



The projection problem in commutative, positively ordered monoids

Gianluca Cassese¹

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Abstract

We examine the problem of projecting subsets of a commutative, positively ordered monoid into an \mathcal{o} -ideal. We prove that to this end one may restrict to a sufficient subset, for whose cardinality we provide an explicit upper bound. Several applications to set functions, vector lattices and other more explicit structures are provided.

Keywords κ -domain · κ -ideal · Ordered monoid · Prime \mathcal{o} -ideal · Projection · Semilattice

1 Introduction

Several problems in analysis are greatly simplified by the possibility of reducing the cardinality of the set under scrutiny from arbitrary to finite – or at least countable – which follows from compactness or separability. In this paper we explore the possibility of a similar simplification arising from a notion of a purely set theoretic nature, κ -ideals, a concept originally introduced by Tarski [15] that we adapt to the study of commutative, positively ordered monoids (or semigroups). In the context of this mathematical structure we define, in Sect. 2, ideals and projections and investigate, in Sect. 4, the projection problem, that is the problem of projecting a subset of a positively ordered monoid into a given ideal. We show in Theorem 4.2 that a set can be projected on an ideal if and only if the same is true for any of its subsets with cardinality less than some explicit bound – often just countable subsets. The proof is elementary and exploits some properties of cardinal numbers. Given the simple mathematical structure of positively ordered monoids, our result is quite general and abstract, although it has almost immediate applications to lattices, Boolean algebras and to families of set functions. The main applications of these results are developed

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✉ Gianluca Cassese
gianluca.cassese@unimib.it

¹ Università Milano Bicocca, via Bicocca degli Arcimboldi 8, 20126 Milano, Italy

in Sect. 5 where we introduce the class $\mathcal{V}(\mathfrak{M})$ of functions of finite variation defined on a p.o. monoid \mathfrak{M} . As an application of Theorem 4.2, we obtain in Theorem 5.3 a necessary and sufficient condition for a subset $V \subseteq \mathcal{V}(\mathfrak{M})$ to admit a strictly positive element. This problem is connected to Maharam problem in the theory of additive functions on Boolean algebras.

To simplify definitions, throughout the paper we assume the commutative property without explicit mentioning, so that a monoid or semigroup is always meant to be commutative.

For the rest of the paper, and without further mention, \mathfrak{M} will be a positively ordered (p.o.) monoid, as defined by Clifford [6, p. 308]. That is, \mathfrak{M} is a monoid (written multiplicatively and with 1 designating its unit) endowed with a partial order \geq that satisfies

$$1 \geq m \quad m \in \mathfrak{M} \quad \text{and} \quad (1a)$$

$$m \geq m' \quad \text{implies} \quad mh \geq m'h \quad m, m', h \in \mathfrak{M}. \quad (1b)$$

Every monoid is a p.o. monoid if we write $m \geq n$ whenever m divides n ¹.

Our results become significantly simpler if we assume, with no loss of generality, the existence of a least element $0 \in \mathfrak{M}$. Two elements $m, n \in \mathfrak{M}$ are disjoint if $mn = 0$ and a set $M \subseteq \mathfrak{M}$ is mutually disjoint if $0 \notin M$ and $mn = 0$ for all distinct pairs $m, n \in M$. When $M \subseteq \mathfrak{M}$ we also use the lattice notation

$$M^\perp = \{n \in \mathfrak{M} : nm = 0 \text{ for all } m \in M\}. \quad (2)$$

At times we shall require in addition that \mathfrak{M} is *normal*, i.e. that 0 is the only nilpotent element of \mathfrak{M} , or even that \mathfrak{M} is idempotent (more precisely, that all elements of \mathfrak{M} are idempotent²).

Many well known mathematical structures, such as lattices and Boolean algebras, are examples of a p.o. monoid or semigroup. If \mathfrak{M}_α is a p.o. monoid for each $\alpha \in \mathfrak{A}$, the product p.o. monoid is the set $\times_{\alpha \in \mathfrak{A}} \mathfrak{M}_\alpha$ with order and composition defined coordinatewise. The space $\mathfrak{F}(Z, \mathfrak{M})$ of all functions defined on some arbitrary set Z and with values in a p.o. monoid \mathfrak{M} is then a product p.o. monoid. If X is an arbitrary set and each $x \in X$ is identified, via the evaluation map, with a function $\hat{x} \in \mathfrak{F}(Z, [0, 1])$ where $Z = \mathfrak{F}(X, [0, 1])$, we obtain the *abstract p.o. monoid* $\hat{X} = \{\hat{x} : x \in X\}$ associated with X .

We denote by $\mathfrak{c}(A)$ the cardinality of a set A and refer to A as an \mathfrak{n} -set if \mathfrak{n} is a cardinal number and $\mathfrak{n} > \mathfrak{c}(A)$. If $F \subseteq \mathfrak{F}(X, Y)$, the image of $A \subseteq X$ under $f \in F$ is written as $f[A]$ and we let $F[A] = \bigcup_{f \in F} f[A]$.

¹ A p.o. semigroup \mathfrak{S} is defined likewise, but replacing (1a) with the condition: $m \geq mn$ for all $m, n \in \mathfrak{S}$. Given that each p.o. semigroup may be embedded into a p.o. monoid, we shall mainly focus on the latter structure.

² Idempotent semigroups are often referred to as *semilattices*. See Birkhoff [3, p. 9], Leader [10] or Blyth [4, p. 19]

2 Preliminary notions: ideals and projections

Given the interaction between algebraic and order properties, several concepts, including ideals and projections, may be given distinct definitions depending if considered in algebraic or in order terms. This section is of limited mathematical content, provides some rigorous definitions and proves some basic facts.

A monoid ideal (or simply an ideal, for short) in \mathfrak{M} is a subset $\mathcal{I} \subseteq \mathfrak{M}$ such that ³

$$m \in \mathcal{I} \text{ and } n \in \mathfrak{M} \text{ imply } mn \in \mathcal{I}; \quad (3)$$

an order ideal (*o*-ideal) is a subset $\mathcal{J} \subseteq \mathfrak{M}$ satisfying the more restrictive condition

$$m \in \mathcal{J}, n \in \mathfrak{M} \text{ and } m \geq n \text{ imply } n \in \mathcal{J}. \quad (4)$$

Given any monoid \mathfrak{M} , even without a partial order, an ideal $\mathcal{I} \subseteq \mathfrak{M}$ induces a reflexive and transitive binary relation $\geq_{\mathcal{I}}$ defined by

$$m \geq_{\mathcal{I}} n \text{ whenever } mh \in \mathcal{I} \text{ implies } nh \in \mathcal{I} \quad h \in \mathfrak{M}. \quad (5)$$

The associated equivalence relation $\sim_{\mathcal{I}}$ (i.e. $m \sim_{\mathcal{I}} n$ if $m \geq_{\mathcal{I}} n$ and $n \geq_{\mathcal{I}} m$), generates a quotient monoid $\mathfrak{M}/\sim_{\mathcal{I}}$, which we write more simply as \mathfrak{M}/\mathcal{I} . Defining multiplication of equivalence classes in the usual way and letting $m/\mathcal{I} \geq n/\mathcal{I}$ if and only if $m \geq_{\mathcal{I}} n$, we obtain a p.o. monoid ⁴. The canonical map $\mathfrak{M} \rightarrow \mathfrak{M}/\mathcal{I}$ is a homomorphism of monoids but, if \mathfrak{M} is a p.o. monoid, it preserves order if and only if \mathcal{I} is an *o*-ideal. If $M \subseteq \mathfrak{M}$ we use the standard notation

$$M/\mathcal{I} = \{m/\mathcal{I} : m \in M\}. \quad (6)$$

If m/\mathcal{I} and n/\mathcal{I} are disjoint we say that m and n are \mathcal{I} -disjoint and this is equivalent to $mn \in \mathcal{I}$. A set $M \subseteq \mathfrak{M}$ is mutually \mathcal{I} -disjoint if M/\mathcal{I} is mutually disjoint.

o-ideals have special importance. Each set $A \subseteq \mathfrak{M}$ generates a corresponding *o*-ideal defined as

$$I(A) = \bigcup_{a \in A} \{m \in \mathfrak{M} : m \leq a\}^5 \quad (7)$$

The map $A \rightarrow I(A)$ is clearly a closure operation which induces the order topology τ_o . A map between two p.o. monoids, each endowed with its own order topology, is continuous if and only if it preserves order. The composition operation is thus a continuous mapping of $\mathfrak{M} \times \mathfrak{M}$ to \mathfrak{M} and (\mathfrak{M}, τ_o) a topological p.o. monoid.

³ Our definition corresponds to that of a semigroup ideal, see e.g. Anderson and Johnson [2] or Rees [13]. We adopt the convention that the empty set is an ideal.

⁴ The factor p.o. monoid \mathfrak{M}/\mathcal{I} should not be confused with other quotients, such as the factor semigroup $\mathfrak{M} - \mathcal{I}$ defined by Rees [13, p. 389]. The latter need not possess an order structure and is induced by the equivalence relation $m = n$ or $m, n \in \mathcal{I}$. Rees congruence implies $\sim_{\mathcal{I}}$ but the converse need not be true.

⁵ We prefer $I(m)$ to $I(\{m\})$, when $m \in \mathfrak{M}$, and $I(p)$ to $I(p[\mathfrak{M}])$, if $p \in \mathfrak{F}(\mathfrak{M}, \mathfrak{M})$.

Further classes of ideals are of interest. \mathcal{I} is a radical ideal if $m \in \mathcal{I}$ whenever $m^j \in \mathcal{I}$ for some $j \in \mathbb{N}$. We write $\sqrt{\mathcal{I}} = \{m \in \mathfrak{M} : m^j \in \mathcal{I} \text{ for some } j \in \mathbb{N}\}$. \mathcal{I} is radical if and only if \mathfrak{M}/\mathcal{I} is normal. We also define a D -ideal to be an o -ideal \mathcal{I} such that $\mathcal{I} \cap I(m)$ admits a greatest element for each $m \in \mathfrak{M}$.

Lemma 2.1 $\mathcal{I} \subseteq \mathfrak{M}$ is a D -ideal if and only if $\mathcal{I} = p[\mathfrak{M}]$ for some order preserving $p \in \mathfrak{F}(\mathfrak{M}, \mathfrak{M})$ satisfying

$$m \geq p(m) \quad m \in \mathfrak{M} \quad \text{and} \quad (8a)$$

$$p(m) \geq n \quad \text{implies} \quad n = p(n) \quad m, n \in \mathfrak{M}. \quad (8b)$$

Proof If p satisfies (8b) its range is necessarily an o -ideal; if, in addition, p preserves order and satisfies (8a), then $p(n) \leq m$ implies $p(n) \leq p(m) \leq m$ so that its range is a D -ideal. Conversely, if \mathcal{I} is a D -ideal and if $p(m)$ is the greatest element in the set $\mathcal{I} \cap I(m)$, then the map p clearly preserves order and satisfies (8a). Moreover, if $n \leq p(m)$ then $n \in \mathcal{I}$ so that $\mathcal{I} \cap I(n) = I(n)$, i.e. $n = p(n)$ so that $I(p) = p[\mathfrak{M}]$. Eventually, $\mathcal{I} = \bigcup_{m \in \mathfrak{M}} \mathcal{I} \cap I(m) = \bigcup_{m \in \mathfrak{M}} I(p(m)) = I(p)$. \square

By analogy with the theory of vector lattices, a map with the properties of Lemma 2.1 is called an o -projection. The family \mathfrak{P} of o -projections forms a p.o., idempotent monoid if endowed with composition. The corresponding algebraic notion is that of a (monoid) projection i.e. an order preserving map $q \in \mathfrak{F}(\mathfrak{M}, \mathfrak{M})$ which satisfies (8a) and

$$mq(n) \leq q(mn) \quad m, n \in \mathfrak{M}. \quad (8c)$$

Denoting by \mathfrak{Q} the family of projections we have $\mathfrak{P} \subseteq \mathfrak{Q}$ ⁶ while the converse holds if and only if \mathfrak{M} is idempotent. More interestingly, each $m \in \mathfrak{M}$ corresponds with a projection m^* on \mathfrak{P} via the identity

$$(m^*(p))(n) = p(nm) \quad p \in \mathfrak{P}, \quad n \in \mathfrak{M}. \quad (9)$$

The main example of a projection is the translate T_m by m , defined as $T_m(n) = mn$.

3 κ -ideals

The following is a generalisation of the classical notion of a prime ideal (due to Tarski [15, Definition 4.1]) and often used in set theory. We adapt the definition to p.o. monoids.

Definition 3.1 Let $\kappa \geq 2$ be a cardinal. A κ -ideal in \mathfrak{M} is an o -ideal \mathcal{I} with the property that every mutually \mathcal{I} -disjoint subset of \mathfrak{M} is a κ -set. A 2-ideal is referred to as a prime ideal. ⁷

⁶ In fact, if $p \in \mathfrak{P}$ then (8b) and $mp(n) \leq p(n)$ imply $mp(n) = p(mp(n))$ while (8a) implies $p(mp(n)) \leq p(mn)$.

⁷ In the terminology of [9], a κ -ideal is a κ -saturated o -ideal. The restriction to o -ideals is only for terminological convenience.

Notice that if \mathcal{I} is a κ -ideal and \mathcal{I}' is an o -ideal contained in \mathcal{I} , then \mathcal{I}' need not be a κ -ideal.

Some properties of prime ideals carry over unchanged from the theory of rings. The family of prime ideals contains \emptyset and is closed with respect to arbitrary unions. Their complements form thus a base for a topology, τ_p . If $q \in \mathfrak{Q}$ and \mathcal{I} is a prime ideal, then $q^{-1}(\mathcal{I})$ is clearly an o -ideal. Moreover, if $m, n \in \mathfrak{M}$ are such that $q(mn) \in \mathcal{I}$, then $q(m)q(n) \in \mathcal{I}$, by (8c) and (8a). We must then have either $q(n) \in \mathcal{I}$ or $q(m) \in \mathcal{I}$. Thus $q^{-1}(\mathcal{I})$ is a prime ideal when \mathcal{I} is so.

Lemma 3.2 (i) (\mathfrak{M}, τ_p) is a topological p.o. monoid in which each $q \in \mathfrak{Q}$ is continuous, (ii) if $\{0\}$ is prime, the annihilator M^\perp of any $M \subseteq \mathfrak{M}$ is τ_p -closed and (iii) each radical o -ideal is the intersection of all prime ideals containing it (and is thus closed).

Proof Radical, o -ideals are closed with respect to arbitrary unions. If \mathcal{I} is such an ideal and if $a \notin \mathcal{I}$, by Zorn's lemma we can form a maximal radical o -ideal \mathcal{I}_a which includes \mathcal{I} but not a . Suppose that $m, n \in \mathfrak{M}$ are such that $mn \in \mathcal{I}_a$. If $n, m \in \mathcal{I}_a^c$ then the radical o -ideals $\mathcal{I}_a \cup \sqrt{I(m)}$ and $\mathcal{I}_a \cup \sqrt{I(n)}$ both contain a , by maximality. There exist then $j, k \in \mathbb{N}$ such that $a^j \leq m$ and $a^k \leq n$ and thus $a^{j+k} \leq mn \in \mathcal{I}_a$. But this implies $a \in \mathcal{I}_a$, a contradiction. Thus \mathcal{I}_a is prime and $\mathcal{I} = \bigcap_{a \notin \mathcal{I}} \mathcal{I}_a$. \square

The well-ordering principle permits the following definition:

Definition 3.3 Given a subset $M \subseteq \mathfrak{M}$ and an ideal \mathcal{I} in \mathfrak{M} we define $\kappa(M, \mathcal{I})$ to be the least cardinal number $> \mathfrak{c}(M_0)$ for any mutually \mathcal{I} -disjoint subset $M_0 \subseteq M$. We write $\kappa(M, \{0\}) = \kappa(M)$.

By definition, every o -ideal is a $\kappa(\mathfrak{M}, \mathcal{I})$ -ideal. In applications, we shall mainly encounter the case $\kappa(M, \mathcal{I}) \leq \aleph_1$. In general, computing $\kappa(M, \mathcal{I})$ may not be easy. We provide some explicit examples.

Example 1 Let \mathfrak{M} be the monoid of real valued, non negative, lower semicontinuous functions on some topological space X with binary operation $fg = f \wedge g$. If X is separable, then $\kappa(\mathfrak{M}) \leq \aleph_1$; if X is compact and totally disconnected then $\kappa(\mathfrak{M}) \leq \aleph_0$.

Example 2 Consider a commutative monoid \mathfrak{M} with its natural order. Then, ideals and o -ideals coincide. Let m_1, \dots, m_N be distinct, irreducible elements of \mathfrak{M} and let $m_0 = \prod_{i=1}^N m_i$. The ideal $I(m_0)$ is clearly seen to be a radical o -ideal. Consider $M \subseteq \mathfrak{M}$ to be mutually $I(m_0)$ -disjoint. Then, for each $h \in M$ there must be an integer $1 \leq i \leq N$ such that m_i does not divide h . At the same time, since $hf \in I(m_0)$ when $h, f \in M$, for each i there is at most one element in M which is not divided by m_i . It follows that $\mathfrak{c}(M) \leq N$.

Example 3 Let L be an AL -space ([1, p. 193]). Fix $x_0 \in L_+$ and consider $\mathfrak{M} = \{x \in L : 0 \leq x \leq x_0\}$ endowed with the binary operation \wedge . If $x_1, \dots, x_n \in \mathfrak{M}$ are mutually disjoint, then

$$\|x_0\| \geq \|x_1 \vee \dots \vee x_n\| = \|x_1\| + \dots + \|x_n\|. \quad (10)$$

This implies that $\kappa(\mathfrak{M}) \leq \aleph_1$.

Erdős and Tarski [8] proved that the general conjecture that $\kappa(M, \mathcal{I})$ may be any cardinal number is false. In the next Lemma we adapt their result to the present setting.

Lemma 3.4 (Erdős and Tarski) *Let \mathcal{I} be a radical \mathcal{o} -ideal in \mathfrak{M} and fix $M \subseteq \mathfrak{M}$. Then $\kappa(M, \mathcal{I}) = \kappa(M/\mathcal{I})$ and $\kappa(M, \mathcal{I})$ cannot be a singular limit cardinal nor \aleph_0 .*

Proof The inequality $\kappa(M/\mathcal{I}) \leq \kappa(M, \mathcal{I})$ is clear. Let $m, n \in \mathcal{I}^c$ be distinct, \mathcal{I} -disjoint elements such that $m/\mathcal{I} = n/\mathcal{I}$. Then necessarily

$$0/\mathcal{I} = (mn/\mathcal{I}) = (m/\mathcal{I})(n/\mathcal{I}) = (m/\mathcal{I})(m/\mathcal{I}) = (m^2/\mathcal{I})$$

i.e. $m^2 \in \mathcal{I}$ which is impossible if \mathcal{I} is radical.

The second claim follows from [8, Theorem 1] once we prove that two elements $x, z \in \mathfrak{M}/\mathcal{I}$ are mutually disjoint if and only if there is no $y \in \mathfrak{M}/\mathcal{I}$ such that $y \leq x$ and $y \leq z$ other than $0/\mathcal{I}$. One implication follows from the fact that \mathfrak{M}/\mathcal{I} is a p.o. monoid so that $xz \leq x$ and $xz \leq z$. Conversely, if $h/\mathcal{I} \leq m/\mathcal{I}$ and $h/\mathcal{I} \leq n/\mathcal{I}$ for some $m, n, h \in \mathfrak{M}$ with $mn \in \mathcal{I}$, we conclude

$$(h^2/\mathcal{I}) = (h/\mathcal{I})(h/\mathcal{I}) \leq (n/\mathcal{I})(m/\mathcal{I}) = 0/\mathcal{I}, \quad (11)$$

i.e. that $h \in \mathcal{I}$ since \mathcal{I} is radical. This shows that when \mathcal{I} is a radical \mathcal{o} -ideal a subset of \mathfrak{M}/\mathcal{I} is mutually disjoint in our definition if and only if it is so in the sense of [8, p. 316]. \square

The simple properties of prime ideals listed in Lemma 3.2 need not be true in the case of κ -ideals.

Lemma 3.5 *If κ is a regular cardinal number then the intersection of a κ -family of κ -ideals is a κ -ideal and, if $\kappa = \aleph_0$, so is their union.*

Proof First of all, $\mathcal{I}_0 = \bigcap_{\alpha} \mathcal{I}_{\alpha}$ and $\mathcal{I}_1 = \bigcup_{\alpha} \mathcal{I}_{\alpha}$ are \mathcal{o} -ideals if \mathcal{I}_{α} is so for each index α in the κ -set \mathfrak{A} . Choose $M \subseteq \mathfrak{M}$ to be mutually \mathcal{I}_0 -disjoint. Then, $M \subseteq \mathcal{I}_0^c$. Write $M_{\alpha} = M \cap \mathcal{I}_{\alpha}^c$. Of course, M_{α} is mutually \mathcal{I}_{α} -disjoint. But then, given that $M = \bigcup_{\alpha} M_{\alpha}$,

$$\mathfrak{c}(M) \leq \sum_{\alpha \in \mathfrak{A}} \mathfrak{c}(M_{\alpha}) \leq \mathfrak{c}(\mathfrak{A}) \cdot \kappa \leq \kappa. \quad (12)$$

However, if $\mathfrak{c}(M) = \kappa$ then κ is a singular, limit cardinal a contradiction. This proves that \mathcal{I}_0 is a κ -ideal.

Concerning union, assume that \mathfrak{A} is a finite set and choose $M \subseteq \mathfrak{M}$ to be mutually \mathcal{I}_1 -disjoint. Suppose that $\mathfrak{c}(M) \geq \aleph_0$. By passing to a subset, we can assume with no loss of generality that $\mathfrak{c}(M) = \aleph_0$. Then, since $mn \in \mathcal{I}_{\alpha}$ for some $\alpha \in \mathfrak{A}$, we conclude from a well known result of Ramsey [12, Theorem A] (see also Erdős and Rado [7, Theorem 1]) that there exist $\alpha_0 \in \mathfrak{A}$ and a subset $M_0 \subseteq M$ such that $\mathfrak{c}(M_0) = \aleph_0$ and that M_0 is \mathcal{I}_{α_0} -disjoint, which contrasts with the assumption that each \mathcal{I}_{α} is a κ -ideal. Thus necessarily $\mathfrak{c}(M) < \aleph_0$. \square

4 The projection problem

Each D -ideal is the range of some o -projection. In general, the question of whether a projection maps a given subset of \mathfrak{M} into some ideal is not trivial and we refer to it as the projection problem. More formally,

Definition 4.1 (*Projection Problem*) Given $M \subseteq \mathfrak{M}$, $Q \subseteq \Omega$ and an ideal \mathcal{I} in \mathfrak{M} , is there $q \in Q$ such that $q[M] \subseteq \mathcal{I}$?

This problem has an interesting structure in the case in which \mathcal{I} is a radical o -ideal. Given the preceding remarks, there is no loss of generality in setting $\mathcal{I} = \{0\}$ and assuming that \mathfrak{M} is normal. The following result establishes the existence of sufficient subsets of bounded cardinality.

Theorem 4.2 Assume that \mathfrak{M} is normal and let $Q \subseteq \Omega$. Each $M \subseteq \mathfrak{M}$ admits a $\kappa(Q[M])$ -subset $M_0 \subseteq M$ which is sufficient for Q , i.e. such that

$$q[M] = \{0\} \text{ if and only if } q[M_0] = \{0\} \quad q \in Q. \quad (13)$$

Moreover, $Q[M] \cap Q[M_0]^\perp \subseteq \{0\}$.

Proof Form the class of all subsets of $Q[M]$ which are mutually disjoint and choose, by virtue of Zorn's lemma, a maximal set $\{q_\alpha(m_\alpha) : \alpha \in \mathfrak{A}\}$ in this class. Write $M_0 = \{m_\alpha : \alpha \in \mathfrak{A}\}$. Then, $\mathfrak{c}(M_0) \leq \mathfrak{c}(\mathfrak{A}) < \kappa(Q[M])$. To prove the last claim first, let $q(n) \in Q[M]$ be such that $q(n)q_\alpha(m_\alpha) = 0$ for all $\alpha \in \mathfrak{A}$. Given that \mathfrak{M} is normal, the collection $\{q_\alpha(m_\alpha) : \alpha \in \mathfrak{A}\} \cup \{q(n)\}$ contains $\{q_\alpha(m_\alpha) : \alpha \in \mathfrak{A}\}$ properly so that $q(n) \neq 0$ would contradict maximality. Let $q_0 \in Q$ satisfy $q_0[M_0] = \{0\}$. By (8c),

$$q_0(m)q_\alpha(m_\alpha) \leq q_0(mm_\alpha) \leq q_0(m_\alpha) = 0 \quad m \in M, \alpha \in \mathfrak{A}. \quad (14)$$

Then $q_0(m) \in \{q_\alpha(m_\alpha) : \alpha \in \mathfrak{A}\}^\perp$ so that necessarily $q_0[M] = \{0\}$. \square

A number of implications of Theorem 4.2 may be obtained right away.

Corollary 4.3 (i) If \mathfrak{M} is normal, then each set $M \subseteq \mathfrak{M}$ admits a $\kappa(I(M))$ -subset M_0 such that

$$M_0^\perp = M^\perp. \quad (15)$$

In particular, M and M_0 have the same $\geq_{[0]}$ -upper bounds.

(ii) Each $P \subseteq \mathfrak{P}$ admits a $\kappa(I(P))$ -subset P_0 such that

$$\bigcap_{p \in P} p^{-1}(0) = \bigcap_{p \in P_0} p^{-1}(0). \quad (16)$$

Proof The first claim follows from Theorem 4.2 upon letting Q consist of all translates by some $m \in \mathfrak{M}$ and noting that, in this case, $Q[M] \subseteq I(M)$. By definition (5) an element $n \in \mathfrak{M}$ is an $\geq_{[0]}$ -upper bound for M_0 if and only if $\{n\}^\perp \subseteq M_0^\perp$ i.e. by (15), if and only if it is a $\geq_{[0]}$ -upper bound for M .

To prove (ii) recall that each $m \in \mathfrak{M}$ acts as an o -projection m^* on the normal p.o. monoid \mathfrak{P} . Then, $m \in \bigcap_{p \in P} p^{-1}(0)$ is equivalent to $m^*[P] = \{0\}$ and $\{m^*p : m \in \mathfrak{M}, p \in P\} \subseteq I(P)$. \square

For a set M of positive elements in a vector lattice X the condition $\kappa(I(M)) \leq \aleph_1$ implies then that $M \subseteq \{m\}^{\perp\perp}$ for some $m \in X_+$.

The projection problem may also be studied locally, by looking at the sets

$$\Omega(m) = \{q \in \Omega : q(m) \neq 0\} \quad m \in M \quad (17)$$

which are open in the order topology of Ω . Theorem 4.2 translates into a compactness statement: if $\{\Omega(m) : m \in M\}$ covers Q then it admits a $\kappa(Q[M])$ -subcover. But then, if Q is a “large” set and if $Q \subseteq \bigcup_{m \in M} \Omega(m)$, then one of such sets must be “large” as well. This version of the pigeonhole principle admits a rigorous formulation.

Theorem 4.4 *Let \mathfrak{M} , M and Q be as in Theorem 4.2. Let the cardinal \mathfrak{n} be the greatest of $\kappa(Q[M])$ and \aleph_0 . If*

$$\mathfrak{c}(Q) \geq \mathfrak{n} > \mathfrak{c}(Q \cap \Omega(m)) \quad m \in M \quad (18)$$

then the set $Q_0 = \{q \in Q : q[M] = \{0\}\}$ has the same cardinality as Q .

Proof If \mathfrak{n} is as in the statement it is then necessarily a regular cardinal number. Let $M_0 \subseteq M$ be a $\kappa(Q[M])$ -subset sufficient for Q . Define $Q_1 = Q \cap \bigcup_{m \in M_0} \Omega(m)$. Clearly, $Q_0 = Q \setminus Q_1$. Moreover, by basic cardinal arithmetic

$$\mathfrak{c}(Q_1) \leq \sum_{m \in M_0} \mathfrak{c}(Q \cap \Omega(m)) \leq \mathfrak{n} \cdot \mathfrak{n} = \mathfrak{n}. \quad (19)$$

However, since \mathfrak{n} is regular by [9, Lemma 3.10] we must have $\mathfrak{c}(Q_1) < \mathfrak{n}$ and $\mathfrak{c}(Q_0) = \mathfrak{c}(Q)$. \square

Theorem 4.4 applies, e.g., when $\mathfrak{c}(Q) = \aleph_1 \geq \kappa(\mathfrak{M})$ (in fact, every successive cardinal is regular, see [9, Corollary 5.3]).

Eventually, Theorem 4.2 translates into a result on partitions.

Theorem 4.5 *Let \mathfrak{M} , M and Q be as in Theorem 4.2 and let \mathfrak{n} be a regular cardinal number $\geq \kappa(Q[M])$. Assume that \mathfrak{M} decomposes as*

$$\mathfrak{M} = \{0\} \cup \bigcup_{\alpha \in \mathfrak{A}} \mathfrak{M}_\alpha \quad (20)$$

in which \mathfrak{A} is a \mathfrak{n} -set and $0 \notin \mathfrak{M}_\alpha$ for each $\alpha \in \mathfrak{A}$. Then, Q admits the decomposition

$$Q = Q_0 \cup \bigcup_{\beta \in \mathfrak{B}} Q_\beta \quad (21)$$

in which \mathfrak{B} is a \mathfrak{n} -set, $Q_0[M] = \{0\}$ while for each $\beta \in \mathfrak{B}$ there exist $\alpha_\beta \in \mathfrak{A}$ and $m_\beta \in M$ such that

$$Q_\beta[m_\beta] \subseteq \mathfrak{M}_{\alpha_\beta}. \quad (22)$$

Proof Define $Q_0 = \{q \in Q : q[M] = \{0\}\}$. According to Theorem 4.2 we can extract a \mathfrak{n} -set $M_0 \subseteq M$ which is sufficient for Q . Then, $q \notin Q_0$ if and only if there exists a pair $(\alpha, m) \in \mathfrak{A} \times M_0$ such that $q(m) \in \mathfrak{M}_\alpha$. Let \mathfrak{B} be the set of all such pairs and write each $\beta \in \mathfrak{B}$ in the form (α_β, m_β) . Given that \mathfrak{A} and M_0 are \mathfrak{n} -sets and that \mathfrak{n} is regular $\mathfrak{c}(\mathfrak{B}) \leq \sum_{\alpha \in \mathfrak{A}} \mathfrak{c}(M_0) < \mathfrak{n}$. The decomposition (21) becomes obvious upon defining $Q_\beta = \{q \in Q : q(m_\beta) \in \mathfrak{M}_{\alpha_\beta}\}$ for all $\beta \in \mathfrak{B}$. \square

5 Functions of bounded variation.

We provide some more explicit applications. If we define the group operation on \mathbb{R}_+ by \wedge we obtain that $\mathfrak{F}(X, \mathbb{R}_+)$ is an idempotent, product p.o. monoid.

Corollary 5.1 *Let X be a non empty set and let \mathcal{F} be an ideal in the p.o. product monoid $\mathfrak{F}(X, \mathbb{R}_+)$. Assume that $\kappa(\mathcal{F}) \leq \aleph_1$. Then there exist $f_1, f_2, \dots \in \mathcal{F}$ such that*

$$\sup_n f_n(x) = 0 \quad \text{if and only if} \quad \sup_{f \in \mathcal{F}} f(x) = 0. \quad (23)$$

Proof Upon replacing f with $f/(1+f)$ there is no loss of generality in assuming that $\mathcal{F} \subseteq \mathfrak{F}(X, [0, 1])$. The map $q_A(f) = f \mathbb{1}_A$ is an o -projection for each $\emptyset \neq A \subseteq X$. Let Q be the corresponding family. We have $\kappa(Q[\mathcal{F}]) \leq \kappa(\mathcal{F}) \leq \aleph_1$. The claim then follows from Theorem 4.2. \square

More interesting conclusions may be obtained with additional structure.

Definition 5.2 An increasing function $f \in \mathfrak{F}(\mathfrak{M}, \mathbb{R})$ is of finite variation, in symbols $f \in \mathcal{V}(\mathfrak{M})$, if

$$f(0) = 0 \quad \text{and} \quad \|f\|_{\mathcal{V}(\mathfrak{M})} = \sup_{M \subseteq \mathfrak{M}} \sum_{m \in M} f(m) < +\infty \quad (24)$$

where M ranges over all finite, mutually $f^{-1}(0)$ -disjoint subsets of \mathfrak{M} .

$\mathcal{V}(\mathfrak{M})$ is a p.o. subsemigroup in $\mathfrak{F}(\mathfrak{M}, \mathbb{R}_+)$ and for each $f \in \mathcal{V}(\mathfrak{M})$ the set $f^{-1}(0)$ is an \aleph_1 -ideal. Thus, each set $M \subseteq \mathfrak{M}$ contains a countable subset $m_1, m_2, \dots \in M$ such that for each $n \in \mathbb{N}$

$$\sup_{m \in M} f(nm) = 0 \quad \text{if and only if} \quad \sup_{i \in \mathbb{N}} f(nm_i) = 0. \quad (25)$$

Example 4 Let \mathfrak{M} be an idempotent, p.o. monoid and let the function $f \in \mathfrak{F}(\mathfrak{M}, \mathbb{R})$ satisfy $f(0) = 0$ and

$$f(m) \geq \sum_{\emptyset \neq M_0 \subseteq M} (-1)^{1+c(M_0)} f\left(\prod_{n \in M_0} n\right) \quad m \in \mathfrak{M}, \quad M \subseteq I(m) \text{ finite.} \quad (26)$$

If \mathfrak{M} is a π system of sets then (26) corresponds to the definition of a supermodular capacity given by Choquet [5, p. 171]. If $M \subseteq \mathfrak{M}$ is mutually $f^{-1}(0)$ -disjoint, then $f(1) \geq \sum_{m \in M} f(m)$ so that $f \in \mathcal{V}(\mathfrak{M})$.

Theorem 5.3 Let $V \subseteq \mathcal{V}(\mathfrak{M})$ be such that $f \mathbb{1}_{I(m)} \in V$ whenever $f \in V$ and $m \in \mathfrak{M}$. The following properties are mutually equivalent:

- (a) for every $\emptyset \neq U \subseteq V$ the set $\bigcap_{f \in U} f^{-1}(0)$ is an \aleph_1 -ideal in \mathfrak{M} ;
- (b) $\kappa(V) \leq \aleph_1$;
- (c) every $\emptyset \neq U \subseteq V$ admits some $f_U \in \mathcal{V}(\mathfrak{M})$ such that $f_U^{-1}(0) = \bigcap_{f \in U} f^{-1}(0)$.

Proof (a) \Rightarrow (b) If $\emptyset \neq U \subseteq V$ is mutually disjoint then any two elements $f, g \in U$ have disjoint support. For each $g \in U$ we can thus select $m_g \in \mathfrak{M}$ such that $g(m_g) > 0$. The family $\{m_g : g \in U\}$ is mutually $\bigcap_{f \in U} f^{-1}(0)$ -disjoint and has the same cardinality as U . If $\bigcap_{f \in U} f^{-1}(0)$ is an \aleph_1 -ideal, then necessarily U is countable. This proves that $\kappa(V) \leq \aleph_1$.

(b) \Rightarrow (c) For each $m \in \mathfrak{M}$ consider the projection $q_m \in \Omega(\mathcal{V}(\mathfrak{M}))$ implicitly defined by $q_m f = f \mathbb{1}_{I(m)}$ and apply Theorem 4.2 with $Q = \{q_m : m \in \mathfrak{M}\}$ and $M = U$. Observe that $q_m f = 0$ if and only if $f(m) = 0$. By assumption, $Q[U] \subseteq V$ so that $\kappa(Q[U]) \leq \aleph_1$. We obtain a countable subset $U_0 \subseteq U$ such that for any $m \in \mathfrak{M}$, $q_m[U_0] = 0$ if and only if $q_m[U] = 0$. If f_1, f_2, \dots is an enumeration of U_0 we can define $f_U \in \mathfrak{F}(\mathfrak{M}, \mathbb{R})$ by letting

$$f_U(m) = \sum_j 2^{-j} f_j(m) / (1 + \|f_j\|_{\mathcal{V}(\mathfrak{M})}) \quad m \in \mathfrak{M}. \quad (27)$$

Notice that $f_U \in \mathcal{V}(\mathfrak{M})$ and that $f_U(m) = 0$ is equivalent to $\sup_{f \in U} f(m) = 0$.

(c) \Rightarrow (a) If $\emptyset \neq U \subseteq V$ and if $\bigcap_{f \in U} f^{-1}(0) = f_U^{-1}(0)$ for some $f_U \in \mathcal{V}(\mathfrak{M})$, then necessarily $\bigcap_{f \in U} f^{-1}(0)$ is certainly an \aleph_1 -ideal by the finite variation property. \square

Two final comments. First, the proof shows that if for given $U \subseteq V$ there exists $f_U \in \mathcal{V}(\mathfrak{M})$ satisfying (c) then it can always be taken to be of the form (27). This is important since if U consists e.g. of supermodular capacities, so will be f_U . Second, Theorem 5.3 may be useful to prove the existence of some $f \in \mathcal{V}(\mathfrak{M})$ which is positive on all $m \in \mathfrak{M}$, $m \neq 0$. This is an important and challenging problem (that goes much beyond the scope of this work). In the case in which \mathfrak{M} is a Boolean algebra and V consists of additive probabilities on \mathfrak{M} , this is just the problem raised by Maharam in [11] (see [14] for a partial, negative answer). The problem makes however sense also in other classes of functions of finite variation e.g. not additive set functions on a given algebra of sets. In applying condition (b) of Theorem 5.3 one should recall that the

disjointness condition formulated for $\mathcal{V}(\mathfrak{M})$ need not coincide with the corresponding condition e.g. for additive set functions.

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References

1. Aliprantis, C.D., Burkinshaw, O.: Positive Operators. Springer, Dordrecht (2006)
2. Anderson, D.D., Johnson, E.W.: Ideal theory in commutative semigroups. *Semigroup Forum* **30**, 127–158 (1984)
3. Birkhoff, G.: Lattice Theory. Amer. Math. Soc. Colloquium Publications, vol. 25. American Mathematical Society, Providence, RI, (1973)
4. Blyth, T.S.: Lattices and Ordered Algebraic Structures. Springer, London (2006)
5. Choquet, G.: Theory of capacities. *Ann. Inst. Fourier* **5**, 131–295 (1954)
6. Clifford, A.H.: Totally ordered commutative semigroups. *Bull. Amer. Math. Soc.* **64**(6), 305–316 (1958)
7. Erdős, P., Rado, R.: Intersection theorems for systems of sets. *J. Lond. Math. Soc.* **35**(1), 85–90 (1960)
8. Erdős, P., Tarski, A.: On families of mutually exclusive sets. *Ann. Math.* **44**, 315–329 (1943)
9. Jech, T.: Set Theory, 3rd ed. Springer, Berlin (2003)
10. Leader, S.: Measures on semilattices. *Pacific J. Math.* **39**(2), 407–423 (1971)
11. Maharam, D.: An algebraic characterization of measure algebras. *Ann. Math.* **48**, 154–167 (1947)
12. Ramsey, F.P.: On a problem of formal logic. *Proc. London Math. Soc.* **30**(2), 264–286 (1930)
13. Rees, D.: On semi-groups. *Math. Proc. Cambridge Phil. Soc.* **36**(4), 387–400 (1940)
14. Talagrand, M.: Maharam's problem. *Ann. Math.* **168**(2), 981–1009 (2008)
15. Tarski, A.: Ideale in vollständigen Mengenkörpern. II. *Fund. Math.* **33**, 51–65 (1945)

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