i \in \mathbb{R}^2$, with $P_1 = (1, 0)$ and $P_2 = (0, 1)$; thus we have

$$J_1(x) = [(1 - x_1)^2 + x_2^2]/2, \qquad J_2(x) = [x_1^2 + (1 - x_2)^2]/2,$$
$$\nabla J(x_1, x_2) = (x_1 - 1, x_2 - 1).$$

The gradient of the game is strictly monotone, the set of Nash equilibria equals the closed line segment between $(-1, 0)$ and $(0, -1)$, whereas the only variational equilibrium is $(-1/2, -1/2)$ (see Figure 2).

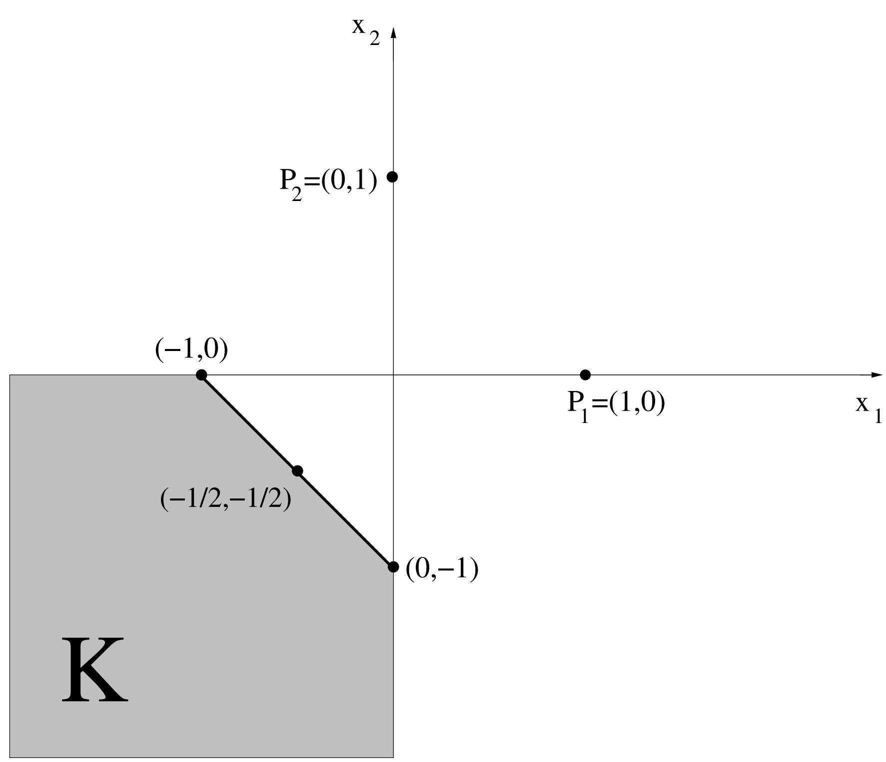

Figure 2: Example of infinitely many Nash equilibria but only one variational equilibrium.

4 The Convex Case. Ky Fan, Performance and Merit Functions

Given a game in normal form, we consider the following function that we call the *Ky Fan function* of the game:

$$\phi(x,y)=\sum_{i=1}^{n}[J_i(y_i,x_{-i})-J_i(x_i,x_{-i})], \qquad \text{where } \ x,y\in\prod_{i\in I}\Sigma_i.$$

The performance function of the game represent the loss of the society of players when they deviate from the planned x to another strategy y. It was introduced in (Ref. 6-7) with a slightly different definition from that considered here.

Definition 4.1. We call performance function of a game in normal form the function

$$\begin{aligned}\phi^\sharp(x) &= \inf_{y\in x+S(x)}\phi(x,y)= \\ &= \inf_{y\in x+S(x)}\textstyle\sum_{i\in I}[J_i(y_i,x_{-i})-J_i(x_i,x_{-i})]= \\ &= \min_{i\in I}\inf_{y_i\in x_i+S_i(x)}[J_i(y_i,x_{-i})-J_i(x_i,x_{-i})] \ \ \forall\ x\in K.\end{aligned}$$

The main properties of the performance function are here recalled shortly. They are:

- $\phi^\sharp(x)\le 0$ for all $x\in K$;
- $\phi^\sharp(x^*)=0$ with $x^*\in K \iff x^*$ is a Nash equilibrium of the game;
- the set of Nash equilibria is non empty $\iff \ \max_{x\in K}\phi^\sharp(x)=0$;

Thus Nash equilibria of the game are maximizers of the performance function. We want to show that the performance function is related to some merit functions for the quasi variational inequality (4) and Minty quasi variational inequality (5) associated to the game.

Define:

$$U(x)=\inf_{y\in x+S(x)}\langle\nabla J(y),y-x\rangle$$

$$L(x)=\inf_{y\in x+I_K(x)}\langle\nabla J(x),y-x\rangle$$

It easy to see that L and U are merit functions for the quasi variational inequality (4) and the Minty quasi variational inequality (5), respectively, i.e.:

- $L(x)\le 0$ for all $x\in K$ and $U(x)\le 0$ for all $x\in K$;
- $L(x^*)=0$ with $x^*\in K \iff x^*$ is a solution to quasi variational inequality (4);
- $U(x^*)=0$ with $x^*\in K \iff x^*$ is a solution to Minty quasi variational inequality (5).

Therefore we prove the following:

Theorem 4.1. Assume that $J_i(\cdot,x_{-i})$ are convex for all $i\in I$. We have:

$$U(x)\ge\phi^\sharp(x)\ge L(x) \quad \forall\ x\in K. \tag{7}$$

Proof. Taking into account convexity we have: for all $x\in K$, for all $y_i\in x_i+S_i(x)$ and for all $i\in I$:

$$\begin{aligned}\langle\partial J_i(y_i,x_{-i})/\partial x_i, y_i-x_i\rangle \ge&\quad J_i(y_i,x_{-i})-J_i(x_i,x_{-i});\\ J_i(y_i,x_{-i})-J_i(x_i,x_{-i}) \ge&\quad \langle\partial J_i(x_i,x_{-i})/\partial x_i, y_i-x_i\rangle.\end{aligned}$$

Finally we obtain the following relationship

$$\begin{aligned}
U(x) = & \inf_{y\in x+S(x)}\langle \nabla J(y), y-x\rangle = \\
= & \inf_{y\in x+S(x)} \sum_{i\in I}\langle \partial J_i(y_i, x_{-i})/\partial x_i, y_i - x_i\rangle \geq \\
\geq & \inf_{y\in x+S(x)} \sum_{i\in I}[J_i(y_i, x_{-i}) - J_i(x_i, x_{-i})] = \phi^\sharp(x) \geq \\
\geq & \inf_{y\in x+S(x)}\langle \nabla J(x), y-x\rangle = \\
= & \inf_{y\in x+I_K(x)}\langle \nabla J(x), y-x\rangle = L(x).
\end{aligned}$$

and therefore we have the thesis. □

(7) allows us to give the following three equivalent formulations of the equilibrium problem in the convex case:

- $\max_{x\in K} \inf_{y\in x+S(x)}\langle \nabla J(y), y-x\rangle = 0$;
- $\max_{x\in K} \inf_{y\in x+S(x)} \phi(x,y) = 0$;
- $\max_{x\in K} \inf_{y\in x+I_K(x)}\langle \nabla J(x), y-x\rangle = 0$.

We discuss briefly the well posedness of Nash equilibrium problem in a special, but significant case, in which things are easily obtained.

Definition 4.2. We say that $\{x_n\}$, with $n \in \mathbb{N}$, is an approximating sequence for a Nash equilibrium when

$$\lim_{n\to+\infty} \phi^\sharp(x_n) = \lim_{n\to+\infty} \inf_{y\in\, x_n+S(x_n)} \phi(x_n, y) = 0$$

and we call x_ε, with $\varepsilon > 0$, an ε-approximate Nash equilibrium if

$$\phi^\sharp(x_\varepsilon) \geq -\varepsilon.$$

Definition 4.3. We say that a Nash equilibrium problem (J,K) is Tykhonov well posed (Ref. 6) if

(i) (J,K) has a unique Nash equilibrium;

(ii) every approximating sequence converges to the unique Nash equilibrium.

Definition 4.4. (Ref. 6) Let us call a game (J,K) coercive if there is an $\varepsilon > 0$ such that

$$\{x \in K : \ \phi^\sharp(x) \geq \varepsilon\} = L_\varepsilon$$

is compact.

With these definitions, we have the following result.

Theorem 4.2. Any coercive game (J,K), with $K = \prod_{i\in I}\Sigma_i$, with continuous payoff J_i and strictly monotone gradient ∇J, is Tykhonov well posed.

Proof. The game has equilibria and because of Theorem 3.4 there is only one, let us call it x^*. We take any approximating sequence x_n and we observe that it must belong to a compact set, by coercivity. We consider a convergent subsequence $x_{k_n} \to \overline{x}$, and it is again an approximating sequence and

$$\phi^\sharp(\overline{x}) \geq \limsup_{n\to+\infty} \phi^\sharp(x_{k_n}) = 0,$$

so that $\overline{x}$ is a Nash equilibrium. We have $\overline{x} = x^*$ and this for all converging subsequences, so $\lim\limits_{n\to+\infty} x_n = x^*$.
□

Remark 4.1. We recall that well posedness is important because any method for generating approximating sequences is a method to approximate the equilibrium. For example any bimatrix game with strictly monotone gradient is well posed. Strictly monotonicity of the gradient is no more sufficient for well posedness when social constraints are considered, as the Example 3.1 shows.

5 Dynamical Systems

After we defined equilibria of the game, we are interested to dynamical systems which describe the evolution of the game from a disequilibrium starting point. One of the natural ways to define the adjustment process is the following: we suppose that all the n players are in the state $x \in K$ at starting point t_0, then each player i wish to move along the antigradient $-\nabla J(x)$, because this direction offers him the steepest cost reduction. If x is an internal point of K, then we can consider this motion direction without problems. However if x stays on the boundary of the feasible set K, the antigradient of the game $-\nabla J(x)$ may have a normal component to K at x, thus we have to project this direction in such a way that obtaining a feasible direction. We can do it in three different ways.

The first one is to use the projection on the tangent cone of K, that is we consider the dynamical system

$$\dot{x}(t) = P_{T_K(x)}(-\nabla J(x)). \tag{8}$$

It would be possible to prove that the steady states of (8) equal the solutions to variational inequality (6), that is the variational equilibria of the game.

The second one is to exploit the projection on the whole set K, and we study the dynamical system

$$\dot{x}(t) = P_K(x - \alpha \nabla J(x)) - x, \tag{9}$$

where α is a fixed positive constant. It is trivial to check that also the steady states of (9) coincide with variational equilibria of the game.

Finally, the third one is to use the projection on the internal cone of K, thus we have the third dynamical system

$$\dot{x}(t) = P_{I_K(x)}(-\nabla J(x)). \tag{10}$$

In this case we remark that its steady states coincide with the solutions to the quasi variational inequality (4), and when the assumption of pseudo-convexity on J_i holds true, they equals Nash equilibria of the game.

We remark that the third dynamical system gives a better interpretation of the adjustment process of the game than two others, in fact in the dynamical systems (8) and (9) a Nash equilibrium is not necessarily a steady state. We can explain this behaviour if we assume an authority unattached to players controlling the evolution of the game. We can see this same situation in the previous example: if we consider $(0, -1)$ as starting point, then the solutions of (8) and (9) move along the set of Nash equilibria and converge to the only variational equilibrium $(-1/2, -1/2)$.

Moreover, we observe that in the dynamical system (9) the vector $P_K(x - \alpha \nabla J(x)) - x$ is, in a certain sense, an approximation of the projection of $-\nabla J(x)$ on the tangent cone $T_K(x)$. Hence on the one hand dynamical system (9) gives a worse interpretation of the adjustment process of the game than (8), on the other hand (9) is, in general, more smooth than (8) (P_K is Lipschitz continuous everywhere on K, but P_{T_K} is not).

We now pass to outline some convergence results for dynamical systems (8).

Theorem 5.1. Let $x^* \in K$ be a stationary point of (8). If the gradient of the game is strictly monotone with modulus ψ on K, that is

$$\langle \nabla J(x) - \nabla J(y), x - y \rangle \geq \psi(\|x - y\|), \quad \forall\, x, y \in K,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a forcing function, namely lower semicontinuous and

$$\psi(x) = 0 \iff x = 0,$$
$$\lim_{n\to+\infty} \psi(x_n) = 0 \implies \lim_{n\to+\infty} x_n = 0,$$

then for every trajectory $x(t)$ solving (8), with $x(0) \in K$, we have

$$\lim_{t\to+\infty} x(t) = x^*.$$

Proof. We denote $N_K(x)$ the normal cone to K at x, and we consider the squared distance function $\lambda(t) = \|x(t) - x^*\|^2/2$. Its derivative is

$$\begin{aligned}\dot\lambda(t) &= \langle x(t) - x^*, P_{T_K(x(t))}(-\nabla J(x(t)))\rangle = \\ &= \langle x(t) - x^*, -\nabla J(x(t)) - P_{N_K(x(t))}(-\nabla J(x(t)))\rangle \le \\ &\le \langle x(t) - x^*, -\nabla J(x(t))\rangle \le \\ &\le -\langle x(t) - x^*, -\nabla J(x(t)) - \nabla J(x^*)\rangle \le \\ &\le -\psi(\|x(t) - x^*\|) < 0.\end{aligned}$$

Thus λ is a decreasing function of t and there exists $\lim_{t\to+\infty} \lambda(t) = l \ge 0$. We suppose by contradiction that $l > 0$, then $\|x(t) - x^*\| > \sqrt{2l} > 0$ for all $t \ge 0$, hence there is $m > 0$ such that $\psi(\|x(t) - x^*\|) \ge m > 0$, because ψ is a forcing function. Therefore $\dot\lambda(t) \le -m$ for all $t \ge 0$ and $\lim_{t\to+\infty} \lambda(t) = -\infty$, but this is impossible, thus $l = 0$ and the proof is complete. □

For the dynamical system (9) we achieve the following result (Ref. 8).

Theorem 5.2. Let $x^* \in K$ be a stationary point of (9). If ∇J is strongly monotone on K with constant $\eta > 0$ and Lipschitz continuous on K with constant $L > 0$, then there is $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$ there exists a constant $C > 0$ such that for each solution $x(t)$ of (9), with $x(0) \in K$, we have

$$\|x(t) - x^*\| \le \|x(0) - x^*\|\, exp\,(-C\,t), \qquad \forall\, t \ge 0,$$

that is $x(t)$ converges exponentially to x^*.

Proof. First we remark that for all $x \in K$ the following holds:

$$\begin{aligned} & \|P_K(x - \alpha\,\nabla J(x)) - x^*\|^2 = \\ = \;& \|P_K(x - \alpha\,\nabla J(x)) - P_K(x^* - \alpha\,\nabla J(x^*))\|^2 \\ \le \;& \|x - x^* - \alpha\,(\nabla J(x) - \nabla J(x^*))\|^2 \\ = \;& \|x - x^*\|^2 - 2\alpha\,\langle \nabla J(x) - \nabla J(x^*), x - x^*\rangle + \alpha^2\,\|\nabla J(x) - \nabla J(x^*)\|^2 \\ \le \;& (1 - 2\,\alpha\,\eta + \alpha^2\,L^2)\|x - x^*\|^2.\end{aligned}$$

We denote $\alpha_0 = 2\eta/L^2$ and let $0 < \alpha < \alpha_0$. Taking into account (Ref. 8) the fact that the solutions of (9) remain in K $\forall t$, for any solution $x(t)$ with $x(0) \in K$ we can consider $\lambda(t) = \|x(t) - x^*\|^2/2$, thus we have:

$$\begin{aligned}\dot\lambda(t) &= \langle x(t) - x^*, P_K(x(t) - \alpha\,F(x(t))) - x(t)\rangle \\ &= \langle x(t) - x^*, P_K(x(t) - \alpha\,\nabla J(x(t))) - x^*\rangle - \|x(t) - x^*\|^2 \\ &\le \|x(t) - x^*\|\,\|P_K(x(t) - \alpha\,\nabla J(x(t))) - x^*\| - \|x(t) - x^*\|^2 \\ &\le (\sqrt{1 - 2\,\alpha\,\eta + \alpha^2\,L^2} - 1)\,\|x(t) - x^*\|^2 \\ &= -2\,C\,\lambda(t)\end{aligned}$$

where $C = 1 - \sqrt{1 - 2\,\alpha\,\eta + \alpha^2\,L^2} = 1 - \sqrt{1 - L^2\,\alpha(\alpha_0 - \alpha)} > 0$. Thus

$$\|x(t) - x^*\| \le \|x(0) - x^*\|\, exp(-C\,t), \qquad \forall\, t \ge 0.$$

□

References

1. KINDERLEHRER, D., and STAMPACCHIA, G., *An Introduction to Variational Inequality and Their Application*, Academic Press, New York, NY, 1980.

2. FUDENBERG, D., TIROLE, J., *Game Theory*, MIT Press, Cambridge Massachusetts, MA, 1993.

3. GIBBONS, R., *Teoria dei Giochi*, Il Mulino, Bologna, Italy, 1992.

4. CAVAZZUTI, E., *Esistenza ed Approssimazione di Equilibri di Nash*, Metodi di ottimizzazione per le decisioni, Edited by G. Di Pillo, Masson, Milano, Italy, 1994.

5. LUENBERGER, D. G., *Complete Stability of Noncooperative Games*, Journal of Optimization Theory and Applications, Vol. 25, pp. 485-505, 1978.

6. CAVAZZUTI, E., *Cobwebs and Something Else*, Decision Processes in Economics, Edited by G. Ricci, Springer, Berlin, Germany, 1991.

7. CAVAZZUTI, E., and MORGAN, J., *Well Posed Saddle Point Problems*, Optimization: Theory and Algorithms, Edited by J.-B. Hiriart-Urruty, W. Oettli and J. Stoer, Dekker, New York, NY, 1983.

8. PAPPALARDO, M. and PASSACANTANDO, M., *Stability for Equilibrium Problems: from Variational Inequalities to Dynamical Systems*, University of Pisa, Department of Applied Mathematics Report No. 13, 2001, accepted for publication in Journal of Optimization Theory and Applications.

9. AUBIN, J. P., and CELLINA, A., *Differential Inclusions. Set-valued Maps and Viability Theory*, Springer-Verlag, Berlin, Germany, 1984.

10. CAVAZZUTI, E., and FLÅM, S. D., *Evolution to Selected Nash Equilibria*, Nonsmooth optimization: Methods and Applications, Edited by F. Giannessi, Gordon and Breach, Philadelphia, PA, 1992.

11. YAO, J.-C., *Variational Inequalities with Generalized Monotone Operators*, Mathematics of Operations Research, Vol. 19, pp. 691-705, 1994.

12. GIANNESSI, F., *On Minty Variational Principle*, New Trends in Mathematical Programming, Applied Optimization Vol. 13, Edited by F. Giannessi, S. Komlosi and T. Rapsak, Kluwer, Boston, MA, 1998.

13. AUBIN, J. P., *Mathematical Methods of Game and Economic Theory*, North Holland, Amsterdam, Holland, 1979.