Fountain-like solutions for nonlinear elliptic equations with critical growth and Hardy potential

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Abstract

We prove the existence of fountain-like solutions, obtained by superposition of bubbles of different blow-up orders, for a nonlinear elliptic equation with critical growth and Hardy-type potential.

Key Words: Hardy potential, Sobolev critical exponent, perturbation methods

MSC classification: 35J60, 35J20, 35B33.

1 Introduction

This paper deals with the following class of nonlinear elliptic problems

$$\begin{cases} -\Delta u - \dfrac{\lambda}{|x|^2}\, u = \big(1+\varepsilon K(x)\big)u^{2^*-1}, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N\setminus\{0\}, \end{cases} \tag{$\mathcal{P}^{\varepsilon}_{\lambda,K}$}$$

where

$$N \geq 3, \quad 2^* = \frac{2N}{N-2}, \quad -\infty < \lambda < \frac{(N-2)^2}{4},$$

K is a continuous bounded function, and ε is a small real perturbation parameter. Here $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the closure space of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^{1/2}$$

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which is equivalent to the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \lambda\,\frac{u^2}{|x|^2}\,dx\right)^{1/2}$$

in view of the Hardy inequality. $\mathcal{D}^{1,2}(\mathbb{R}^N)$ endowed with the scalar product

$$(u,v) = \int_{\mathbb{R}^N} \nabla u\cdot\nabla v\,dx - \lambda\,\frac{uv}{|x|^2}\,dx, \quad u,v\in\mathcal{D}^{1,2}(\mathbb{R}^N)$$

is a Hilbert space.

The main features of the above problem are the critical Sobolev growth and the presence of a singular potential having the same spatial homogeneity. The study of this type of singular problems arises in several fields, such as quantum mechanics, astrophysics, as well as Riemannian geometry. Indeed equation $(\mathcal{P}^{\varepsilon}_{\lambda,K})$ is related to the nonlinear Schrödinger equation with a potential which exhibits a singular behavior at the origin. For discussion on Schrödinger operators we refer to [7]. Moreover $(\mathcal{P}^{\varepsilon}_{\lambda,K})$ can be seen as a simplified prototype of the nonlinear Wheeler-De Witt equation which appears in quantum cosmological models (for more details we remind to [6] and references therein). Let us finally remark that the study of $(\mathcal{P}^{\varepsilon}_{\lambda,K})$ has also a geometric motivation, since it is related to the scalar curvature problem on the sphere $\mathbb{S}^N$. Indeed, if we identify $\mathbb{R}^N$ with $\mathbb{S}^N$ through the stereographic projection and endow $\mathbb{S}^N$ with a metric whose scalar curvature is singular at the north and the south poles, then the problem of finding a conformal metric with prescribed scalar curvature $1+\varepsilon K(x)$ leads to solve equation $(\mathcal{P}^{\varepsilon}_{\lambda,K})$, where the unknown u has the meaning of a conformal factor (see [5] and [22]).

For $\lambda < (N-2)^2/4$, the unperturbed problem, namely the problem with $\varepsilon = 0$, admits a one-dimensional manifold of radial solutions

$$Z^{\lambda} = \left\{ z^{\lambda}_{\mu} = \mu^{-\frac{N-2}{2}} z^{\lambda}_1\left(\tfrac{\cdot}{\mu}\right) \,\Big|\, \mu > 0\right\} \tag{1.1}$$

where

$$z^{\lambda}_1(x) = A(N,\lambda)\left[|x|^{\frac{2a_\lambda}{N-2}} + |x|^{\frac{2(N-2-a_\lambda)}{N-2}}\right]^{-\frac{N-2}{2}}, \quad A(N,\lambda) = \left[\frac{N(N-2-2a_\lambda)^2}{N-2}\right]^{\frac{N-2}{4}} \tag{1.2}$$

and $a_\lambda = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda}$, see [26]. Moreover in [26] it is proved that if $\lambda > 0$ such solutions are the unique positive solutions whereas if λ is sufficiently negative also nonradial solutions exist.

Positive solutions to $(\mathcal{P}^{\varepsilon}_{\lambda,K})$ can be found as critical points in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ of the functional

$$f^K_\varepsilon(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N}\big(1+\varepsilon K(x)\big)\,u_+^{2^*}, \quad u\in\mathcal{D}^{1,2}(\mathbb{R}^N),$$

where $u_+ := \max\{u,0\}$. A key role in the variational approach to the problem is the study of nondegeneracy properties of the unperturbed functional, i.e. of the functional

$$f_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} u_+^{2^*}.$$

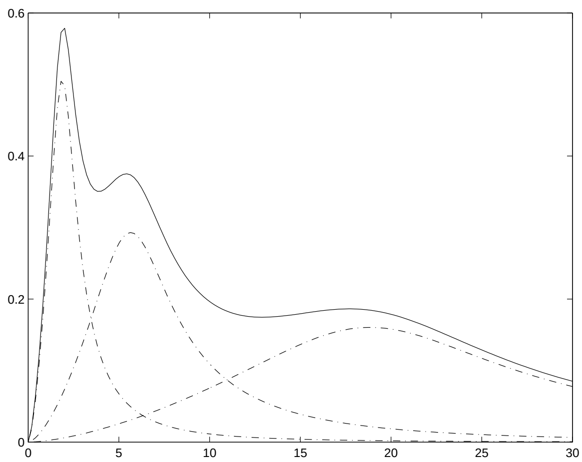
We say that the critical manifold Z^λ is nondegenerate if

$$\ker D^2 f_0(z) = T_z Z^\lambda \quad \text{for all } z \in Z^\lambda, \tag{1.3}$$

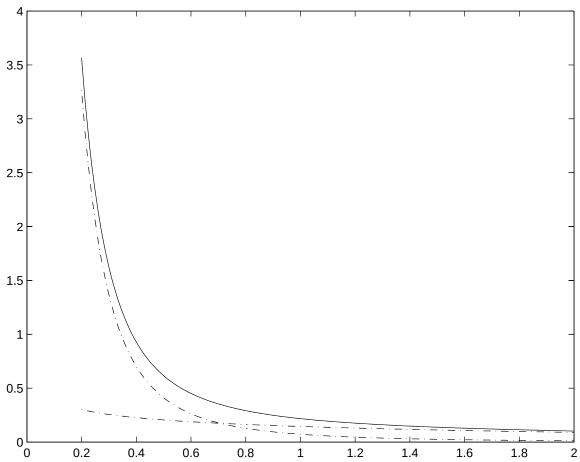
where $D^2 f_0(z)$ denotes the second Fréchet derivative of f_0 at z, which is considered as an element of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ in view of the canonical identification of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ with its dual. A complete answer to the question of nondegeneracy of Z^λ is given in [16], where it is proved that degeneracy occurs only along a sequence of values of λ; more precisely Z^λ is nondegenerate if and only if

$$\lambda \neq \frac{(N-2)^2}{4}\left[1 - \frac{j(N+j-2)}{N-1}\right] \quad \text{for any } j \in \mathbb{N} \setminus \{0\}, \tag{1.4}$$

see Theorem 2.1 below. For all values of $\lambda < (N-2)^2/4$ satisfying (1.4), it is possible to perform a finite dimensional reduction, using a perturbative method developed by Ambrosetti and Badiale, see [3]. Using this procedure, in [16] existence of solutions to problem $(\mathcal{P}^{\varepsilon}_{\lambda,K})$ close to Z^λ was proved for $|\varepsilon|$ small, provided K vanishes at 0 and at infinity and (1.4) is satisfied, even for a more general class of operators related to Caffarelli-Kohn-Nirenberg inequalities. Such solutions have a single bubble profile wich is singular at zero when $\lambda > 0$ and vanishes at zero when $\lambda < 0$. The purpose of this paper is to prove the existence of solutions with a multi-bubbling profile, see figures 1 and 2.



1. 3-bubbling solution profile ($\lambda = -4$ and $N = 3$)



2. 2-bubbling solution profile ($\lambda = 1$ and $N = 5$)

To this aim we will assume that K is of the form

$$K(x) = \sum_{i=1}^{\ell} K_i\big(\tfrac{x}{\nu_i}\big) \tag{1.5}$$

where $\nu_i > 0$,

$$K_i(\infty) := \lim_{|x|\to\infty} K_i(x) \quad \text{exists and} \quad K_i(\infty) = K_i(0) = 0 \tag{1.6}$$

for any $i = 1, \dots, \ell$, and

$$\min_{i\neq j}\left[\frac{\nu_i}{\nu_j}+\frac{\nu_j}{\nu_i}\right] \quad \text{is large,} \tag{1.7}$$

and discuss the existence of fountain-like solutions, obtained by superposition of ℓ bubbles of different blow-up orders. The main result of the present paper is the following existence theorem.

Main Theorem. *Let $\lambda < (N-2)^2/4$ satisfying* (1.4) *and assume* (1.5) *and* (1.6) *hold. Suppose that for each $i = 1, 2, \dots, \ell$, $K_i \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ satisfies*

$$\textit{either} \quad \textit{there exists} \quad r_i > 0 \quad \textit{such that} \quad \int_{\mathbb{S}^{N-1}} K_i(r_i\theta)\, d\theta \neq 0, \tag{1.8$_i$}$$

$$\textit{or} \quad K_i \not\equiv 0 \quad \textit{and} \quad K_i \quad \textit{has a fixed sign (i.e. either } K_i \geq 0 \textit{ or } K_i \leq 0). \tag{1.9$_i$}$$

Then there exist $0 < a_i < b_i$, $i = 1, 2, \dots, \ell$, $C = C(\lambda, N, a_i, b_i, \|K_i\|_{L^\infty}, \ell) > 0$ and some $\bar\varepsilon$ sufficiently small such that if $|\varepsilon| \leq \bar\varepsilon$ and

$$\varepsilon^2 \min_{i\neq j} g\Big(\frac{\nu_i}{\nu_j}\Big)^{-\max\left\{\frac12,\frac{N-2}{N+2}\right\}} \geq C \tag{1.10}$$

where

$$g(\nu) := \max\left\{\left[\nu^{1-\frac{2a_\lambda}{N-2}} + \nu^{\frac{2a_\lambda}{N-2}-1}\right]^{-\max\left\{2,\frac{N-2}{2}\right\}}, \big(f_1(\nu)+f_2(\nu)\big)\right\} \tag{1.11}$$

and f_1, f_2 are defined in (2.13) below, there exists a solution u_ε to problem $(\mathcal{P}^\varepsilon_{\lambda,K})$ close to $\sum_{i=1}^{\ell} z^\lambda_{\mu_i}$ for some $\mu_i \in (\nu_i a_i, \nu_i b_i)$.

The proof of the above theorem is given in detail for $\ell = 2$, the general case requiring only simple modifications.

We will construct fountain-like solutions to $(\mathcal{P}^\varepsilon_{\lambda,K})$ using the perturbative method of [3]. This method allows to find critical points of a perturbed functional of the type $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$ by studying a finite dimensional problem. More precisely, if the unperturbed functional f_0 has a finite dimensional manifold of critical points Z which satisfies the nondegeneracy condition (1.3), it is possible to prove, for $|\varepsilon|$ sufficiently small, the existence of a small perturbation function $w_\varepsilon(z) : Z \to (T_zZ)^\perp$ such that any critical point $\bar z \in Z$ of the function

$$\Phi_\varepsilon : Z \to \mathbb{R}, \quad \Phi_\varepsilon(z) = f_\varepsilon\big(z + w_\varepsilon(z)\big)$$

gives rise to a critical point $u_\varepsilon = \bar z + w_\varepsilon(\bar z)$ of f_ε. Moreover the reduced function Φ_ε can be expanded as

$$\Phi_\varepsilon(z) = b_0 - \varepsilon\Gamma(z) + o(\varepsilon) \quad \text{as } |\varepsilon| \to 0$$

for some constant b_0 and for some function $\Gamma : Z \to \mathbb{R}$, see Theorem 2.1, so that critical points of Γ which are stable in a suitable sense correspond to critical points of f_ε which are close to Z. In order to prove the existence of ℓ-bubbling solutions, we mean to construct solutions close to $\sum_{i=1}^{\ell} z^\lambda_{\mu_i}$. We remark that functions of the type $\sum_{i=1}^{\ell} z^\lambda_{\mu_i}$ are pseudo-critical points of f_0, in the sense that $f_0'(\sum_{i=1}^{\ell} z^\lambda_{\mu_i})$ vanishes as the interactions between different bubbles tend to zero. We will show that when the rescaling factors ν_i satisfy condition (1.7), the interaction is small and it is possible to construct a natural constraint for the functional f_ε^K close to the ℓ-dimensional manifold $Z_\varepsilon = \big\{ \sum_{i=1}^{\ell} z_{\mu_i}^\lambda,\ \mu_i \in \mathbb{R},\ i = 1, \dots, \ell \big\}$. Figure 3 represents the function $\sum_{i=1}^{\ell} z_{\mu_i}^\lambda$ when $\lambda = -4$, $N = 3$, $\ell = 3$, $\mu_1 = 2$, $\mu_2 = 6$, and $\mu_3 = 20$.

The finite dimensional reduction described above leads us to look for critical points of a function defined on $\mathbb{R}^\ell$. The study of such a finite dimensional function will be performed by a topological degree argument based on the Miranda's Theorem, see Theorem 7.2.

We mention that a similar perturbative argument was used in [9] to construct multi-bump solutions for the Yamabe problem on $\mathbb{S}^N$ and in [8] to find multi-bump and infinite-bump solutions to a perturbed dynamical second order system. Moreover multi-bubbling phenomenon at a single point was observed for the scalar curvature problem in [12], where a sequence of solutions blowing up with infinite energy was found. The existence of radial solutions which behave like superposition of bubbles was also proved in [13] for the supercritical Brezis-Nirenberg problem and in [14] for an elliptic equation involving the p-laplacian and an exponential nonlinearity.

Let us now recall some results concerning elliptic equations with singular potential which can be found in the literature. In [25] Smets considers the nonperturbative problem

$$-\Delta u - \frac{\lambda}{|x|^2}\, u = F(x) u^{2^*-1} \tag{1.12}$$

in the case $N = 4$, proving, by minimax methods, that, if F is a C^2 positive function such that $F(0) = \lim_{|x|\to+\infty} F(x)$, then for any $\lambda \in (0, 1)$ there exists at least one solution.

In [1], existence of solutions to problem (1.12) blowing-up at global maximum points of F as the parameter λ goes to zero is proved under some suitable assumption about the local behavior of F close to such maximum points. In [15], it is studied the existence of solutions to problem (1.12) blowing-up at a suitable critical point (not necessarily a maximum point) of the function F, as λ goes to zero. Let us mention that some related singular equations with Hardy type potential were also studied in [2, 18, 20, 21, 24].

The change of variable $v(x) = |x|^{a_\lambda} u(x)$ transforms problem (1.12) into the following degenerate elliptic equation with the same critical growth

$$-\mathrm{div}\,(|x|^{-2a_\lambda} \nabla v) = F(x) \frac{v^{2^*-1}}{|x|^{2^* a_\lambda}} \tag{1.13}$$

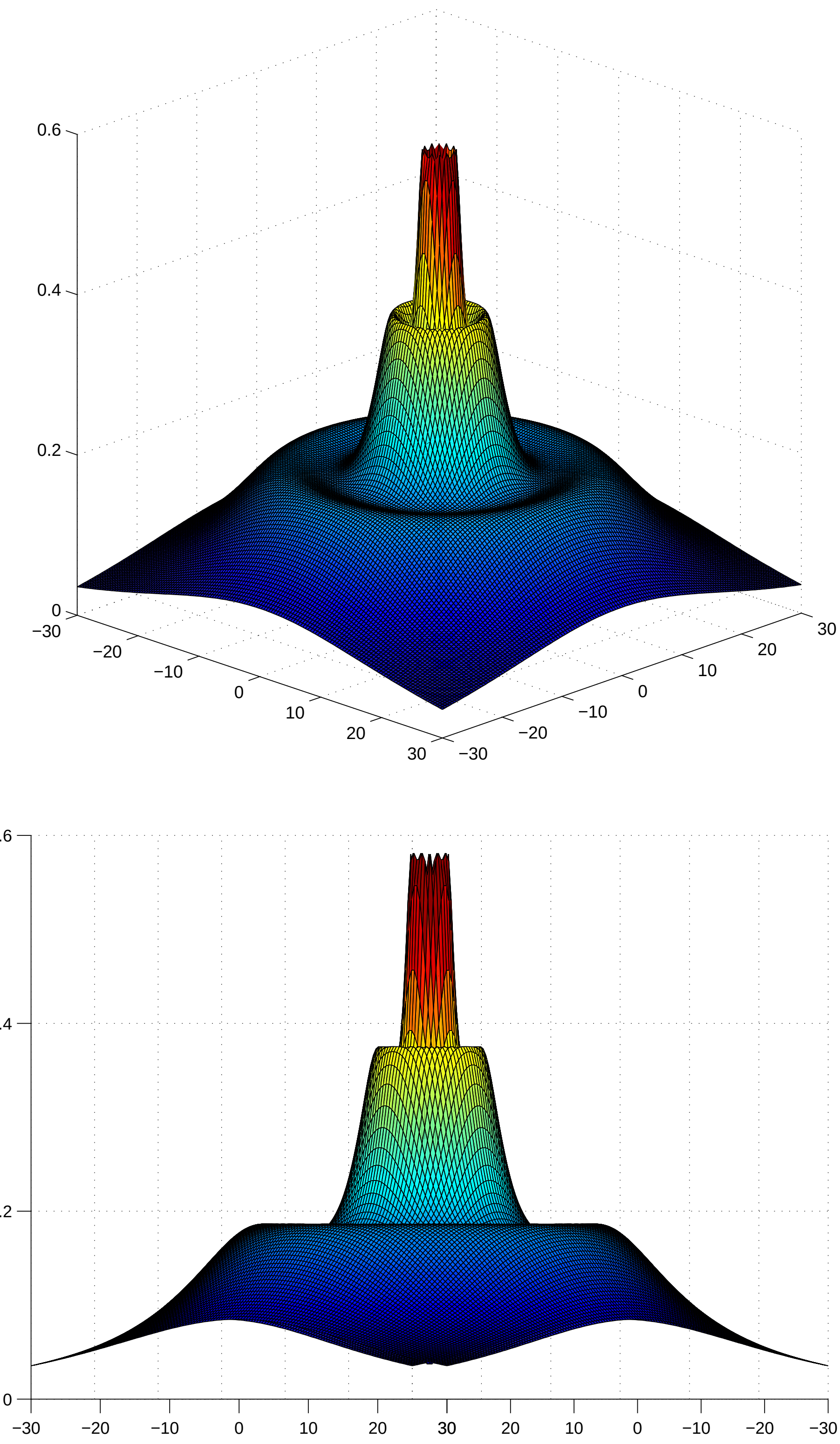
which is related to Caffarelli-Kohn-Nirenberg inequalities, see [10] and [11]. The estimates of the behavior of single bubbles given in Section 3 are essentially based on the regularity results for equation (1.13) contained in [17].

To construct fountain-like solutions to $\mathcal{P}_{\lambda,K}^\varepsilon$ is equivalent to built multi-bump solutions for a transformed problem on a cylinder. Indeed the Emden transformation

$$v(x) = |x|^{-\frac{N-2-2a_\lambda}{2}} \varphi\Big(-\ln|x|, \tfrac{x}{|x|}\Big)$$

turns equation (1.13) into the following problem on the cylinder $\mathcal{C} := \mathbb{R} \times \mathbb{S}^{N-1}$

$$-\varphi_{tt} - \Delta_\theta \varphi + \Big(\tfrac{N-2-2\,a_\lambda}{2}\Big)^2 \varphi = F(\theta e^t) \varphi^{2^*-1}, \quad (t, \theta) \in \mathcal{C} \tag{1.14}$$



3. 3-bubbling fountain ($\lambda = -4$ and $N = 3$)

where Δ_θ denotes the Laplace-Beltrami operator on the sphere $\mathbb{S}^{N-1}$. We remark that fountain-like solutions to $\mathcal{P}^{\varepsilon}_{\lambda,K}$ correspond through the above transformations to multi-bump solutions to equation (1.14).

The paper is organized as follows. In section 2 we introduce some notation, recall some known facts and state the existence theorem for two-bubble solutions. In section 3 we provide estimates of the behavior of the one-bubble solutions while section 4 contains estimates of the interaction between different bubbles. Section 5 is devoted to the construction of the natural constraint for the problem. In section 6 we expand the Jacobian of the reduced function up to the first order and in section 7 we give the proof of the existence of critical points of the reduced function by topological degree arguments. In the appendix we collect some technical lemmas.

2 Two bubble fountain solutions

For any $k \in L^\infty(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$, let us consider the functional $f^k_\varepsilon : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$f^k_\varepsilon(u) = f_0(u) - \varepsilon\, G_k(u)$$

where

$$f_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N} u_+^{2^*}\, dx \quad \text{and} \quad G_k(u) = \frac{1}{2^*}\int_{\mathbb{R}^N} k(x) u_+^{2^*}\, dx.$$

The functional f^k_ε is of class C^2 and its critical points are solutions of the problem

$$\begin{cases} -\Delta u - \dfrac{\lambda}{|x|^2}\, u = \big(1 + \varepsilon k(x)\big) u^{2^*-1}, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{$\mathcal{P}^{\varepsilon}_{\lambda,k}$}$$

The following theorem ensures that for λ satisfying (1.4) a finite dimensional reduction is possible. For simplicity of notation, in the sequel we write z_μ instead of z^λ_μ and Z instead of Z^λ if there is no possibility of confusion.

Theorem 2.1. *Let $\lambda < (N-2)^2/4$ satisfying (1.4). Then the critical manifold $Z = Z^\lambda$ defined in (1.1) satisfies the following nondegeneracy condition*

$$T_{z_\mu} Z = \ker D^2 f_0(z_\mu) \quad \textit{for all } \mu > 0. \tag{2.1}$$

Moreover for any $k \in L^\infty(\mathbb{R}^N)\cap C^0(\mathbb{R}^N)$ there exist ε_k, $C_k = C_k(\|k\|_{L^\infty}, \lambda, N) > 0$, and a unique couple of smooth functions $w^k : (0, +\infty) \times (-\varepsilon_k, \varepsilon_k) \to \mathcal{D}^{1,2}(\mathbb{R}^N)$, $\alpha^k : (0, +\infty) \times (-\varepsilon_k, \varepsilon_k) \to \mathbb{R}$ such that for any $\mu > 0$ and $\varepsilon \in (-\varepsilon_k, \varepsilon_k)$

$$w^k(\mu, \varepsilon) \textit{ is orthogonal to } T_{z_\mu} Z, \tag{2.2}$$

$$D f^k_\varepsilon\big(z_\mu + w^k(\mu,\varepsilon)\big) = \alpha^k(\mu,\varepsilon)\dot{\xi}_\mu, \tag{2.3}$$

$$\|w^k(\mu,\varepsilon)\| + |\alpha^k(\mu,\varepsilon)| \le C_k|\varepsilon|, \tag{2.4}$$

$$\|\partial_\mu w^k(\mu,\varepsilon)\| \le C_k \mu^{-1} |\varepsilon|^{\min\{1, 4/(N-2)\}}, \tag{2.5}$$

where $\dot\xi_\mu$ denotes the normalized tangent vector

$$\dot\xi_\mu := \frac{\dot z_\mu}{\|\dot z_\mu\|}, \quad \dot z_\mu = \frac{d}{d\mu} z_\mu. \tag{2.6}$$

Finally

$$f^k_\varepsilon\big(z_\mu + w^k(\mu,\varepsilon)\big) = f_0(z_1) - \varepsilon\,\Gamma^k(\mu) + o(\varepsilon) \tag{2.7}$$

as $|\varepsilon| \to 0$ uniformly with respect to $\mu > 0$, where

$$\Gamma^k(\mu) = G_k(z_\mu) = \frac{1}{2^*}\int_{\mathbb{R}^N} k(x) z_\mu^{2^*}\,dx. \tag{2.8}$$

Proof. We refer to [16, Theorem 1.1, Lemma 3.4, and Lemma 4.1] for the proof of nondegeneracy, existence, uniqueness, estimate (2.4), and expansion (2.7). To prove estimate (2.5) we observe that w^k and α^k are implicitly defined by $h(\mu, w, \alpha, \varepsilon) = (0,0)$ where

$$\begin{aligned} &h : (0,\infty)\times\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathbb{R}\times\mathbb{R}\to\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathbb{R}\\ &h(\mu,w,\alpha,\varepsilon) := \big(Df_\varepsilon(z_\mu + w) - \alpha\,\dot\xi_\mu,\ (w,\dot\xi_\mu)\big).\end{aligned}$$

It is possible to show (see [16]) that (2.1) implies the existence of a positive constant C_* such that for any $\mu > 0$

$$\left\|\left(\frac{\partial h}{\partial(w,\alpha)}(\mu,0,0,0)\right)^{-1}\right\| < C_*.$$

Since $\partial_\mu w^k(\mu,\varepsilon)$ satisfies

$$\begin{pmatrix}\partial_\mu w^k(\mu,\varepsilon)\\ \partial_\mu \alpha^k(\mu,\varepsilon)\end{pmatrix} = -\left(\frac{\partial h}{\partial(w,\alpha)}\right)^{-1}\Bigg|_{(\mu,w^k,\alpha^k,\varepsilon)}\cdot\frac{\partial h}{\partial\mu}\Bigg|_{(\mu,w^k,\alpha^k,\varepsilon)}$$

we have

$$\begin{aligned}\|\partial_\mu w^k(\mu,\varepsilon)\| &\le C_*\left\|\frac{\partial h}{\partial\mu}\Big|_{(\mu,w^k,\alpha^k,\varepsilon)}\right\|\\ &\le C_*\left[\left\|D^2 f^k_\varepsilon(z_\mu + w^k(\mu,\varepsilon))\dot z_\mu - \alpha(\mu,\varepsilon)\frac{d}{d\mu}\dot\xi_\mu\right\| + \left|\left(w^k(\mu,\varepsilon),\frac{d}{d\mu}\dot\xi_\mu\right)\right|\right]\\ &\le C_*\Bigg[\left\|\big(D^2 f_0(z_\mu + w^k(\mu,\varepsilon)) - D^2 f_0(z_\mu)\big)\dot z_\mu\right\| + |\varepsilon|\left\|D^2 G_k(z_\mu + w^k(\mu,\varepsilon))\dot z_\mu\right\|\\ &\qquad + |\alpha(\mu,\varepsilon)|\left\|\frac{d}{d\mu}\dot\xi_\mu\right\| + \|w^k(\mu,\varepsilon)\|\left\|\frac{d}{d\mu}\dot\xi_\mu\right\|\Bigg].\end{aligned} \tag{2.9}$$

Since $\|z_\mu\| = \|z_1\|$, $\|\dot z_\mu\| = \mu^{-1}\|\dot z_1\|$, $\big\|\frac{d}{d\mu}\dot\xi_\mu\big\| \le c(\lambda,N)\mu^{-1}$ for some positive constant $c(\lambda,N)$ depending only on λ and N (see (A.10) and (A.18)), using (A.8) of the appendix, (2.4), and the estimate

$$\left\|D^2 G_k(z_\mu + w^k(\mu,\varepsilon))\dot z_\mu\right\| = \sup_{\|v\|\le 1}\left|(2^*-1)\int_{\mathbb{R}^N} k(x)(z_\mu + w^k(\mu,\varepsilon))_+^{2^*-2}\dot z_\mu v\right| \le \text{const}\,\|\dot z_\mu\|$$

which follows easily from Hölder inequality, (2.9) yields estimate (2.5). This ends the proof. □

Let us set

$$z^k_{\mu,\varepsilon} := z_\mu + w^k(\mu,\varepsilon). \tag{2.10}$$

Hereafter we assume

$$K(x) = K_1(x) + K_2(x/\nu) \tag{2.11}$$

where $\nu > 0$ and $K_1, K_2 \in L^\infty(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ satisfy (1.6). We will also use the notation

$$k_1(x) = K_1(x) \quad \text{and} \quad k_2(x) = K_2(x/\nu). \tag{2.12}$$

Let us set

$$f_1(\nu) = \left[\int_{\mathbb{R}^N} |K_1(\nu x)| z_1^{2^*}(x)\,dx\right]^{\frac{2^*-1}{2^*}}, \quad f_2(\nu) = \left[\int_{\mathbb{R}^N} \left|K_2\Big(\frac{x}{\nu}\Big)\right| z_1^{2^*}(x)\,dx\right]^{\frac{2^*-1}{2^*}}. \tag{2.13}$$

Lemma 2.2. *Assume that* (1.6) *holds. Then*

$$\lim_{\nu\to 0^+} f_i(\nu) = \lim_{\nu\to+\infty} f_i(\nu) = 0, \qquad i = 1,2.$$

Proof. It follows from (1.6) and the Dominated Convergence Theorem. □

In order to ensure that the functions Γ^{k_1} and Γ^{k_2} are not identically equal to zero, we require that K_i, $i = 1,2$, satisfies either $(1.8)_i$ or $(1.9)_i$.

Lemma 2.3. *Let* $i = 1,2$. *Assume that* $K_i \in L^\infty(\mathbb{R}^N)$ *satisfies either* $(1.8)_i$ *or* $(1.9)_i$. *Then* $\Gamma^{k_i} \not\equiv 0$.

Proof. Let $i = 1$ and assume $(1.8)_i$. For any $g \in L^1([0,\infty), dr/r)$ let us define the Mellin transform of g as

$$\mathcal{M}[g](s) = \int_0^\infty r^{-is} g(r)\,\frac{dr}{r},$$

see [19] and [4, Theorem 4.3]. The associated convolution is defined by

$$(g \times h)(s) = \int_0^\infty g(r) h(s/r)\,\frac{dr}{r}.$$

There holds $\mathcal{M}[g \times h] = \mathcal{M}[g]\cdot\mathcal{M}[h]$. Let η be a smooth cut-off function such that $\eta(x) \equiv 1$ for $|x| \le 1$, $\eta(x) \equiv 0$ for $|x| \ge 2$, and $0 \le \eta \le 1$ in $\mathbb{R}^N$. Using polar coordinates and the above notation we can write Γ^{k_1} as

$$\begin{aligned}
\Gamma^{k_1}(\mu) &= \frac{1}{2^*}\int_{\mathbb{R}^N} K_1(x) z_\mu^{2^*}\,dx = \frac{1}{2^*}\int_0^\infty \left[\int_{\mathbb{S}^{N-1}} K_1(r\theta)\,d\theta\right] z_\mu^{2^*}(r)\, r^{N-1}\,dr \\
&= \frac{1}{2^*}\mu^{-\alpha_1}\int_0^\infty g_1(r) z_1^{2^*}\Big(\frac{r}{\mu}\Big)\Big(\frac{r}{\mu}\Big)^{N-\alpha_1}\frac{dr}{r} + \frac{1}{2^*}\mu^{-\alpha_2}\int_0^\infty g_2(r) z_1^{2^*}\Big(\frac{r}{\mu}\Big)\Big(\frac{r}{\mu}\Big)^{N-\alpha_2}\frac{dr}{r} \\
&= \frac{1}{2^*}\mu^{-\alpha_1}(g_1 \times h_1)(\mu) + \frac{1}{2^*}\mu^{-\alpha_2}(g_2 \times h_2)(\mu)
\end{aligned}$$

where

$$g_1(r) := r^{\alpha_1}\int_{\mathbb{S}^{N-1}} \eta(r\theta)K_1(r\theta)\,d\theta, \quad g_2(r) := r^{\alpha_2}\int_{\mathbb{S}^{N-1}} (1-\eta)(r\theta)K_1(r\theta)\,d\theta,$$
$$h_1(r) := z_1^{2^*}\Big(\frac{1}{r}\Big)\Big(\frac{1}{r}\Big)^{N-\alpha_1}, \quad h_2(r) := z_1^{2^*}\Big(\frac{1}{r}\Big)\Big(\frac{1}{r}\Big)^{N-\alpha_2},$$

and α_1, α_2 are chosen in such a way that

$$0<\alpha_1<N-2^*a_\lambda, \quad 2^*a_\lambda - N < \alpha_2 < 0. \tag{2.14}$$

Note that the choice of η, (1.2), and (2.14) imply that $g_1, g_2, h_1, h_2 \in L^1([0,\infty), dr/r)$. If, by contradiction, $\Gamma^{k_1}\equiv 0$, then $\mu^{-\alpha_1}(g_1\times h_1)(\mu)+\mu^{-\alpha_2}(g_2\times h_2)(\mu)\equiv 0$, and hence

$$\mathcal{M}[g_1\times h_1]\Big(s+\frac{\alpha_1}{i}\Big)+\mathcal{M}[g_2\times h_2]\Big(s+\frac{\alpha_2}{i}\Big)=0, \quad \text{for any } s\in\mathbb{R}.$$

From multiplication property of convolution we obtain

$$\mathcal{M}[g_1]\Big(s+\frac{\alpha_1}{i}\Big)\cdot\mathcal{M}[h_1]\Big(s+\frac{\alpha_1}{i}\Big)+\mathcal{M}[g_2]\Big(s+\frac{\alpha_2}{i}\Big)\cdot\mathcal{M}[h_2]\Big(s+\frac{\alpha_2}{i}\Big)=0, \quad \text{for any } s\in\mathbb{R}.$$

Since $\mathcal{M}[h_1]$ is real analytic, it has a discrete number of zeroes. Moreover from a direct computation we have

$$\mathcal{M}[h_1]\Big(s+\frac{\alpha_1}{i}\Big)=\mathcal{M}[h_2]\Big(s+\frac{\alpha_2}{i}\Big)=\int_0^\infty r^{-is}z_1^{2^*}\Big(\frac{1}{r}\Big)\Big(\frac{1}{r}\Big)^N\frac{dr}{r}.$$

Hence by continuity if follows that

$$\mathcal{M}[g_1]\Big(s+\frac{\alpha_1}{i}\Big)+\mathcal{M}[g_2]\Big(s+\frac{\alpha_2}{i}\Big)=0, \quad \text{for any } s\in\mathbb{R}. \tag{2.15}$$

On the other hand a direct computation yields

$$\mathcal{M}[g_1]\Big(s+\frac{\alpha_1}{i}\Big)+\mathcal{M}[g_2]\Big(s+\frac{\alpha_2}{i}\Big)=\mathcal{M}[\tilde g](s) \tag{2.16}$$

where $\tilde g(r) := \int_{\mathbb{S}^{N-1}} K_1(r\theta)\,d\theta$. From (2.15) and (2.16) we deduce that $\mathcal{M}[\tilde g]\equiv 0$. Then $\tilde g\equiv 0$, which contradicts assumption $(1.8)_i$. The proof for $i=2$ is analogous. The proof under assumption $(1.9)_i$ is elementary. □

Lemma 2.4. *Let $i=1,2$. Assume that $K_i\in L^\infty(\mathbb{R}^N)\cap C^0(\mathbb{R}^N)$ satisfies (1.6). Then*

$$\lim_{\mu\to 0^+}\Gamma^{k_i}(\mu)=\lim_{\mu\to+\infty}\Gamma^{k_i}(\mu)=0.$$

Proof. It follows from (2.8), the change of variable $y=x/\mu$, (1.6), and the Dominated Convergence Theorem. □

An easy consequence of Lemmas 2.3 and 2.4 is the following result.

Corollary 2.5. *Assume that* $K_1 \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ *satisfies either* $(1.8)_i$ *or* $(1.9)_i$*,* $K_2 \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ *satisfies either* $(1.8)_i$ *or* $(1.9)_i$*, and* (1.6) *holds. Then there exist* $0 < a_1 < b_1$*,* $0 < a_2 < b_2$ *depending only on* K_1*, respectively* K_2*, (and independent of* ν*), such that*

$$(\Gamma^{k_1})'(a_1) \cdot (\Gamma^{k_1})'(b_1) < 0,$$
$$(\Gamma^{k_2})'(\nu\, a_2) \cdot (\Gamma^{k_2})'(\nu\, b_2) < 0.$$

Proof. Lemmas 2.3 and 2.4 imply that the functions

$$\mu \mapsto \int_{\mathbb{R}^N} K_i(\mu x) z_1^{2^*}(x)\, dx, \quad i = 1, 2$$

vanish at 0 and at ∞ and are not identically zero. Hence we can find $0 < a_1 < b_1$, $0 < a_2 < b_2$ such that

$$\left(\int_{\mathbb{R}^N} \nabla K_i(a_i x) \cdot x\, z_1^{2^*}(x)\, dx\right)\left(\int_{\mathbb{R}^N} \nabla K_i(b_i x) \cdot x\, z_1^{2^*}(x)\, dx\right) < 0, \quad i = 1, 2.$$

The result follows from above and the definition of Γ^{k_i}, see (2.8) and (2.12). □

For ν either large or small enough, we will construct a natural constraint for the functional $f_\varepsilon^{k_1+k_2}$ close to the 2-dimensional manifold

$$Z_\varepsilon = \left\{ z_{\mu_1,\varepsilon}^{k_1+k_2} + z_{\mu_2,\varepsilon}^{k_1+k_2} \,\middle|\, \mu_1 \in (a_1, b_1),\ \mu_2 \in (\nu a_2, \nu b_2) \right\}$$

where $0 < a_1 < b_1$ and $0 < a_2 < b_2$ are as in Corollary 2.5.

We will give a proof of our main theorem in the case $\ell = 2$, i.e. of the following theorem.

Theorem 2.6. *Let* $\lambda < (N-2)^2/4$ *satisfy* (1.4) *and assume* (1.6) *and* (2.11) *hold. Suppose that* $K_1 \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ *satisfies either* $(1.8)_i$ *or* $(1.9)_i$*,* $K_2 \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ *satisfies either* $(1.8)_i$ *or* $(1.9)_i$*, and that* a_1, a_2, b_1, b_2 *are as in Corollary 2.5. Then there exist a constant* $C = C(\lambda, N, a_1, a_2, b_1, b_2, \|K_1\|_{L^\infty}, \|K_2\|_{L^\infty}) > 0$ *and some* $\bar\varepsilon$ *sufficiently small such that if* $|\varepsilon| \le \bar\varepsilon$ *and*

$$\varepsilon^2 g(\nu)^{-\max\left\{\frac{1}{2}, \frac{N-2}{N+2}\right\}} \ge C \tag{2.17}$$

there exists a solution u_ε *to problem* $(\mathcal{P}_{\lambda,K}^\varepsilon)$ *close to* $z_{\mu_1} + z_{\mu_2}$ *for some* $\mu_1 \in (a_1, b_1)$ *and* $\mu_2 \in (\nu a_2, \nu b_2)$*.*

The general case $\ell > 2$ requires just simple modifications.

3 Estimates of the behaviour of $z_{\mu_i,\varepsilon}^{k_i}$

Lemma 3.1. *There exists* $C = C(\lambda, N, a_1, a_2, b_1, b_2, \|K_1\|_{L^\infty}, \|K_2\|_{L^\infty}) > 0$ *such that for any* $\mu_1 \in (a_1, b_1)$*,* $\mu_2 \in (\nu a_2, \nu b_2)$*, and* $|\varepsilon| \le \varepsilon_0 = \min\{\varepsilon_{k_1}, \varepsilon_{k_2}\}$

$$(i) \qquad \left|z_{\mu_1,\varepsilon}^{k_1}(x)\right| \le C\,|x|^{-(N-2-a_\lambda)} \qquad \text{for all} \quad |x| > 1,$$

$$(ii) \qquad \left|z_{\mu_2,\varepsilon}^{k_2}(x)\right| \le C\,\nu^{\frac{N-2-2a_\lambda}{2}}|x|^{-(N-2-a_\lambda)} \qquad \text{for all} \quad |x| > \nu,$$

$$(iii) \qquad \left|z_{\mu_1,\varepsilon}^{k_1}(x)\right| \le C\,|x|^{-a_\lambda} \qquad \text{for all} \quad |x| < 1,$$

$$(iv) \qquad \left|z_{\mu_2,\varepsilon}^{k_2}(x)\right| \le C\,\nu^{-\frac{N-2-2a_\lambda}{2}}|x|^{-a_\lambda} \qquad \text{for all} \quad |x| < \nu.$$

In order to prove the above estimates we will use the following elliptic estimate which is an easy consequence of [17, Theorem 1.1].

Theorem 3.2. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in H^1(\Omega)$ weakly solves*

$$-\Delta u - \frac{\lambda}{|x|^2} u = f.$$

Assume that

$$\int_\Omega |x|^{-a_\lambda(2^* - 2^* s + s)} |f(x)|^s \, dx < \infty$$

for some $s > N/2$. Then for any $\Omega' \Subset \Omega$ there is a constant $C = C(N, \Omega, \operatorname{dist}(\Omega', \Omega), s)$ such that

$$\sup_{\Omega'} \big| |x|^{a_\lambda} u(x) \big| \leq C \left\{ \|u\|_{L^2(\Omega)} + \left(\int_\Omega |x|^{-a_\lambda(2^* - 2^* s + s)} |f(x)|^s \, dx \right)^{1/s} \right\}.$$

Proof. It follows from [17, Theorem 1.1] after making the change of variable $v(x) = |x|^{a_\lambda} u(x)$. □

Let us denote as $\mathcal{D}^{1,2}_{a_\lambda}(\mathbb{R}^N)$ the space obtained by completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the weighted norm

$$\|v\|_{\mathcal{D}^{1,2}_{a_\lambda}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |x|^{-2a_\lambda} |\nabla v|^2 \, dx \right)^{1/2},$$

and set

$$S(\lambda, N) := \inf_{\mathcal{D}^{1,2}_{a_\lambda}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2a_\lambda} |\nabla v|^2 \, dx}{\left(\int_{\mathbb{R}^N} |x|^{-2^* a_\lambda} |v(x)|^{2^*} \, dx \right)^{2/2^*}}. \tag{3.1}$$

We have that $S(\lambda, N) > 0$; moreover $S(\lambda, N)$ is attained if $a_\lambda \geq 0$ (i.e. if $\lambda \geq 0$) and not attained if $a_\lambda < 0$ (i.e. if $\lambda < 0$), see [11].

The following Brezis-Kato type Lemma will be also used to prove Lemma 3.1. We refer to [17] for a proof (see also [25, Theorem 2.3]).

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^N$ be open and $q > 2$. Assume that $v \in \mathcal{D}^{1,2}_{a_\lambda}(\mathbb{R}^N)$,*

$$\int_\Omega |x|^{-2^* a_\lambda} |v(x)|^q < +\infty$$

and v is a weak solution of

$$-\operatorname{div}(|x|^{-2a_\lambda} \nabla v) - \frac{V(x)}{|x|^{2^* a_\lambda}} v = \frac{f(x)}{|x|^{2^* a_\lambda}} \quad \text{in } \Omega,$$

where

$$\int_\Omega |x|^{-2^* a_\lambda} |f(x)|^q < +\infty$$

and V satisfies for some $\sigma > 0$

$$\int_{|V(x)| \geq \sigma} |x|^{-2^* a_\lambda} |V(x)|^{\frac{N}{2}} + \int_{\Omega \setminus B_\sigma(0)} |x|^{-2^* a_\lambda} |V(x)|^{\frac{N}{2}} \leq \min \left\{ \frac{1}{8} S(\lambda, N), \frac{2}{q+4} S(\lambda, N) \right\}^{\frac{N}{2}} \tag{3.2}$$

where $S(\lambda, N)$ is defined in (3.1). Then for any $\Omega' \Subset \Omega$ there is a constant $C = C(\sigma, q, \Omega')$ such that

$$\left(\int_{\Omega'} |x|^{-2^* a_\lambda} |v(x)|^{\frac{2^* q}{2}}\, dx\right)^{\frac{2}{2^* q}} \le C(\sigma, q, \Omega')\left(\left(\int_{\Omega} |x|^{-2^* a_\lambda} |v(x)|^q\right)^{1/q} + \left(\int_{\Omega} |x|^{-2^* a_\lambda} |f(x)|^q\right)^{1/q}\right).$$

Proof of Lemma 3.1. Let us set $u_1 = z^{k_1}_{\mu_1,\varepsilon}$. From (2.3) we have that u_1 solves

$$-\Delta u_1 - \frac{\lambda}{|x|^2} u_1 = \big(1 + \varepsilon k_1(x)\big) u_1^{2^*-1} - \alpha^{k_1}_{\mu_1,\varepsilon}\Big(\Delta \dot\xi_{\mu_1} + \frac{\lambda}{|x|^2}\dot\xi_{\mu_1}\Big).$$

Since $\dot\xi_{\mu_1}$ solves the linearized problem $-\Delta \dot\xi_{\mu_1} - \frac{\lambda}{|x|^2}\dot\xi_{\mu_1} = (2^*-1) z_{\mu_1}^{2^*-2}\dot\xi_{\mu_1}$, we obtain

$$-\Delta u_1 - \frac{\lambda}{|x|^2} u_1 = \big(1 + \varepsilon k_1(x)\big) u_1^{2^*-1} + \alpha^{k_1}_{\mu_1,\varepsilon}(2^*-1) z_{\mu_1}^{2^*-2}\dot\xi_{\mu_1}. \tag{3.3}$$

For any function $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ let us denote by $u^\star \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ its Kelvin transform

$$u^\star(x) = |x|^{-(N-2)} u(x/|x|^2). \tag{3.4}$$

Let us set $w_1 := u_1^\star$. Since u_1 satisfies (3.3) in $\mathbb{R}^N \setminus B_{1/2}(0)$, we have that w_1 satisfies

$$-\Delta w_1 - \frac{\lambda}{|x|^2} w_1 = \big(1 + \varepsilon k_1(x/|x|^2)\big) w_1^{2^*-1} + (2^*-1)\alpha^{k_1}_{\mu_1,\varepsilon}(z^\star_{\mu_1})^{2^*-2}(\dot\xi_{\mu_1})^\star \quad \text{in } B_2(0). \tag{3.5}$$

The weighted function $v_1(x) = |x|^{a_\lambda} w_1(x)$ satisfies

$$-\mathrm{div}\,(|x|^{-2a_\lambda}\nabla v_1) - \frac{V(x)}{|x|^{2^* a_\lambda}} v_1 = \frac{f(x)}{|x|^{2^* a_\lambda}} \quad \text{in } B_2(0)$$

where

$$V(x) = (1 + \varepsilon k_1(x/|x|^2))|x|^{(2^*-2)a_\lambda} w_1(x)^{2^*-2}$$

and

$$f(x) = (2^*-1)\alpha^{k_1}_{\mu_1,\varepsilon}|x|^{(2^*-1)a_\lambda}(z^\star_{\mu_1})^{2^*-2}(\dot\xi_{\mu_1})^\star.$$

We claim that the function V defined above satisfies (3.2) for some σ independent of $\mu_1 \in (a_1, b_1)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Indeed since the map w^{k_1} depends continuously on μ_1 and ε and the Kelvin transform defined in (3.4) is an isomorphism of $\mathcal{D}^{1,2}(\mathbb{R}^N)$, it is easy to check that the family of functions

$$\{|{z^{k_1}_{\mu_1,\varepsilon}}^\star|^{2^*} :\ \mu_1 \in (a_1, b_1),\ \varepsilon \in (-\varepsilon_0, \varepsilon_0)\}$$

is relatively compact in $L^1(\mathbb{R}^N)$, hence from the Dunford-Pettis Theorem such a family is equi-integrable, i.e. for any $\eta > 0$ there exists $\delta > 0$ such that for any measurable set A with measure less than δ there holds

$$\int_A |{z^{k_1}_{\mu_1,\varepsilon}}^\star|^{2^*} < \eta \quad \text{for all} \quad \mu_1 \in (a_1, b_1) \quad \text{and} \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Set $A^\sigma_{\mu_1,\varepsilon}=\{x\in B_2(0):\ |V(x)|\geq\sigma\}$ and let $0<\gamma<\min\big\{2^*,\frac{2^*N}{N-2^*a_\lambda}\big\}$. Then for some positive constants c_i independent of $\mu_1\in(a_1,b_1)$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, Hölder inequality yields

$$\sigma^{\frac{\gamma}{2^*-2}}|A^\sigma_{\mu_1,\varepsilon}|\leq\int_{A^\sigma_{\mu_1,\varepsilon}}|V(x)|^{\frac{\gamma}{2^*-2}}\,dx\leq c_1\int_{A^\sigma_{\mu_1,\varepsilon}}|x|^{a_\lambda\gamma}|{z^{k_1}_{\mu_1,\varepsilon}}^\star|^\gamma\leq c_2\bigg(\int_{A^\sigma_{\mu_1,\varepsilon}}|{z^{k_1}_{\mu_1,\varepsilon}}^\star|^{2^*}\bigg)^{\frac{\gamma}{2^*}}\leq c_3$$

hence we can choose σ large enough independently of $\mu_1\in(a_1,b_1)$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ in order to have $|A^\sigma_{\mu_1,\varepsilon}|$ as small as we need. From this and the equi-integrability of $(z^{k_1}_{\mu_1,\varepsilon})^{2^*}$, condition (3.2) is proved to hold for some σ large enough independently of $\mu_1\in(a_1,b_1)$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$. Lemma A.5 and Lemma 3.3 with $q=2^*$ imply that for any $r<2$

$$\int_{B_r(0)}|x|^{-2^*a_\lambda}|v_1(x)|^{\frac{(2^*)^2}{2}}\,dx\leq c(N,\lambda,r).$$

Iterating the argument above a finite number of times, it is possible to show that for any $\tau>2^*$ there exists some constant $c=c(N,\lambda,\tau)$ independent of $\mu_1\in(a_1,b_1)$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ such that

$$\int_{B_{\frac{3}{2}}(0)}|x|^{-2^*a_\lambda}|v_1(x)|^\tau\,dx=\int_{B_{\frac{3}{2}}(0)}|x|^{-2^*a_\lambda+a_\lambda\tau}|w_1(x)|^\tau\,dx\leq c(N,\lambda,\tau).\tag{3.6}$$

Estimate (3.6) with some fixed $\tau>\frac{(2^*-1)N}{2}$, Lemma A.5 and (2.4) ensure that w_1 satisfies the assumptions of Theorem 3.2 with $s=\frac{\tau}{2^*-1}>\frac{N}{2}$. Then Theorem 3.2 yields

$$\sup_{B_1(0)}\bigg||x|^{a_\lambda-(N-2)}z^{k_1}_{\mu_1,\varepsilon}\Big(\frac{x}{|x|^2}\Big)\bigg|\leq C$$

for some positive constant C independent of $\mu_1\in(a_1,b_1)$ and $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, hence

$$\sup_{|y|>1}\big||x|^{N-2-a_\lambda}z^{k_1}_{\mu_1,\varepsilon}(y)\big|\leq C.$$

Estimate (i) is thereby proved. To prove (ii) we set $u_2=z^{k_2}_{\mu_2,\varepsilon}$. From (2.3) we have that u_2 solves

$$-\Delta u_2-\frac{\lambda}{|x|^2}u_2=\big(1+\varepsilon k_2(x)\big)u_2^{2^*-1}+\alpha^{k_2}_{\mu_2,\varepsilon}(2^*-1)z_{\mu_2}^{2^*-2}\dot\xi_{\mu_2}.\tag{3.7}$$

Let us set $\bar w_2:=u_2^\star$. Since u_2 satisfies (3.7) in $\mathbb{R}^N\setminus B_{\nu/2}(0)$, we have that $\bar w_2$ satisfies

$$-\Delta\bar w_2-\frac{\lambda}{|x|^2}\bar w_2=\big(1+\varepsilon k_2(x/|x|^2)\big)\bar w_2^{2^*-1}+(2^*-1)\alpha^{k_2}_{\mu_2,\varepsilon}(z^\star_{\mu_2})^{2^*-2}(\dot\xi_{\mu_2})^\star\quad\text{in }B_{2/\nu}(0).$$

The rescaled function $w_2(x)=\nu^{-\frac{N-2}{2}}\bar w_2(\nu^{-1}x)$ satisfies

$$-\Delta w_2-\frac{\lambda}{|x|^2}w_2=\big(1+\varepsilon k_2(\nu x/|x|^2)\big)w_2^{2^*-1}+(2^*-1)\alpha^{k_2}_{\mu_2,\varepsilon}\big(z^\star_{\frac{\mu_2}{\nu}}\big)^{2^*-2}\big(\dot\xi_{\frac{\mu_2}{\nu}}\big)^\star\quad\text{in }B_2(0).\tag{3.8}$$

Since $\frac{\mu_2}{\nu}\in(a_2,b_2)$ we can argue as in the proof of (i) above to conclude

$$\sup_{B_1(0)}\big||x|^{a_\lambda}w_2(x)\big|\leq C\tag{3.9}$$

for some positive constant C independent of $\mu_2 \in (a_2, b_2)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ (depending on a_2, b_2, λ, N, $\|k_2\|_{L^\infty}$). Estimate (ii) hence follows from (3.9) and

$$w_2(x) = \nu^{\frac{N-2}{2}} |x|^{-(N-2)} z^{k_2}_{\mu_2,\varepsilon}\Big(\nu \frac{x}{|x|^2}\Big).$$

Let us now prove (iii). Set $\tilde v_1(x) = |x|^{a_\lambda} u_1(x)$, where $u_1 = z^{k_1}_{\mu_1,\varepsilon}$ solves (3.3). We have that $\tilde v_1$ solves equation

$$-\mathrm{div}\,(|x|^{-2a_\lambda} \nabla \tilde v_1) - \frac{\widetilde V(x)}{|x|^{2^* a_\lambda}} \tilde v_1 = \frac{\tilde f(x)}{|x|^{2^* a_\lambda}} \quad \text{in } B_2(0)$$

where

$$\widetilde V(x) = (1 + \varepsilon k_1(x)) |x|^{(2^*-2)a_\lambda} u_1(x)^{2^*-2}$$

and

$$\tilde f(x) = (2^* - 1) \alpha^{k_1}_{\mu_1,\varepsilon} |x|^{(2^*-1)a_\lambda} z^{2^*-2}_{\mu_1} \dot\xi_{\mu_1}.$$

Since $\{z^{k_1}_{\mu_1,\varepsilon};\ \mu_1 \in (a_1, b_1)\}$ is equi-integrable and (A.17) holds, we can argue as the proof of (i) and apply Lemma 3.3 a finite number of times to deduce that for any $\tau > 2^*$ there exists some constant $c = c(N, \lambda, \tau)$ independent of $\mu_1 \in (a_1, b_1)$ such that

$$\int_{B_{\frac32}(0)} |x|^{-2^* a_\lambda} |\tilde v_1(x)|^\tau \, dx = \int_{B_{\frac32}(0)} |x|^{-2^* a_\lambda + a_\lambda \tau} |u_1(x)|^\tau \, dx \le c(N, \lambda, \tau). \tag{3.10}$$

Estimate (3.10) with some fixed $\tau > \frac{(2^*-1)N}{2}$, (A.17) and (2.4) ensure that u_1 satisfies the assumptions of Theorem 3.2 with $s = \frac{\tau}{2^*-1} > \frac{N}{2}$ hence we obtain

$$\sup_{B_1(0)} \big| |x|^{a_\lambda} z^{k_1}_{\mu_1,\varepsilon}(x) \big| \le C$$

for some positive constant C independent of $\mu_1 \in (a_1, b_1)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ thus proving (iii). Similarly (iv) follows by applying Theorem 3.2 to the function $\nu^{\frac{N-2}{2}} z^{k_2}_{\mu_2,\varepsilon}(\nu x)$. □

As a consequence of Lemma 3.1 we obtain the following result.

Lemma 3.4. *There exists* $C = C(\lambda, N, a_1, a_2, b_1, b_2, \|K_1\|_{L^\infty}, \|K_2\|_{L^\infty}) > 0$ *such that for any* $\mu_1 \in (a_1, b_1)$*,* $\mu_2 \in (\nu a_2, \nu b_2)$*, and* $|\varepsilon| \le \varepsilon_0$

$$\big| z^{k_1}_{\mu_1,\varepsilon}(x) \big| \le C z_1(x), \tag{3.11}$$

$$\big| z^{k_2}_{\mu_2,\varepsilon}(x) \big| \le C z_\nu(x). \tag{3.12}$$

Proof. A direct calculation gives

$$z_1(x) = \frac{A(N,\lambda)|x|^{-a_\lambda}}{\big(1 + |x|^{2 - \frac{4a_\lambda}{N-2}}\big)^{\frac{N-2}{2}}} = \frac{A(N,\lambda)|x|^{-(N-2-a_\lambda)}}{\big(1 + |x|^{\frac{4a_\lambda}{N-2} - 2}\big)^{\frac{N-2}{2}}}$$

hence

$$z_1(x) \ge \begin{cases} c(\lambda, N)|x|^{-a_\lambda} & \text{if } |x| < 1, \\ c(\lambda, N)|x|^{-(N-2-2a_\lambda)} & \text{if } |x| > 1. \end{cases} \tag{3.13}$$

(3.11) follows from (3.13) and $(i),(iii)$ of Lemma 3.1. To prove (3.12) we observe that

$$z_\nu(x)=\frac{A(N,\lambda)\nu^{-\frac{N-2-2a_\lambda}{2}}|x|^{-a_\lambda}}{\left(1+|x/\nu|^{2-\frac{4a_\lambda}{N-2}}\right)^{\frac{N-2}{2}}}=\frac{A(N,\lambda)\nu^{\frac{N-2-2a_\lambda}{2}}|x|^{-(N-2-a_\lambda)}}{\left(1+|x/\nu|^{\frac{4a_\lambda}{N-2}-2}\right)^{\frac{N-2}{2}}}$$

hence

$$z_\nu(x)\geq\begin{cases}c(\lambda,N)\nu^{-\frac{N-2-2a_\lambda}{2}}|x|^{-a_\lambda} & \text{if } |x|<\nu,\\ c(\lambda,N)\nu^{\frac{N-2-2a_\lambda}{2}}|x|^{-(N-2-2a_\lambda)} & \text{if } |x|>\nu.\end{cases}\tag{3.14}$$

(3.12) follows from (3.14) and $(ii),(iv)$ of Lemma 3.1. □

Remark 3.5. *The same argument used in the proof of Lemmas 3.1 and 3.4 can be performed to prove analogous estimates for $z_{\mu_1,\varepsilon}^{k_1+k_2}$ and $z_{\mu_2,\varepsilon}^{k_1+k_2}$, namely it is possible to prove*

$$\left|z_{\mu_1,\varepsilon}^{k_1+k_2}(x)\right|\leq Cz_1(x),\tag{3.15}$$

$$\left|z_{\mu_2,\varepsilon}^{k_1+k_2}(x)\right|\leq Cz_\nu(x),\tag{3.16}$$

for some positive constant $C=C(\lambda,N,a_1,a_2,b_1,b_2,\|K_1\|_{L^\infty},\|K_2\|_{L^\infty})>0$.

4 Interaction estimates

Lemma 4.1. *For any $0<\beta<2^*$ there exists $C=C(\beta,\lambda,N,a_1,a_2,b_1,b_2,\|K_1\|_{L^\infty},\|K_2\|_{L^\infty})>0$ such that for any $\mu_1\in(a_1,b_1)$, $\mu_2\in(\nu a_2,\nu b_2)$, and $|\varepsilon|\leq\varepsilon_0$*

$$\int_{\mathbb{R}^N}|z_{\mu_1,\varepsilon}^{k_1}|^{2^*-\beta}|z_{\mu_2,\varepsilon}^{k_2}|^\beta\leq C\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-\beta\frac{N-2}{2}},\tag{4.1}$$

$$\int_{\mathbb{R}^N}|z_{\mu_2,\varepsilon}^{k_2}|^{2^*-\beta}|z_{\mu_1,\varepsilon}^{k_1}|^\beta\leq C\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-\beta\frac{N-2}{2}},\tag{4.2}$$

$$\int_{\mathbb{R}^N}|z_{\mu_1,\varepsilon}^{k_1+k_2}|^{2^*-\beta}|z_{\mu_2,\varepsilon}^{k_1+k_2}|^\beta\leq C\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-\beta\frac{N-2}{2}},\tag{4.3}$$

$$\int_{\mathbb{R}^N}|z_{\mu_2,\varepsilon}^{k_1+k_2}|^{2^*-\beta}|z_{\mu_1,\varepsilon}^{k_1+k_2}|^\beta\leq C\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-\beta\frac{N-2}{2}},\tag{4.4}$$

where $\gamma_\lambda=1-\frac{2a_\lambda}{N-2}$.

Proof. We claim that

$$\int_{\mathbb{R}^N}z_1^{2^*-\beta}z_\mu^\beta\leq c\big[\mu^{\gamma_\lambda}+\mu^{-\gamma_\lambda}\big]^{-\beta\frac{N-2}{2}}\quad\text{for any }\mu>0\tag{4.5}$$

for some positive constant c depending only on N, β, and λ. Indeed if $\beta<2^*/2$, for any $\mu>0$ we have

$$\int_{\mathbb{R}^N}z_1^{2^*-\beta}z_\mu^\beta=A(N,\lambda)^{2^*}|\mathbb{S}^{N-1}|\int_0^\infty r^{-1}\big(r^{\gamma_\lambda}+r^{-\gamma_\lambda}\big)^{-(2^*-\beta)\frac{N-2}{2}}\left(\left|\frac{r}{\mu}\right|^{\gamma_\lambda}+\left|\frac{r}{\mu}\right|^{-\gamma_\lambda}\right)^{-\beta\frac{N-2}{2}}dr$$

where $|\mathbb{S}^{N-1}|$ denotes the measure of the unit $(N-1)$-dimensional sphere. Performing the change of variable $t = \ln r$ we obtain

$$
\begin{aligned}
&\int_0^\infty r^{-1}\big(r^{\gamma_\lambda}+r^{-\gamma_\lambda}\big)^{-(2^*-\beta)\frac{N-2}{2}}\bigg(\Big|\frac{r}{\mu}\Big|^{\gamma_\lambda}+\Big|\frac{r}{\mu}\Big|^{-\gamma_\lambda}\bigg)^{-\beta\frac{N-2}{2}}\,dr\\
&=2^{-N}\int_{\mathbb{R}}\big(\cosh(\gamma_\lambda t)\big)^{-(2^*-\beta)\frac{N-2}{2}}\big[\cosh(\gamma_\lambda(t-\ln\mu))\big]^{-\beta\frac{N-2}{2}}\,dt\\
&=\frac{\big[\cosh(\gamma_\lambda\ln\mu)\big]^{-\beta\frac{N-2}{2}}}{2^N}\int_{\mathbb{R}}\big[\cosh(\gamma_\lambda t)\big]^{-(2^*-\beta)\frac{N-2}{2}}\big[\cosh(\gamma_\lambda t)-\tanh(\gamma_\lambda\ln\mu)\sinh(\gamma_\lambda t)\big]^{-\beta\frac{N-2}{2}}\,dt\\
&\le \mathrm{const}\big[\cosh(\gamma_\lambda\ln\mu)\big]^{-\beta\frac{N-2}{2}}\int_{\mathbb{R}}\big[\cosh(\gamma_\lambda t)\big]^{-(2^*-\beta)\frac{N-2}{2}}\big[\cosh(\gamma_\lambda t)-\operatorname{sign}t\sinh(\gamma_\lambda t)\big]^{-\beta\frac{N-2}{2}}\,dt\\
&\le \mathrm{const}\big[\cosh(\gamma_\lambda\ln\mu)\big]^{-\beta\frac{N-2}{2}}=\mathrm{const}\big[\mu^{\gamma_\lambda}+\mu^{-\gamma_\lambda}\big]^{-\beta\frac{N-2}{2}}
\end{aligned}
$$

proving (4.5) for $\beta < 2^*/2$. Estimate (4.5) for $2^* > \beta > 2^*/2$ follows from above and

$$\int_{\mathbb{R}^N} z_\mu^{2^*-\beta} z_1^\beta = \int_{\mathbb{R}^N} z_1^{2^*-\beta} z_{\frac{1}{\mu}}^\beta. \tag{4.6}$$

If $\beta = 2^*/2 = N/(N-2)$ from Hölder's inequality and (4.5) with $\beta = \frac{4N}{3(N-2)} \neq \frac{2^*}{2}$ we have that

$$
\begin{aligned}
\int_{\mathbb{R}^N} z_1^{\frac{N}{N-2}} z_\mu^{\frac{N}{N-2}} &= \int_{\mathbb{R}^N} z_1^{\frac{N}{2(N-2)}} z_1^{\frac{N}{2(N-2)}} z_\mu^{\frac{N}{N-2}}\\
&\le \bigg(\int_{\mathbb{R}^N} z_1^{2^*}\bigg)^{1/4}\bigg(\int_{\mathbb{R}^N} z_1^{\frac{2N}{3(N-2)}} z_\mu^{\frac{4N}{3(N-2)}}\bigg)^{3/4} \le \mathrm{const}\,\big[\mu^{\gamma_\lambda}+\mu^{-\gamma_\lambda}\big]^{-\frac{N}{2}}
\end{aligned}
\tag{4.7}
$$

thus proving (4.5) in the case $\beta = \frac{2^*}{2}$. Estimate (4.1) follows from Lemma 3.4 and (4.5). Estimate (4.2) follows from Lemma 3.4, (4.6), and (4.5). The proof of (4.3) and (4.4) is analogous taking into account Remark 3.5. □

Lemma 4.2. *There exists $C = C(\lambda, N, a_1, a_2, b_1, b_2, \|K_1\|_{L^\infty}, \|K_2\|_{L^\infty}) > 0$ such that for any $\mu_1 \in (a_1, b_1)$, $\mu_2 \in (\nu a_2, \nu b_2)$, and $|\varepsilon| \le \varepsilon_0$*

$$\|Df_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1}) - Df_\varepsilon^{k_1}(z_{\mu_1,\varepsilon}^{k_1})\| \le C|\varepsilon| f_2(\nu), \tag{4.8}$$

$$\|Df_\varepsilon^{k_1+k_2}(z_{\mu_2,\varepsilon}^{k_2}) - Df_\varepsilon^{k_2}(z_{\mu_2,\varepsilon}^{k_2})\| \le C|\varepsilon| f_1(\nu), \tag{4.9}$$

where f_1, f_2 are defined in (2.13).

Proof. Hölder and Sobolev inequalities and estimate (3.11) yield

$$
\begin{aligned}
\big|\big(Df_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1}) - Df_\varepsilon^{k_1}(z_{\mu_1,\varepsilon}^{k_1}), v\big)\big| &= \bigg|\varepsilon\int_{\mathbb{R}^N} k_2(x)\big(z_{\mu_1,\varepsilon}^{k_1}\big)_+^{2^*-1} v\,dx\bigg|\\
&\le \mathrm{const}\,|\varepsilon|\|v\|\bigg(\int_{\mathbb{R}^N}\Big|K_2\Big(\frac{x}{\nu}\Big)\Big| z_1^{2^*}(x)\,dx\bigg)^{\frac{2^*-1}{2^*}}.
\end{aligned}
$$

In a similar way using Hölder and Sobolev inequalities and estimate (3.12) we obtain

$$\begin{aligned}\big|\big(Df_\varepsilon^{k_1+k_2}(z^{k_2}_{\mu_2,\varepsilon})-Df_\varepsilon^{k_2}(z^{k_2}_{\mu_2,\varepsilon}),v\big)\big| &= \left|\varepsilon\int_{\mathbb{R}^N}k_1(x)\big(z^{k_2}_{\mu_2,\varepsilon}\big)_+^{2^*-1}v\,dx\right|\\ \leq \text{const}\,|\varepsilon|\|v\|\bigg(\int_{\mathbb{R}^N}\Big|K_1(x)\Big|z_\nu^{2^*}(x)\,dx\bigg)^{\frac{2^*-1}{2^*}} &= \text{const}\,|\varepsilon|\|v\|\bigg(\int_{\mathbb{R}^N}\Big|K_1(\nu x)\Big|z_1^{2^*}(x)\,dx\bigg)^{\frac{2^*-1}{2^*}}.\end{aligned}$$

The lemma is thereby established. □

Lemma 4.3. *Let $\lambda<(N-2)^2/4$ satisfying (1.4). There exist C_1, $\varepsilon_1>0$, $\varepsilon_1\leq\varepsilon_0$ such that for all $|\varepsilon|\leq\varepsilon_1$ and $\mu>0$ there holds*

$$\|w^{k_1+k_2}(\mu,\varepsilon)-w^{k_1}(\mu,\varepsilon)\|\leq C_1\|Df_\varepsilon^{k_1+k_2}(z^{k_1}_{\mu,\varepsilon})-Df_\varepsilon^{k_1}(z^{k_1}_{\mu,\varepsilon})\| \tag{4.10}$$

and

$$\|w^{k_1+k_2}(\mu,\varepsilon)-w^{k_2}(\mu,\varepsilon)\|\leq C_1\|Df_\varepsilon^{k_1+k_2}(z^{k_2}_{\mu,\varepsilon})-Df_\varepsilon^{k_2}(z^{k_2}_{\mu,\varepsilon})\|. \tag{4.11}$$

Proof. Let us define the map

$$\Psi:\mathbb{R}^+\times\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathbb{R}\times\mathbb{R}\to\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathbb{R}$$

with components $\Psi_1\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\Psi_2\in\mathbb{R}$ given by

$$\begin{cases}\Psi_1(\mu,w,\alpha,\varepsilon) &= Df_\varepsilon^{k_1+k_2}\big(z_\mu+w^{k_1}(\mu,\varepsilon)+w\big)-\big(\alpha^{k_1}(\mu,\varepsilon)+\alpha\big)\dot{\xi}_\mu\\ \Psi_2(\mu,w,\alpha,\varepsilon) &= \big(w,\dot{\xi}_\mu\big).\end{cases} \tag{4.12}$$

We have that $\Psi(\mu,w,\alpha,\varepsilon)=0$ if and only if $(w,\alpha)=\Phi_{\varepsilon,\mu}(w,\alpha)$ where

$$\Phi_{\varepsilon,\mu}(w,\alpha)=-\left[\frac{\partial\Psi}{\partial(w,\alpha)}(\mu,0,0,\varepsilon)\right]^{-1}\Psi(\mu,w,\alpha,\varepsilon)+(w,\alpha).$$

Using non-degeneracy property (2.1) and (A.8), we can easily obtain that for ε sufficiently small $\frac{\partial\Psi}{\partial(w,\alpha)}(\mu,0,0,\varepsilon)$ is invertible and

$$\left\|\left(\frac{\partial\Psi}{\partial(w,\alpha)}(\mu,0,0,\varepsilon)\right)^{-1}\right\|\leq\text{const}\quad\text{uniformly with respect to }\mu>0$$

hence for some positive constant c

$$\begin{aligned}\|\Phi_{\varepsilon,\mu}(w,\alpha)\| &\leq c\left\|\Psi(\mu,w,\alpha,\varepsilon)-\frac{\partial\Psi}{\partial(w,\alpha)}(\mu,0,0,\varepsilon)(w,\alpha)\right\|\\ &= c\Big\|Df_\varepsilon^{k_1+k_2}\big(z_\mu+w^{k_1}(\mu,\varepsilon)+w\big)-Df_\varepsilon^{k_1}\big(z_\mu+w^{k_1}(\mu,\varepsilon)\big)-D^2f_\varepsilon^{k_1+k_2}\big(z_\mu+w^{k_1}(\mu,\varepsilon)\big)w\Big\|\\ &\leq c\Big\|Df_\varepsilon^{k_1+k_2}\big(z^{k_1}_{\mu,\varepsilon}+w\big)-Df_\varepsilon^{k_1+k_2}\big(z^{k_1}_{\mu,\varepsilon}\big)-D^2f_\varepsilon^{k_1+k_2}\big(z^{k_1}_{\mu,\varepsilon}\big)w\Big\|\\ &\qquad+c\Big\|Df_\varepsilon^{k_1+k_2}\big(z^{k_1}_{\mu,\varepsilon}\big)-Df_\varepsilon^{k_1}\big(z^{k_1}_{\mu,\varepsilon}\big)\Big\|\\ &= c\left|\int_0^1\Big[D^2f_\varepsilon^{k_1+k_2}\big(z^{k_1}_{\mu,\varepsilon}+tw\big)-D^2f_\varepsilon^{k_1+k_2}\big(z^{k_1}_{\mu,\varepsilon}\big)\Big]w\,dt\right|+c\Big\|Df_\varepsilon^{k_1+k_2}\big(z^{k_1}_{\mu,\varepsilon}\big)-Df_\varepsilon^{k_1}\big(z^{k_1}_{\mu,\varepsilon}\big)\Big\|.\end{aligned} \tag{4.13}$$

Therefore if $\|w\| \leq \rho < 1$, (4.13) and (A.8) yield for some positive constant c_1

$$\|\Phi_{\varepsilon,\mu}(w,\alpha)\| \leq c_1 \Big[\Big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu,\varepsilon}^{k_1}\big) - Df_\varepsilon^{k_1}\big(z_{\mu,\varepsilon}^{k_1}\big)\Big\| + \rho^{\min\left\{2,\frac{N+2}{N-2}\right\}}\Big]. \tag{4.14}$$

Similarly from (A.8), we have for some positive constant c_2 that for any $w_1,\ w_2 \in B_\rho(0) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\begin{aligned}
&\|\Phi_{\varepsilon,\mu}(w_1,\alpha_1) - \Phi_{\varepsilon,\mu}(w_2,\alpha_2)\| \\
&\quad \leq c\Big\| Df_\varepsilon^{k_1+k_2}\big(z_\mu + w^{k_1}(\mu,\varepsilon) + w_1\big) - Df_\varepsilon^{k_1+k_2}\big(z_\mu + w^{k_1}(\mu,\varepsilon) + w_2\big) \\
&\qquad\qquad - D^2 f_\varepsilon^{k_1+k_2}\big(z_\mu + w^{k_1}(\mu,\varepsilon)\big)(w_1 - w_2)\Big\| \\
&\quad \leq c\int_0^1 \Big\| D^2 f_\varepsilon^{k_1+k_2}\big(z_{\mu,\varepsilon}^{k_1} + t(w_1-w_2) + w_2\big) - D^2 f_\varepsilon^{k_1+k_2}\big(z_{\mu,\varepsilon}^{k_1}\big)\Big\| \|w_1 - w_2\|\,dt \\
&\quad \leq c_2\, \rho^{\min\left\{1,\frac{4}{N-2}\right\}} \|w_1 - w_2\|.
\end{aligned} \tag{4.15}$$

Let us choose $\rho = \rho(\mu,\varepsilon) = 2c_1 \Big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu,\varepsilon}^{k_1}\big) - Df_\varepsilon^{k_1}\big(z_{\mu,\varepsilon}^{k_1}\big)\Big\|$ and note that in view of Lemma 4.2 there exists $0 < \varepsilon_1 < \varepsilon_0$ such that for all $|\varepsilon| \leq \varepsilon_1$ and for all $\mu > 0$

$$\rho(\mu,\varepsilon)^{\min\left\{1,\frac{4}{N-2}\right\}} \leq \min\Big\{\frac{1}{2c_1}, \frac{1}{2c_2}\Big\}. \tag{4.16}$$

From (4.14), (4.15) and (4.16), we deduce that $\Phi_{\varepsilon,\mu}$ for $|\varepsilon| \leq \varepsilon_1$ maps the ball of radius $\rho(\mu,\varepsilon)$ into itself and it is a contraction there. From the Contraction Mapping Theorem we have that $\Phi_{\varepsilon,\mu}$ has a unique fixed point in the ball of radius $\rho(\mu,\varepsilon)$, namely there exists a unique couple of functions $\big(\bar{w}(\mu,\varepsilon), \bar{\alpha}(\mu,\varepsilon)\big) \in \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathbb{R}$ such that for all $\mu > 0$ and $|\varepsilon| \leq \varepsilon_1$

$$\begin{aligned}
&\big(\bar{w}(\mu,\varepsilon), \dot{\xi}_\mu\big) = 0, \\
&Df_\varepsilon^{k_1+k_2}\big(z_\mu + w^{k_1}(\mu,\varepsilon) + \bar{w}(\mu,\varepsilon)\big) = \big(\alpha^{k_1}(\mu,\varepsilon) + \bar{\alpha}(\mu,\varepsilon)\big)\dot{\xi}_\mu, \\
&\|\bar{w}(\mu,\varepsilon)\| \leq 2c_1 \Big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu,\varepsilon}^{k_1}\big) - Df_\varepsilon^{k_1}\big(z_{\mu,\varepsilon}^{k_1}\big)\Big\|.
\end{aligned}$$

By the uniqueness statement of Theorem 2.1 there must be

$$w^{k_1}(\mu,\varepsilon) + \bar{w}(\mu,\varepsilon) = w^{k_1+k_2}(\mu,\varepsilon)$$

and hence

$$\|w^{k_1+k_2}(\mu,\varepsilon) - w^{k_1}(\mu,\varepsilon)\| = \|\bar{w}(\mu,\varepsilon)\| \leq 2c_1 \Big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu,\varepsilon}^{k_1}\big) - Df_\varepsilon^{k_1}\big(z_{\mu,\varepsilon}^{k_1}\big)\Big\|.$$

(4.10) is thereby proved. The proof of (4.11) is analogous. □

5 Natural constraint for two bubble fountain solutions

Let us consider the function

$$H : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R}$$

with components $H_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $H_2 \in \mathbb{R} \times \mathbb{R}$ given by

$$\begin{cases} H_1(\mu_1, \mu_2, w, \alpha_1, \alpha_2, \varepsilon) & = Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2} + z_{\mu_2,\varepsilon}^{k_1+k_2} + w\big) \\ & \quad -\big(\alpha_1 + \alpha^{k_1+k_2}(\mu_1, \varepsilon)\big)\dot{\xi}_{\mu_1} - \big(\alpha_2 + \alpha^{k_1+k_2}(\mu_2, \varepsilon)\big)\dot{\xi}_{\mu_2} \\ H_2(\mu_1, \mu_2, w, \alpha_1, \alpha_2, \varepsilon) & = \big((w, \dot{\xi}_{\mu_1}), (w, \dot{\xi}_{\mu_2})\big). \end{cases} \tag{5.1}$$

We endow the space $\mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathbb{R}\times\mathbb{R}$ with the norm $\|(v, \beta_1, \beta_2)\| = \max\{\|v\|, |\beta_1|, |\beta_2|\}$. The following lemma ensures that if ν is either large or small enough, the operator $\frac{\partial H}{\partial(w,\alpha_1,\alpha_2)}\big|_{(\mu_1,\mu_2,0,0,0,\varepsilon)}$ is invertible for ε small and the norm of the inverse is uniformly bounded.

Lemma 5.1. *There exist $C_2, \varepsilon_2, L_1 > 0$ such that for any $\nu \in (0, 1/L_1)\cup(L_1, +\infty)$, $\mu_1 \in (a_1, b_1)$, $\mu_2 \in (\nu a_2, \nu b_2)$ and for $|\varepsilon| \leq \varepsilon_2$ there holds*

$$\left\| \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}\bigg|_{(\mu_1,\mu_2,0,0,0,\varepsilon)} (v, \beta_1, \beta_2) \right\| \geq C_2 \|(v, \beta_1, \beta_2)\| \tag{5.2}$$

for any $(v, \beta_1, \beta_2) \in \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R}$.

Proof. Since from Lemma A.3 and (2.4)

$$\begin{aligned} &\left\| \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}\bigg|_{(\mu_1,\mu_2,0,0,0,\varepsilon)} - \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}\bigg|_{(\mu_1,\mu_2,0,0,0,0)} \right\| \\ &\quad = \big\|D^2 f_\varepsilon^{k_1+k_2}(z_{\mu_1} + z_{\mu_2} + w^{k_1+k_2}(\mu_1, \varepsilon) + w^{k_1+k_2}(\mu_2, \varepsilon)) - D^2 f_0(z_{\mu_1} + z_{\mu_2})\big\| \\ &\quad \leq \big\|D^2 f_\varepsilon^{k_1+k_2}(z_{\mu_1} + z_{\mu_2} + w^{k_1+k_2}(\mu_1, \varepsilon) + w^{k_1+k_2}(\mu_2, \varepsilon)) - D^2 f_\varepsilon^{k_1+k_2}(z_{\mu_1} + z_{\mu_2})\big\| \\ &\qquad + \big\|D^2 f_\varepsilon^{k_1+k_2}(z_{\mu_1} + z_{\mu_2}) - D^2 f_0(z_{\mu_1} + z_{\mu_2})\big\| \leq \text{const}\, |\varepsilon|^{\min\{1,4/(N-2)\}} \end{aligned}$$

to prove (5.2) it is enough to prove that for some positive constant C

$$\big\|\big(D^2 f_0(z_{\mu_1} + z_{\mu_2})v - \beta_1\dot{\xi}_{\mu_1} - \beta_2\dot{\xi}_{\mu_2}, (v, \dot{\xi}_{\mu_1}), (v, \dot{\xi}_{\mu_2})\big)\big\| \geq C\|(v, \beta_1, \beta_2)\| \tag{5.3}$$

for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $\beta_1, \beta_2 \in \mathbb{R}$, $\mu_1 \in (a_1, b_1)$, $\mu_2 \in (\nu a_2, \nu b_2)$, provided ν is either large or small enough. Arguing by contradiction, let us assume that (5.3) is not verified, namely that there exist sequences $\{\nu_n\}_n$, $\{\beta_n^1\}_n$, $\{\beta_n^2\}_n$, $\{\mu_n^1\}_n$, $\{\mu_n^2\}_n \subset \mathbb{R}$ and $\{v_n\}_n \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that

$$\text{either} \quad \nu_n < 1/n \quad \text{or} \quad \nu_n > n, \quad a_1 < \mu_n^1 < b_1, \quad \nu_n a_2 < \mu_n^2 < \nu_n b_2, \tag{5.4}$$

$$\|v_n\| + |\beta_n^1| + |\beta_n^2| = 1, \tag{5.5}$$

$$\big\|\big(D^2 f_0(z_{\mu_n^1} + z_{\mu_n^2})v_n - \beta_n^1\dot{\xi}_{\mu_n^1} - \beta_n^2\dot{\xi}_{\mu_n^2}, (v_n, \dot{\xi}_{\mu_n^1}), (v_n, \dot{\xi}_{\mu_n^2})\big)\big\| \underset{n\to\infty}{\longrightarrow} 0, \tag{5.6}$$

$$|(v_n, \dot{\xi}_{\mu_n^1})| + |(v_n, \dot{\xi}_{\mu_n^2})| \underset{n\to\infty}{\longrightarrow} 0. \tag{5.7}$$

From (5.4) either there exists a subsequence of $\{\nu_n\}_n$ tending to 0 or there exists a subsequence tending to ∞, therefore there is no restriction assuming that either $\lim_{n\to\infty}\nu_n = 0$ or $\lim_{n\to\infty}\nu_n = +\infty$. For $\mu > 0$, we denote as $U_\mu : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathcal{D}^{1,2}(\mathbb{R}^N)$ the rescaling map defined by

$$U_\mu(u) := \mu^{-\frac{N-2}{2}} u\big(x/\mu\big). \tag{5.8}$$

It is easy to check that U_μ conserves the norms $\|\cdot\|$ and $\|\cdot\|_{L^{2^*}(\mathbb{R}^N)}$, thus for every $\mu > 0$

$$(U_\mu)^{-1} = (U_\mu)^t = U_{\mu^{-1}} \quad \text{and} \quad f_0 = f_0 \circ U_\mu \tag{5.9}$$

where $(U_\mu)^t$ denotes the adjoint of U_μ. Twice differentiating the identity $f_0 = f_0 \circ U_\mu$ we obtain for all $h_1, h_2, v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$(D^2 f_0(v)h_1, h_2) = (D^2 f_0(U_\mu(v))U_\mu(h_1), U_\mu(h_2)). \tag{5.10}$$

Set $u_n := U_{1/\mu_n^1} v_n$. From (5.10) we have that for any $h \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\big(D^2 f_0(z_{\mu_n^1} + z_{\mu_n^2})v_n, h\big) = \big(D^2 f_0(z_1 + z_{\mu_n^2/\mu_n^1})u_n, U_{1/\mu_n^1} h\big). \tag{5.11}$$

From (A.18) it follows

$$\dot\xi_\mu = \frac{U_\mu \dot z_1}{\|\dot z_1\|} \quad \text{for all } \mu > 0. \tag{5.12}$$

From (5.9) and (5.12) we have

$$(h, \dot\xi_{\mu_n^1}) = (U_{1/\mu_n^1} h, \dot\xi_1) \quad \text{and} \quad (h, \dot\xi_{\mu_n^2}) = (U_{1/\mu_n^1} h, \dot\xi_{\mu_n^2/\mu_n^1}). \tag{5.13}$$

From (5.11) and (5.13) we deduce

$$D^2 f_0(z_{\mu_n^1} + z_{\mu_n^2})v_n - \beta_n^1 \dot\xi_{\mu_n^1} - \beta_n^2 \dot\xi_{\mu_n^2} = (U_{1/\mu_n^1})^t \Big(D^2 f_0(z_1 + z_{\mu_n^2/\mu_n^1})u_n - \beta_n^1 \dot\xi_1 - \beta_n^2 \dot\xi_{\mu_n^2/\mu_n^1}\Big)$$

hence from (5.6) we obtain

$$\Big\| D^2 f_0(z_1 + z_{\mu_n^2/\mu_n^1})u_n - \beta_n^1 \dot\xi_1 - \beta_n^2 \dot\xi_{\mu_n^2/\mu_n^1} \Big\| \underset{n\to\infty}{\longrightarrow} 0. \tag{5.14}$$

On the other hand from (5.12) and (5.9) we have $(v_n, \dot\xi_{\mu_n^1}) = (u_n, \dot\xi_1)$ and $(v_n, \dot\xi_{\mu_n^2}) = (u_n, \dot\xi_{\mu_n^2/\mu_n^1})$ hence (5.7) yields

$$|(u_n, \dot\xi_1)| + |(u_n, \dot\xi_{\mu_n^2/\mu_n^1})| \underset{n\to\infty}{\longrightarrow} 0. \tag{5.15}$$

Moreover (5.5) and invariance of norm under rescaling imply

$$\|u_n\| + |\beta_n^1| + |\beta_n^2| = 1. \tag{5.16}$$

If $\nu_n \to \infty$, from (5.4) we have that

$$\mu_n := \frac{\mu_n^2}{\mu_n^1} \underset{n\to\infty}{\longrightarrow} +\infty.$$

From (5.16), there exist subsequences of $\{u_n\}_n$, $\{\beta_n^1\}_n$, $\{\beta_n^2\}_n$ (still denoted by $\{u_n\}_n$, $\{\beta_n^1\}_n$, $\{\beta_n^2\}_n$) such that

$$\beta_n^1 \underset{n\to\infty}{\longrightarrow} \beta_1, \quad \beta_n^2 \underset{n\to\infty}{\longrightarrow} \beta_2, \quad u_n \underset{n\to\infty}{\rightharpoonup} u \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N).$$

For any $h \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ there holds

$$(D^2 f_0(z_1 + z_{\mu_n})u_n, h) = (u_n, h) - (2^* - 1)\int_{\mathbb{R}^N} (z_1 + z_{\mu_n})_+^{2^*-2} u_n h \, dx. \tag{5.17}$$

Since z_{μ_n} converges to 0 poitwise a.e., by Vitali's convergence Theorem, we can pass to the limit in (5.17) and thus find

$$(D^2 f_0(z_1 + z_{\mu_n})u_n, h) \underset{n\to\infty}{\longrightarrow} (u, h) - (2^* - 1)\int_{\mathbb{R}^N} z_1^{2^*-2} u h \, dx. \tag{5.18}$$

From boundedness and pointwise convergence of $\dot\xi_{\mu_n}$, we deduce that $\dot\xi_{\mu_n}$ weakly converges to 0 in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, hence from (5.14) and (5.18) we get

$$(u, h) - (2^* - 1)\int_{\mathbb{R}^N} z_1^{2^*-2} u h \, dx - \beta_1(\dot\xi_1, h) = 0 \quad \text{for any } h \in \mathcal{D}^{1,2}(\mathbb{R}^N)$$

i.e. $D^2 f_0(z_1)u = \beta_1 \dot\xi_1$. Hence $\beta_1 = (D^2 f_0(z_1)u, \dot\xi_1) = (D^2 f_0(z_1)\dot\xi_1, u) = 0$. Then $D^2 f_0(z_1)u = 0$. From (2.1) we deduce that $u = \alpha\dot\xi_1$ for some $\alpha \in \mathbb{R}$. From (5.15) we obtain

$$0 = \lim_{n\to\infty} (u_n, \dot\xi_1) = (u, \dot\xi_1)$$

which implies $\alpha = 0$. Hence $u = 0$. We have thus proved that $u_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\beta_n^1 \to 0$ as $n \to \infty$. In a similar way, we define $w_n := U_{1/\mu_n^2} v_n$. Arguing as above we can prove that $w_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\beta_n^2 \to 0$ as $n \to \infty$. As a consequence, from (5.14) we find that $D^2 f_0(z_1 + z_{\mu_n})u_n \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and hence

$$\|u_n\|^2 - (2^* - 1)\int_{\mathbb{R}^N} (z_1 + z_{\mu_n})_+^{2^*-2} u_n^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{5.19}$$

Since $\int_{\mathbb{R}^N} z_{\mu_n}^{2^*-2} u_n^2 = \int_{\mathbb{R}^N} z_1^{2^*-2} w_n^2$ from Vitali's convergence Theorem we get $\int_{\mathbb{R}^N} z_{\mu_n}^{2^*-2} u_n^2 \to 0$. Using Lemma A.1 with $s = 2^* - 2$ and again Vitali's convergence Theorem, one can easily prove that

$$\int_{\mathbb{R}^N} |(z_1 + z_{\mu_n})_+^{2^*-2} - z_{\mu_n}^{2^*-2}| u_n^2 \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore

$$\int_{\mathbb{R}^N} (z_1 + z_{\mu_n})_+^{2^*-2} u_n^2 \le \int_{\mathbb{R}^N} |(z_1 + z_{\mu_n})_+^{2^*-2} - z_{\mu_n}^{2^*-2}| u_n^2 + \int_{\mathbb{R}^N} z_{\mu_n}^{2^*-2} u_n^2 \underset{n\to\infty}{\longrightarrow} 0$$

and hence from (5.19) we deduce that $u_n \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $n \to \infty$, which is in contradiction with (5.16). As the proof in the case $\nu_n \to 0$ is analogous, we omit it. □

Proposition 5.2. *There exist* $C = C(\lambda, N, a_1, a_2, b_1, b_2, \|K_1\|_{L^\infty}, \|K_2\|_{L^\infty}) > 0$, $L > 0$ *and* $\bar\varepsilon > 0$ *such that for all* $\nu \in (0, 1/L) \cup (L, +\infty)$ *there exist* C^1*-functions*

$$\begin{aligned} w :\ & (a_1, b_1) \times (\nu a_2, \nu b_2) \times (-\bar\varepsilon, \bar\varepsilon) \longrightarrow \mathcal{D}^{1,2}(\mathbb{R}^N), \\ \alpha_i :\ & (a_1, b_1) \times (\nu a_2, \nu b_2) \times (-\bar\varepsilon, \bar\varepsilon) \longrightarrow \mathbb{R}, \quad i = 1, 2, \end{aligned}$$

such that for all $\mu_1 \in (a_1, b_1)$, $\mu_2 \in (\nu a_2, \nu b_2)$, *and for all* $|\varepsilon| \le \bar\varepsilon$, *there hold*

$$\begin{aligned} &(i) && w(\mu_1, \mu_2, \varepsilon) \in \langle \dot\xi_{\mu_1}, \dot\xi_{\mu_2} \rangle^\perp, \\ &(ii) && Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2} + z_{\mu_2,\varepsilon}^{k_1+k_2} + w(\mu_1, \mu_2, \varepsilon)\big) \\ &&& \quad = \big(\alpha^{k_1+k_2}(\mu_1, \varepsilon) + \alpha_1(\mu_1, \mu_2, \varepsilon)\big)\dot\xi_{\mu_1} + \big(\alpha^{k_1+k_2}(\mu_2, \varepsilon) + \alpha_2(\mu_1, \mu_2, \varepsilon)\big)\dot\xi_{\mu_2}, \\ &(iii) && \|w(\mu_1, \mu_2, \varepsilon)\| + \sum_{i=1}^{2} |\alpha_i(\mu_1, \mu_2, \varepsilon)| \le g(\nu)^{\max\left\{\frac12, \frac{N-2}{N+2}\right\}}, \\ &(iv) && \|\partial_{\mu_1} w(\mu_1, \mu_2, \varepsilon)\| \le C\Big[|\varepsilon|^{\min\left\{1, \frac{4}{N-2}\right\}} + g(\nu)^{\min\left\{\frac12, \frac{2}{N-2}\right\}}\Big], \\ &(v) && \|\partial_{\mu_2} w(\mu_1, \mu_2, \varepsilon)\| \le C\,\nu^{-1}\Big[|\varepsilon|^{\min\left\{1, \frac{4}{N-2}\right\}} + g(\nu)^{\min\left\{\frac12, \frac{2}{N-2}\right\}}\Big], \end{aligned}$$

where g *is defined in* (1.11).

Proof. Let H be the function defined in (5.1). If $H(\mu_1, \mu_2, w, \alpha_1, \alpha_2, \varepsilon) = 0$ then w, α_1, and α_2 satisfy $(i - ii)$ and $H(\mu_1, \mu_2, w, \alpha_1, \alpha_2, \varepsilon) = 0$ if and only if $(w, \alpha_1, \alpha_2) = F_{\varepsilon,\mu_1,\mu_2}(w, \alpha_1, \alpha_2)$ where

$$F_{\varepsilon,\mu_1,\mu_2}(w, \alpha_1, \alpha_2) = -\left[\frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}(\mu_1, \mu_2, 0, 0, 0, \varepsilon)\right]^{-1} H(\mu_1, \mu_2, w, \alpha_1, \alpha_2, \varepsilon) + (w, \alpha_1, \alpha_2).$$

Suppose that $(w, \alpha_1, \alpha_2) \in \bar B_\rho(0) = \{(x, \beta_1, \beta_1) \in \mathcal{D}^{1,2}(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R} :\ \|(x, \beta_1, \beta_1)\| \le \rho\}$ with $\rho < 1$ to be detemined. From (5.2) we have

$$\begin{aligned} \|F_{\varepsilon,\mu_1,\mu_2}(w, \alpha_1, \alpha_2)\| &\le \frac{1}{C_2}\left\| H(\mu_1, \mu_2, w, \alpha_1, \alpha_2, \varepsilon) - \frac{\partial H}{\partial(w, \alpha_1, \alpha_2)}(\mu_1, \mu_2, 0, 0, 0, \varepsilon)(w, \alpha_1, \alpha_2)\right\| \\ &= \frac{1}{C_2}\big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2} + z_{\mu_2,\varepsilon}^{k_1+k_2} + w\big) - D^2 f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2} + z_{\mu_2,\varepsilon}^{k_1+k_2}\big) w \\ &\qquad - \alpha^{k_1+k_2}(\mu_1, \varepsilon)\dot\xi_{\mu_1} - \alpha^{k_1+k_2}(\mu_2, \varepsilon)\dot\xi_{\mu_2}\big\| \end{aligned}$$

where C_2 is as in Lemma 5.1. From above and (2.3) we deduce

$$
\begin{aligned}
&\|F_{\varepsilon,\mu_1,\mu_2}(w,\alpha_1,\alpha_2)\| \\
&\quad\leq \frac{1}{C_2}\big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w\big)-Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1}+w^{k_1+k_2}(\mu_1,\varepsilon)\big) \\
&\qquad\qquad -Df_\varepsilon^{k_1+k_2}\big(z_{\mu_2}+w^{k_1+k_2}(\mu_2,\varepsilon)\big)-D^2f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)w\big\| \\
&\quad\leq \frac{1}{C_2}\big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w\big)-Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big) \\
&\qquad\qquad -D^2f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)w\big\| \\
&\quad+\frac{1}{C_2}\big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)-Df_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2})-Df_\varepsilon^{k_1+k_2}(z_{\mu_2,\varepsilon}^{k_1+k_2})\big\|. \qquad (5.20)
\end{aligned}
$$

From Lemma A.3 it follows that

$$
\begin{aligned}
&\big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w\big)-Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)-D^2f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)w\big\| \\
&\leq \|w\|\int_0^1 \big\|D^2f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+tw\big)-D^2f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)\big\|\,dt \\
&\leq \left\{\begin{array}{ll} \text{const}\,\|w\|^2\Big(1+\|w\|^{\frac{6-N}{N-2}}\Big), & \text{if } N<6, \\ \text{const}\,\|w\|^{\frac{N+2}{N-2}}, & \text{if } N\geq 6, \end{array}\right\} \leq \text{const}\,\rho^{\min\{2,(N+2)/(N-2)\}}. \qquad (5.21)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)-Df_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2})-Df_\varepsilon^{k_1+k_2}(z_{\mu_2,\varepsilon}^{k_1+k_2})\big\| \\
&\quad\leq \big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)-Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1}+z_{\mu_2,\varepsilon}^{k_2}\big)\big\| \\
&\qquad+\big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}\big)-Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1}\big)\big\|+\big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_2,\varepsilon}^{k_1+k_2}\big)-Df_\varepsilon^{k_1+k_2}\big(z_{\mu_2,\varepsilon}^{k_2}\big)\big\| \\
&\qquad+\big\|Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1}+z_{\mu_2,\varepsilon}^{k_2}\big)-Df_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1})-Df_\varepsilon^{k_1+k_2}(z_{\mu_2,\varepsilon}^{k_2})\big\|. \qquad (5.22)
\end{aligned}
$$

From Hölder inequality and estimate $(A.4)_s$ with $s=2^*-1$ it follows that for any $h\in\mathcal{D}^{1,2}(\mathbb{R}^N)$

$$
\begin{aligned}
&\big|\big(Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1}+z_{\mu_2,\varepsilon}^{k_2}\big)-Df_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1})-Df_\varepsilon^{k_1+k_2}(z_{\mu_2,\varepsilon}^{k_2}),h\big)\big| \\
&\quad=\left|\int_{\mathbb{R}^N}\big(1+\varepsilon(k_1+k_2)\big)\Big[\big(z_{\mu_1,\varepsilon}^{k_1}+z_{\mu_2,\varepsilon}^{k_2}\big)_+^{2^*-1}-\big(z_{\mu_1,\varepsilon}^{k_1}\big)_+^{2^*-1}-\big(z_{\mu_2,\varepsilon}^{k_2}\big)_+^{2^*-1}\Big]h\right| \\
&\quad\leq \text{const}\,\|h\|\bigg(\int_{\mathbb{R}^N}\Big|\big(z_{\mu_1,\varepsilon}^{k_1}+z_{\mu_2,\varepsilon}^{k_2}\big)_+^{2^*-1}-\big(z_{\mu_1,\varepsilon}^{k_1}\big)_+^{2^*-1}-\big(z_{\mu_2,\varepsilon}^{k_2}\big)_+^{2^*-1}\Big|^{\frac{2N}{N+2}}\bigg)^{\frac{N+2}{2N}} \\
&\quad\leq \text{const}\,\|h\|\bigg(\int_{\mathbb{R}^N}\Big||z_{\mu_1,\varepsilon}^{k_1}|^{\frac{4}{N-2}}|z_{\mu_2,\varepsilon}^{k_2}|+|z_{\mu_2,\varepsilon}^{k_2}|^{\frac{4}{N-2}}|z_{\mu_1,\varepsilon}^{k_1}|\Big|^{\frac{2N}{N+2}}\bigg)^{\frac{N+2}{2N}} \\
&\quad\leq \text{const}\,\|h\|\bigg(\int_{\mathbb{R}^N}|z_{\mu_1,\varepsilon}^{k_1}|^{\frac{8N}{N^2-4}}|z_{\mu_2,\varepsilon}^{k_2}|^{\frac{2N}{N+2}}+|z_{\mu_2,\varepsilon}^{k_2}|^{\frac{8N}{N^2-4}}|z_{\mu_1,\varepsilon}^{k_1}|^{\frac{2N}{N+2}}\bigg)^{\frac{N+2}{2N}}. \qquad (5.23)
\end{aligned}
$$

From (5.23) and Lemma 4.1 we deduce that

$$\big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1}+z_{\mu_2,\varepsilon}^{k_2}\big) - Df_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1}) - Df_\varepsilon^{k_1+k_2}(z_{\mu_2,\varepsilon}^{k_2})\big\| \\ \leq \text{const}\big[\nu^{1-\frac{2a_\lambda}{N-2}}+\nu^{\frac{2a_\lambda}{N-2}-1}\big]^{-\max\left\{2,\frac{N-2}{2}\right\}}. \tag{5.24}$$

From (A.7), Lemma 4.3, and Lemma 4.2 we find that

$$\begin{aligned} &\big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big) - Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1}+z_{\mu_2,\varepsilon}^{k_2}\big)\big\| \\ &\quad + \big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}\big) - Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1}\big)\big\| + \big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu_2,\varepsilon}^{k_1+k_2}\big) - Df_\varepsilon^{k_1+k_2}\big(z_{\mu_2,\varepsilon}^{k_2}\big)\big\| \\ &\quad\leq \text{const}\Big[\|w^{k_1+k_2}(\mu_1,\varepsilon)-w^{k_1}(\mu_1,\varepsilon)\| + \|w^{k_1+k_2}(\mu_2,\varepsilon)-w^{k_2}(\mu_2,\varepsilon)\|\Big] \\ &\quad\leq \text{const}\,|\varepsilon|\big[f_1(\nu)+f_2(\nu)\big]. \end{aligned} \tag{5.25}$$

From (5.20), (5.21), (5.22), (5.24), and (5.25) we deduce the existence of a positive constant c_4 such that

$$\|F_{\varepsilon,\mu_1,\mu_2}(w,\alpha_1,\alpha_2)\| \leq c_4\Big[\rho^{\min\{2,(N+2)/(N-2)\}}+g(\nu)\Big] \tag{5.26}$$

where $g(\nu)$ is defined in (1.11). On the other hand from (A.8) we obtain for some positive constant c_5

$$\begin{aligned} &\|F_{\varepsilon,\mu_1,\mu_2}(w,\alpha_1,\alpha_2) - F_{\varepsilon,\mu_1,\mu_2}(w',\mu_1',\mu_2')\| \\ &\leq \text{const}\,\big\| Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w\big) - Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w'\big) \\ &\qquad\qquad - D^2f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)(w-w')\big\| \\ &\leq \text{const}\,\|w-w'\|\int_0^1 \big\|D^2f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w'+t(w-w')\big) \\ &\qquad\qquad - D^2f_\varepsilon^{k_1+k_2}\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}\big)\big\|\,dt \\ &\leq c_5\,\|w-w'\|\,\rho^{\min\{1,4/(N-2)\}}. \end{aligned} \tag{5.27}$$

From Lemma 2.2 we get that

$$\lim_{\nu\to 0} g(\nu) = \lim_{\nu\to+\infty} g(\nu) = 0$$

hence there exists $L>0$ such that

$$g(\nu)^{\min\left\{\frac12,\frac{4}{N+2}\right\}} \leq \min\left\{\frac{1}{2c_4},\frac{1}{2c_5}\right\} \quad \text{for all } \nu\in(0,1/L)\cup(L,+\infty). \tag{5.28}$$

Let us choose $\rho=\rho(\nu)=g(\nu)^{\max\left\{\frac12,\frac{N-2}{N+2}\right\}}$. With this choice of ρ from (5.26) and (5.27) it follows that $F_{\varepsilon,\mu_1,\mu_2}$ maps the ball of radius $\rho(\nu)$ into itself and it is a contraction there. From the Contraction Mapping Theorem we have that $F_{\varepsilon,\mu_1,\mu_2}$ has a unique fixed point in the ball of radius $\rho(\nu)$, namely there exists a unique triplet of functions $\big(w(\mu_1,\mu_2,\varepsilon),\alpha_1(\mu_1,\mu_2,\varepsilon),\alpha_2(\mu_1,\mu_2,\varepsilon)\big)\in \mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathbb{R}\times\mathbb{R}$ such that $(i-iii)$ are satisfied.

To prove estimates $(iv-v)$ we observe that w and α_i, $i=1,2$, are implicitly defined by $H(\mu_1,\mu_2,w,\alpha_1,\alpha_2,\varepsilon)=0$. From Lemma 5.1, for any $\mu_1\in(a_1,b_1)$, $\mu_2\in(\nu a_2,\nu b_2)$ and $|\varepsilon|<\bar\varepsilon$

$$\left\|\left(\frac{\partial H}{\partial(w,\alpha_1,\alpha_2)}\right)^{-1}\bigg|_{(\mu_1,\mu_2,0,0,0,\varepsilon)}\right\|\leq\frac{1}{C_2}$$

and hence there exists a positive constant $\tilde C$ such that

$$\left\|\left(\frac{\partial H}{\partial(w,\alpha_1,\alpha_2)}\right)^{-1}\bigg|_{(\mu_1,\mu_2,w(\mu_1,\mu_2,\varepsilon),\alpha_1(\mu_1,\mu_2,\varepsilon),\alpha_2(\mu_1,\mu_2,\varepsilon),\varepsilon)}\right\|\leq\tilde C$$

for ν either large or small enough. Since $\partial_{\mu_1}w(\mu_1,\mu_2,\varepsilon)$ satisfies

$$\begin{pmatrix}\partial_{\mu_1}w(\mu_1,\mu_2,\varepsilon)\\ \partial_{\mu_1}\alpha_1(\mu_1,\mu_2,\varepsilon)\\ \partial_{\mu_1}\alpha_2(\mu_1,\mu_2,\varepsilon)\end{pmatrix}=-\left(\frac{\partial H}{\partial(w,\alpha_1,\alpha_2)}\right)^{-1}\bigg|_{(\mu_1,\mu_2,w,\alpha_1,\alpha_2,\varepsilon)}\cdot\frac{\partial H}{\partial\mu_1}\bigg|_{(\mu_1,\mu_2,w,\alpha_1,\alpha_2,\varepsilon)}$$

we have

$$\begin{aligned}\|\partial_{\mu_1}w(\mu_1,\mu_2,\varepsilon)\|&\leq\tilde C\left\|\frac{\partial H}{\partial\mu_1}\bigg|_{(\mu_1,\mu_2,w,\alpha_1,\alpha_2)}\right\|\\&\leq\tilde C\bigg[\Big\|D^2f_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w(\mu_1,\mu_2,\varepsilon))\partial_{\mu_1}z_{\mu_1,\varepsilon}^{k_1+k_2}-\alpha_1(\mu_1,\mu_2,\varepsilon)\frac{d}{d\mu_1}\dot\xi_{\mu_1}\\&\qquad-\partial_{\mu_1}\alpha^{k_1+k_2}(\mu_1,\varepsilon)\dot\xi_{\mu_1}-\alpha^{k_1+k_2}(\mu_1,\varepsilon)\frac{d}{d\mu_1}\dot\xi_{\mu_1}\Big\|+\Big|\Big(w(\mu_1,\mu_2,\varepsilon),\frac{d}{d\mu_1}\dot\xi_{\mu_1}\Big)\Big|\bigg].\end{aligned}\tag{5.29}$$

From (2.3) we have that

$$Df_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2})=\alpha^{k_1+k_2}(\mu_1,\varepsilon)\dot\xi_{\mu_1}$$

which, differentiating with respect to μ_1, yields

$$D^2f_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2})\partial_{\mu_1}z_{\mu_1,\varepsilon}^{k_1+k_2}=\partial_{\mu_1}\alpha^{k_1+k_2}(\mu_1,\varepsilon)\dot\xi_{\mu_1}+\alpha^{k_1+k_2}(\mu_1,\varepsilon)\frac{d}{d\mu_1}\dot\xi_{\mu_1}$$

hence from (5.29) we have

$$\begin{aligned}&\|\partial_{\mu_1}w(\mu_1,\mu_2,\varepsilon)\|\\&\quad\leq\tilde C\bigg[\Big\|\big(D^2f_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w(\mu_1,\mu_2,\varepsilon))-D^2f_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2})\big)\partial_{\mu_1}z_{\mu_1,\varepsilon}^{k_1+k_2}\\&\qquad\qquad-\alpha_1(\mu_1,\mu_2,\varepsilon)\frac{d}{d\mu_1}\dot\xi_{\mu_1}\Big\|+\Big|\Big(w(\mu_1,\mu_2,\varepsilon),\frac{d}{d\mu_1}\dot\xi_{\mu_1}\Big)\Big|\bigg].\end{aligned}\tag{5.30}$$

We have that

$$\begin{aligned}&\Big|\Big(\big(D^2f_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w(\mu_1,\mu_2,\varepsilon))-D^2f_\varepsilon^{k_1+k_2}(z_{\mu_1,\varepsilon}^{k_1+k_2})\big)\partial_{\mu_1}z_{\mu_1,\varepsilon}^{k_1+k_2},v\Big)\Big|\\&=(2^*-1)\bigg|\int_{\mathbb R^N}(1+\varepsilon(k_1+k_2))\big[\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w(\mu_1,\mu_2,\varepsilon)\big)_+^{2^*-2}-\big(z_{\mu_1,\varepsilon}^{k_1+k_2}\big)_+^{2^*-2}\big]\partial_{\mu_1}z_{\mu_1,\varepsilon}^{k_1+k_2}v\bigg|\\&\leq\text{const}\bigg[\int_{\mathbb R^N}\Big|\big(z_{\mu_1,\varepsilon}^{k_1+k_2}+z_{\mu_2,\varepsilon}^{k_1+k_2}+w(\mu_1,\mu_2,\varepsilon)\big)_+^{2^*-2}-\big(z_{\mu_1,\varepsilon}^{k_1+k_2}\big)_+^{2^*-2}-\big(z_{\mu_2,\varepsilon}^{k_1+k_2}\big)_+^{2^*-2}\Big||\partial_{\mu_1}z_{\mu_1,\varepsilon}^{k_1+k_2}||v|\\&\qquad+\int_{\mathbb R^N}\big|z_{\mu_2,\varepsilon}^{k_1+k_2}\big|^{2^*-2}|\partial_{\mu_1}z_{\mu_1,\varepsilon}^{k_1+k_2}||v|\bigg].\end{aligned}\tag{5.31}$$

If $N \geq 6$, then $2^* - 2 \leq 1$, hence using $(A.6)_s$ with $s = 2^* - 2$ and $r = q = 2/(N-2)$, Hölder and Sobolev inequalities we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big| \big(z^{k_1+k_2}_{\mu_1,\varepsilon} + z^{k_1+k_2}_{\mu_2,\varepsilon} + w(\mu_1,\mu_2,\varepsilon)\big)^{2^*-2}_+ - \big(z^{k_1+k_2}_{\mu_1,\varepsilon}\big)^{2^*-2}_+ - \big(z^{k_1+k_2}_{\mu_2,\varepsilon}\big)^{2^*-2}_+ \Big| |\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}||v| \\
&\quad \leq \text{const} \left[\int_{\mathbb{R}^N} |w(\mu_1,\mu_2,\varepsilon)|^{2^*-2} |\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}||v| + \int_{\mathbb{R}^N} |z^{k_1+k_2}_{\mu_1,\varepsilon}|^{\frac{2}{N-2}} |z^{k_1+k_2}_{\mu_2,\varepsilon}|^{\frac{2}{N-2}} |\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}||v| \right] \\
&\quad \leq \text{const}\, \|v\| \|\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}\| \left[\|w(\mu_1,\mu_2,\varepsilon)\|^{2^*-2} + \left(\int_{\mathbb{R}^N} |z^{k_1+k_2}_{\mu_1,\varepsilon}|^{\frac{N}{N-2}} |z^{k_1+k_2}_{\mu_2,\varepsilon}|^{\frac{N}{N-2}} \right)^{\frac{2}{N}} \right]. \qquad (5.32)
\end{aligned}$$

From (A.10) and (2.5) it follows that

$$\|\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}\| \leq \|\dot{z}_{\mu_1}\| + \|\partial_{\mu_1} w^{k_1+k_2}(\mu_1,\varepsilon)\| \leq c \qquad (5.33)$$

for some constant $c > 0$ depending only on $a_1, b_1, \lambda, N, K_1, K_2$. From (5.32), (5.33), (4.3) and (iii) we deduce for $N \geq 6$

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big| \big(z^{k_1+k_2}_{\mu_1,\varepsilon} + z^{k_1+k_2}_{\mu_2,\varepsilon} + w(\mu_1,\mu_2,\varepsilon)\big)^{2^*-2}_+ - \big(z^{k_1+k_2}_{\mu_1,\varepsilon}\big)^{2^*-2}_+ - \big(z^{k_1+k_2}_{\mu_2,\varepsilon}\big)^{2^*-2}_+ \Big| |\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}||v| \\
&\qquad \leq \text{const}\, \|v\|\, g(\nu)^{\frac{2}{N-2}}. \qquad (5.34)
\end{aligned}$$

For $N < 6$, from $(A.5)_s$ with $s = 2^* - 2$, Lemma 4.1, and (iii) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big| \big(z^{k_1+k_2}_{\mu_1,\varepsilon} + z^{k_1+k_2}_{\mu_2,\varepsilon} + w(\mu_1,\mu_2,\varepsilon)\big)^{2^*-2}_+ - \big(z^{k_1+k_2}_{\mu_1,\varepsilon}\big)^{2^*-2}_+ - \big(z^{k_1+k_2}_{\mu_2,\varepsilon}\big)^{2^*-2}_+ \Big| |\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}||v| \\
&\qquad \leq \text{const}\, \|v\|\, g(\nu)^{\frac{1}{2}}. \qquad (5.35)
\end{aligned}$$

From Hölder inequality, (A.11), and (3.16) we obtain

$$\int_{\mathbb{R}^N} |z^{k_1+k_2}_{\mu_2,\varepsilon}|^{2^*-2} |\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}||v| \leq \text{const}\, \|v\| \left[\|\partial_{\mu_1} w^{k_1+k_2}(\mu_1,\varepsilon)\| + \left(\int_{\mathbb{R}^N} |z_\nu|^{\frac{8N}{N^2-4}} |z_1|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}} \right].$$

From above, (4.5), (4.6), and (2.5) we deduce

$$\int_{\mathbb{R}^N} |z^{k_1+k_2}_{\mu_2,\varepsilon}|^{2^*-2} |\partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon}||v| \leq \text{const}\, \|v\| \left[|\varepsilon|^{\min\left\{1,\frac{4}{N-2}\right\}} + g(\nu) \right]. \qquad (5.36)$$

From (5.31), (5.34), (5.35), and (5.36) we obtain

$$\begin{aligned}
&\big\| \big(D^2 f^{k_1+k_2}_\varepsilon (z^{k_1+k_2}_{\mu_1,\varepsilon} + z^{k_1+k_2}_{\mu_2,\varepsilon} + w(\mu_1,\mu_2,\varepsilon)) - D^2 f^{k_1+k_2}_\varepsilon (z^{k_1+k_2}_{\mu_1,\varepsilon}) \big) \partial_{\mu_1} z^{k_1+k_2}_{\mu_1,\varepsilon} \big\| \\
&\qquad \leq \text{const} \left[|\varepsilon|^{\min\left\{1,\frac{4}{N-2}\right\}} + g(\nu)^{\min\left\{\frac{1}{2},\frac{2}{N-2}\right\}} \right]. \qquad (5.37)
\end{aligned}$$

From (A.13) we deduce that for any $\mu > 0$

$$\frac{d}{d\mu} \dot{\xi}_\mu = \mu^{-1} \frac{U_\mu(\dot{z}_1 + \ddot{z}_1)}{\|\dot{z}_1\|}$$

hence

$$\Big\| \frac{d}{d\mu}\dot\xi_\mu \Big\| = \mu^{-1}\frac{\|\dot z_1 + \ddot z_1\|}{\|\dot z_1\|} = c(\lambda, N)\mu^{-1}. \tag{5.38}$$

for some constant $c(\lambda, N)$ depending only on λ and N. From (iii) and (5.38) we deduce

$$\Big\| \alpha_1(\mu_1,\mu_2,\varepsilon)\frac{d}{d\mu_1}\dot\xi_{\mu_1}\Big\| + \Big|\Big(w(\mu_1,\mu_2,\varepsilon)), \frac{d}{d\mu_1}\dot\xi_{\mu_1}\Big)\Big| \leq \text{const}\, g(\nu)^{\max\left\{\frac12, \frac{N-2}{N+2}\right\}}. \tag{5.39}$$

From (5.30), (5.37), (5.39) we get (iv). The proof of (v) is analogous. □

Proposition 5.3. *Under the assumptions of Proposition 5.2 we may choose $\bar\varepsilon$ small and L large enough such that for all $\nu \in (0, 1/L) \cup (L, +\infty)$ and $|\varepsilon| \leq \bar\varepsilon$ the manifold*

$$\widetilde Z_\varepsilon = \big\{ z^{k_1+k_2}_{\mu_1,\varepsilon} + z^{k_1+k_2}_{\mu_2,\varepsilon} + w(\mu_1,\mu_2,\varepsilon) : \ \mu_1 \in (a_1, b_1),\ \mu_2 \in (\nu a_2, \nu b_2)\big\}$$

is a natural constraint for $f^{k_1+k_2}_\varepsilon$.

Proof. Let $u = z^{k_1+k_2}_{\mu_1,\varepsilon} + z^{k_1+k_2}_{\mu_2,\varepsilon} + w(\mu_1,\mu_2,\varepsilon) \in \widetilde Z_\varepsilon$ be a critical point of $f^{k_1+k_2}_\varepsilon\big|_{\widetilde Z_\varepsilon}$, i.e.

$$\big(f^{k_1+k_2}_\varepsilon(u), \dot z_{\mu_1} + \partial_{\mu_1} w^{k_1+k_2}(\mu_1,\varepsilon) + \partial_{\mu_1} w(\mu_1,\mu_2,\varepsilon)\Big) = 0, \tag{5.40}$$

$$\big(f^{k_1+k_2}_\varepsilon(u), \dot z_{\mu_2} + \partial_{\mu_2} w^{k_1+k_2}(\mu_2,\varepsilon) + \partial_{\mu_2} w(\mu_1,\mu_2,\varepsilon)\Big) = 0. \tag{5.41}$$

We have to prove that $Df^{k_1+k_2}_\varepsilon(u) = 0$. From statement (ii) of Proposition 5.2 we have that

$$Df^{k_1+k_2}_\varepsilon(u) = c_1(\mu_1,\mu_1,\varepsilon)\dot z_{\mu_1} + c_2(\mu_1,\mu_1,\varepsilon)\dot z_{\mu_2} \tag{5.42}$$

for some $c_1(\mu_1,\mu_1,\varepsilon), c_2(\mu_1,\mu_1,\varepsilon) \in \mathbb{R}$. From (5.40), (5.41), and (5.42) it follows that

$$\begin{aligned} &c_1(\mu_1,\mu_1,\varepsilon)\big[\|\dot z_{\mu_1}\|^2 + \big(\dot z_{\mu_1}, \partial_{\mu_1} w^{k_1+k_2}(\mu_1,\varepsilon)\big) + \big(\dot z_{\mu_1}, \partial_{\mu_1} w(\mu_1,\mu_2,\varepsilon)\big)\big] \\ &\quad + c_2(\mu_1,\mu_1,\varepsilon)\big[(\dot z_{\mu_1}, \dot z_{\mu_2}) + \big(\dot z_{\mu_2}, \partial_{\mu_1} w^{k_1+k_2}(\mu_1,\varepsilon)\big) + \big(\dot z_{\mu_2}, \partial_{\mu_1} w(\mu_1,\mu_2,\varepsilon)\big)\big] = 0 \end{aligned} \tag{5.43}$$

and

$$\begin{aligned} &c_2(\mu_1,\mu_1,\varepsilon)\big[\|\dot z_{\mu_2}\|^2 + \big(\dot z_{\mu_2}, \partial_{\mu_2} w^{k_1+k_2}(\mu_2,\varepsilon)\big) + \big(\dot z_{\mu_2}, \partial_{\mu_2} w(\mu_1,\mu_2,\varepsilon)\big)\big] \\ &\quad + c_1(\mu_1,\mu_1,\varepsilon)\big[(\dot z_{\mu_1}, \dot z_{\mu_2}) + \big(\dot z_{\mu_1}, \partial_{\mu_2} w^{k_1+k_2}(\mu_2,\varepsilon)\big) + \big(\dot z_{\mu_1}, \partial_{\mu_2} w(\mu_1,\mu_2,\varepsilon)\big)\big] = 0. \end{aligned} \tag{5.44}$$

From statement (i) of Proposition 5.2 we have

$$\big(\dot z_{\mu_2}, w(\mu_1,\mu_2,\varepsilon)\big) = 0 \quad \text{and} \tag{5.45}$$

$$\big(\dot z_{\mu_1}, w(\mu_1,\mu_2,\varepsilon)\big) = 0 \tag{5.46}$$

and differentiating (5.45) with respect to μ_1 and (5.46) with respect to μ_2 we get

$$\big(\dot z_{\mu_2}, \partial_{\mu_1} w(\mu_1,\mu_2,\varepsilon)\big) = 0 \quad \text{and} \quad \big(\dot z_{\mu_1}, \partial_{\mu_2} w(\mu_1,\mu_2,\varepsilon)\big) = 0. \tag{5.47}$$

Differentiating (5.45) with respect to μ_2 and (5.46) with respect to μ_1 we get

$$\begin{aligned}\big(\dot z_{\mu_2}, \partial_{\mu_2} w(\mu_1,\mu_2,\varepsilon)\big) &= -\big(\ddot z_{\mu_2}, w(\mu_1,\mu_2,\varepsilon)\big) \quad \text{and}\\ \big(\dot z_{\mu_1}, \partial_{\mu_1} w(\mu_1,\mu_2,\varepsilon)\big) &= -\big(\ddot z_{\mu_1}, w(\mu_1,\mu_2,\varepsilon)\big)\end{aligned}$$

and hence in view of (A.12) and (iii) of Proposition 5.2 we obtain

$$\big|\big(\dot z_{\mu_2}, \partial_{\mu_2} w(\mu_1,\mu_2,\varepsilon)\big)\big| \le c\,\nu^{-2} g(\nu)^{\max\left\{\frac12,\frac{N-2}{N+2}\right\}} \quad \text{and} \tag{5.48}$$

$$\big|\big(\dot z_{\mu_1}, \partial_{\mu_1} w(\mu_1,\mu_2,\varepsilon)\big)\big| \le c\, g(\nu)^{\max\left\{\frac12,\frac{N-2}{N+2}\right\}} \tag{5.49}$$

for some positive constant c depending only on λ, N, a_1, a_2, b_1, b_2. Since $\dot z_{\mu_1}$ solves the linearized problem

$$-\Delta \dot z_{\mu_1} - \frac{\lambda}{|x|^2}\dot z_{\mu_1} = (2^*-1) z_{\mu_1}^{2^*-2}\dot z_{\mu_1}, \tag{5.50}$$

multiplying by $\dot z_{\mu_2}$ we obtain

$$(\dot z_{\mu_1}, \dot z_{\mu_2}) = (2^*-1)\int_{\mathbb{R}^N} z_{\mu_1}^{2^*-2}\dot z_{\mu_1}\dot z_{\mu_2}$$

and hence using (A.11) and (4.5) we get

$$|(\dot z_{\mu_1}, \dot z_{\mu_2})| \le c\,\nu^{-1}\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-\frac{N-2}{2}}. \tag{5.51}$$

From (5.43), (5.44), (A.10), (2.5), (5.47),(5.48), (5.49), and (5.51) we deduce

$$\begin{aligned} c_1(\mu_1,\mu_1,\varepsilon)&\Big[\|\dot z_{\mu_1}\|^2 + O\Big(|\varepsilon|^{\min\left\{1,\frac{4}{N-2}\right\}}\Big) + O\Big(g(\nu)^{\max\left\{\frac12,\frac{N-2}{N+2}\right\}}\Big)\Big]\\ &+ c_2(\mu_1,\mu_1,\varepsilon)\nu^{-1}\Big[O\Big(\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-\frac{N-2}{2}}\Big) + O\Big(|\varepsilon|^{\min\left\{1,\frac{4}{N-2}\right\}}\Big)\Big] = 0 \end{aligned} \tag{5.52}$$

and

$$\begin{aligned} c_2(\mu_1,\mu_1,\varepsilon)&\Big[\|\dot z_{\mu_2}\|^2 + \nu^{-2}O\Big(|\varepsilon|^{\min\left\{1,\frac{4}{N-2}\right\}}\Big) + \nu^{-2}O\Big(g(\nu)^{\max\left\{\frac12,\frac{N-2}{N+2}\right\}}\Big)\Big]\\ &+ c_1(\mu_1,\mu_1,\varepsilon)\nu^{-1}\Big[O\Big(\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-\frac{N-2}{2}}\Big) + O\Big(|\varepsilon|^{\min\left\{1,\frac{4}{N-2}\right\}}\Big)\Big] = 0. \end{aligned} \tag{5.53}$$

From (5.53) we deduce that for $|\varepsilon|$ sufficiently small and ν either sufficiently small or sufficiently large

$$|c_2(\mu_1,\mu_1,\varepsilon)| \le |c_1(\mu_1,\mu_1,\varepsilon)|\,\nu\,\Big[O\Big(\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-\frac{N-2}{2}}\Big) + O\Big(|\varepsilon|^{\min\left\{1,\frac{4}{N-2}\right\}}\Big)\Big] \tag{5.54}$$

which together with (5.52) yield

$$\begin{aligned} |c_1(\mu_1,\mu_1,\varepsilon)|&\Big[\|\dot z_{\mu_1}\|^2 + O\Big(|\varepsilon|^{\min\left\{1,\frac{4}{N-2}\right\}}\Big) + O\Big(g(\nu)^{\max\left\{\frac12,\frac{N-2}{N+2}\right\}}\Big)\Big]\\ &\le |c_1(\mu_1,\mu_1,\varepsilon)|\Big[O\Big(\big[\nu^{\gamma_\lambda}+\nu^{-\gamma_\lambda}\big]^{-(N-2)}\Big) + O\Big(|\varepsilon|^{\min\left\{2,\frac{8}{N-2}\right\}}\Big)\Big]. \end{aligned}$$

Therefore for $\bar\varepsilon$ sufficiently small and L sufficiently large, the number $c_1(\mu_1,\mu_1,\varepsilon)$ must be zero and hence from (5.54) also $c_2(\mu_1,\mu_1,\varepsilon)=0$. Then from (5.42) $Df_\varepsilon^{k_1+k_2}(u)=0$. □

6 Expansion of the constrained functional

In view of Proposition 5.3, when $|\varepsilon|$ is sufficiently small and ν is either large or small enough, to get critical points of the functional $f_\varepsilon^{k_2+k_2}$ it is enough to find critical points of the two variable function Φ_ε defined by

$$\Phi_\varepsilon:\ (a_1,b_1)\times(\nu a_2,\nu b_2)\longrightarrow\mathbb{R},\qquad \Phi_\varepsilon(\mu_1,\mu_2):=f_\varepsilon^{k_2+k_2}\big(z^{k_1+k_2}_{\mu_1,\varepsilon}+z^{k_1+k_2}_{\mu_2,\varepsilon}+w(\mu_1,\mu_2,\varepsilon)\big).$$

Proposition 6.1. *Let $\lambda<(N-2)^2/4$ satisfying* (1.4) *and assume* (1.6) *and* (2.11) *hold. Then there exists a constant $C=C(\lambda,N,a_1,a_2,b_1,b_2,\|K_1\|_{L^\infty},\|K_2\|_{L^\infty})>0$ and some $\bar\varepsilon$ sufficiently small such that if $|\varepsilon|\le\bar\varepsilon$ and*

$$\varepsilon^2 g(\nu)^{-\max\left\{\frac12,\frac{N-2}{N+2}\right\}}\ge C, \tag{6.1}$$

then the functions $\partial_{\mu_i}\Phi_\varepsilon$, $i=1,2$, admit the following expansions

$$\partial_{\mu_1}\Phi_\varepsilon(\mu_1,\mu_2)=-\varepsilon\partial_{\mu_1}\widetilde\Gamma(\mu_1,\mu_2)+o(\varepsilon), \tag{6.2}$$

$$\partial_{\mu_2}\Phi_\varepsilon(\mu_1,\mu_2)=-\varepsilon\partial_{\mu_2}\widetilde\Gamma(\mu_1,\mu_2)+\nu^{-1}o(\varepsilon) \tag{6.3}$$

as $\varepsilon\to0$ uniformly with respect to $(\mu_1,\mu_2)\in(a_1,b_1)\times(\nu a_2,\nu b_2)$, where

$$\widetilde\Gamma(\mu_1,\mu_2):=\Gamma^{k_1}(\mu_1)+\Gamma^{k_2}(\mu_2) \tag{6.4}$$

and Γ^{k_i} is defined in (2.8).

Proof. From (*ii*) of Proposition 5.2, we have that

$$\begin{aligned}
\partial_{\mu_1}\Phi_\varepsilon(\mu_1,\mu_2)&=\big(Df_\varepsilon^{k_2+k_2}\big(z^{k_1+k_2}_{\mu_1,\varepsilon}+z^{k_1+k_2}_{\mu_2,\varepsilon}+w(\mu_1,\mu_2,\varepsilon)\big),\partial_{\mu_1}\big(z^{k_1+k_2}_{\mu_1,\varepsilon}+w(\mu_1,\mu_2,\varepsilon)\big)\big)\\
&=(\alpha^{k_1+k_2}(\mu_1,\varepsilon)\dot\xi_{\mu_1},\dot z_{\mu_1})+\alpha_1(\mu_1,\mu_2,\varepsilon)(\dot\xi_{\mu_1},\dot z_{\mu_1})\\
&\quad+\big(\alpha^{k_1+k_2}(\mu_1,\varepsilon)+\alpha_1(\mu_1,\mu_2,\varepsilon)\big)\big(\dot\xi_{\mu_1},\partial_{\mu_1}\big(w^{k_1+k_2}(\mu_1,\varepsilon)+w(\mu_1,\mu_2,\varepsilon)\big)\big)\\
&\quad+\big(\alpha^{k_1+k_2}(\mu_2,\varepsilon)+\alpha_2(\mu_1,\mu_2,\varepsilon)\big)\big(\dot\xi_{\mu_2},\partial_{\mu_1}\big(w^{k_1+k_2}(\mu_1,\varepsilon)+w(\mu_1,\mu_2,\varepsilon)\big)\big)\\
&\quad+\big(\alpha^{k_1+k_2}(\mu_2,\varepsilon)+\alpha_2(\mu_1,\mu_2,\varepsilon)\big)(\dot\xi_{\mu_2},\dot z_{\mu_1}).
\end{aligned} \tag{6.5}$$

From (*iii*) of Proposition 5.2, (6.1), and (A.10) we deduce that

$$\big|\alpha_1(\mu_1,\mu_2,\varepsilon)(\dot\xi_{\mu_1},\dot z_{\mu_1})\big|\le\|\dot z_{\mu_1}\|g(\nu)^{\max\left\{\frac12,\frac{N-2}{N+2}\right\}}\le\text{const}\,|\varepsilon|^2=o(\varepsilon). \tag{6.6}$$

From (2.4), (*iii* − *iv*) of Proposition 5.2, (6.1), and (2.5) we get

$$\begin{aligned}
&\big|\big(\alpha^{k_1+k_2}(\mu_1,\varepsilon)+\alpha_1(\mu_1,\mu_2,\varepsilon)\big)\big(\dot\xi_{\mu_1},\partial_{\mu_1}\big(w^{k_1+k_2}(\mu_1,\varepsilon)+w(\mu_1,\mu_2,\varepsilon)\big)\big)\big|\\
&\qquad\le\text{const}\left(|\varepsilon|+g(\nu)^{\max\left\{\frac12,\frac{N-2}{N+2}\right\}}\right)\big\|\partial_{\mu_1}w^{k_1+k_2}(\mu_1,\varepsilon)+\partial_{\mu_1}w(\mu_1,\mu_2,\varepsilon)\big\|=o(\varepsilon).
\end{aligned} \tag{6.7}$$

Similarly

$$\big|\big(\alpha^{k_1+k_2}(\mu_2,\varepsilon)+\alpha_2(\mu_1,\mu_2,\varepsilon)\big)\big(\dot\xi_{\mu_2},\partial_{\mu_1}\big(w^{k_1+k_2}(\mu_1,\varepsilon)+w(\mu_1,\mu_2,\varepsilon)\big)\big)\big|=o(\varepsilon). \tag{6.8}$$

Testing equation (5.50) with $\dot{\xi}_{\mu_2}$ we get

$$(\dot{\xi}_{\mu_2}, \dot{z}_{\mu_1}) = (2^*-1)\int_{\mathbb{R}^N} z_{\mu_1}^{2^*-2}\dot{z}_{\mu_1}\dot{\xi}_{\mu_2}$$

hence in view of (A.19), (A.11), and (4.5)

$$\big|(\dot{\xi}_{\mu_2}, \dot{z}_{\mu_1})\big| \le \text{const}\int_{\mathbb{R}^N} z_{\mu_1}^{2^*-1} z_{\mu_2} \le \text{const}\int_{\mathbb{R}^N} z_1^{2^*-1} z_\nu \le \text{const}\, g(\nu)^{\min\left\{\frac{N+2}{4},\frac{N+2}{N-2}\right\}}. \tag{6.9}$$

From (2.4), (iii) of Proposition 5.2, (6.1), and (6.9) we find

$$\big(\alpha^{k_1+k_2}(\mu_2,\varepsilon) + \alpha_2(\mu_1,\mu_2,\varepsilon)\big)(\dot{\xi}_{\mu_2}, \dot{z}_{\mu_1}) = o(\varepsilon). \tag{6.10}$$

Multiplying $-\Delta z_{\mu_1} - \frac{\lambda}{|x|^2} z_{\mu_1} = z_{\mu_1}^{2^*-1}$ by $\dot{z}_{\mu_1}$, we obtain

$$(\dot{z}_{\mu_1}, z_{\mu_1}) - \int_{\mathbb{R}^N} z_{\mu_1}^{2^*-1}\dot{z}_{\mu_1} = 0 \tag{6.11}$$

while testing (5.50) with $w^{k_1+k_2}(\mu_1,\varepsilon)$ and taking into account (2.2) we get

$$(2^*-1)\int_{\mathbb{R}^N} z_{\mu_1}^{2^*-2} w^{k_1+k_2}(\mu_1,\varepsilon)\dot{z}_{\mu_1} = \big(\dot{z}_{\mu_1}, w^{k_1+k_2}(\mu_1,\varepsilon)\big) = 0. \tag{6.12}$$

From (2.3), (2.2), (6.11), and (6.12) we deduce

$$\begin{aligned}
&(\alpha^{k_1+k_2}(\mu_1,\varepsilon)\dot{\xi}_{\mu_1}, \dot{z}_{\mu_1}) = \big(Df_\varepsilon^{k_1+k_2}\big(z_{\mu_1} + w^{k_1+k_2}(\mu_1,\varepsilon)\big), \dot{z}_{\mu_1}\big)\\
&= -\varepsilon\int_{\mathbb{R}^N}(k_1+k_2) z_{\mu_1}^{2^*-1}\dot{z}_{\mu_1} - \varepsilon\int_{\mathbb{R}^N}(k_1+k_2)\big((z_{\mu_1} + w^{k_1+k_2}(\mu_1,\varepsilon))_+^{2^*-1} - z_{\mu_1}^{2^*-1}\big)\dot{z}_{\mu_1}\\
&\quad - \int_{\mathbb{R}^N}\big((z_{\mu_1} + w^{k_1+k_2}(\mu_1,\varepsilon))_+^{2^*-1} - z_{\mu_1}^{2^*-1} - (2^*-1) z_{\mu_1}^{2^*-2} w^{k_1+k_2}(\mu_1,\varepsilon)\big)\dot{z}_{\mu_1}\\
&\quad + \left[(\dot{z}_{\mu_1}, z_{\mu_1}) - \int_{\mathbb{R}^N} z_{\mu_1}^{2^*-1}\dot{z}_{\mu_1}\right] - (2^*-1)\int_{\mathbb{R}^N} z_{\mu_1}^{2^*-2} w^{k_1+k_2}(\mu_1,\varepsilon)\dot{z}_{\mu_1}\\
&= -\varepsilon\int_{\mathbb{R}^N}(k_1+k_2) z_{\mu_1}^{2^*-1}\dot{z}_{\mu_1} - \varepsilon\int_{\mathbb{R}^N}(k_1+k_2)\big((z_{\mu_1} + w^{k_1+k_2}(\mu_1,\varepsilon))_+^{2^*-1} - z_{\mu_1}^{2^*-1}\big)\dot{z}_{\mu_1}\\
&\quad - \int_{\mathbb{R}^N}\big((z_{\mu_1} + w^{k_1+k_2}(\mu_1,\varepsilon))_+^{2^*-1} - z_{\mu_1}^{2^*-1} - (2^*-1) z_{\mu_1}^{2^*-2} w^{k_1+k_2}(\mu_1,\varepsilon)\big)\dot{z}_{\mu_1}.
\end{aligned} \tag{6.13}$$

Using $(A.3)_s$ with $s = 2^*-1$, (A.11), (2.4) and Hölder and Sobolev inequalities we have

$$\begin{aligned}
&\left|\varepsilon\int_{\mathbb{R}^N}(k_1+k_2)\big((z_{\mu_1} + w^{k_1+k_2}(\mu_1,\varepsilon))_+^{2^*-1} - z_{\mu_1}^{2^*-1}\big)\dot{z}_{\mu_1}\right|\\
&\quad \le \text{const}\,|\varepsilon|\int_{\mathbb{R}^N}\big(|w^{k_1+k_2}(\mu_1,\varepsilon)| z_{\mu_1}^{2^*-2} + |w^{k_1+k_2}(\mu_1,\varepsilon)|^{2^*-1}\big) z_{\mu_1}\\
&\quad \le \text{const}\,|\varepsilon|\big(\|w^{k_1+k_2}(\mu_1,\varepsilon)\| + \|w^{k_1+k_2}(\mu_1,\varepsilon)\|^{2^*-1}\big) \le \text{const}\,|\varepsilon|^2 = o(\varepsilon).
\end{aligned} \tag{6.14}$$

Estimate $(A.1)_s$ with $s = 2^* - 1$, (A.11), Hölder and Sobolev inequalities, and (2.4) yield

$$\left| \int_{\mathbb{R}^N} \big((z_{\mu_1} + w^{k_1+k_2}(\mu_1,\varepsilon))_+^{2^*-1} - z_{\mu_1}^{2^*-1} - (2^*-1) z_{\mu_1}^{2^*-2} w^{k_1+k_2}(\mu_1,\varepsilon) \big) \dot z_{\mu_1} \right|$$
$$\leq \text{const} \int_{\mathbb{R}^N} |w^{k_1+k_2}(\mu_1,\varepsilon)|^{2^*-1} z_{\mu_1} \leq \text{const}\, \|w^{k_1+k_2}(\mu_1,\varepsilon)\|^{2^*-1} \leq \text{const}\, |\varepsilon|^{2^*-1} = o(\varepsilon) \quad (6.15)$$

if $N \geq 6$, and

$$\left| \int_{\mathbb{R}^N} \big((z_{\mu_1} w^{k_1+k_2}(\mu_1,\varepsilon))_+^{2^*-1} - z_{\mu_1}^{2^*-1} - (2^*-1) z_{\mu_1}^{2^*-2} w^{k_1+k_2}(\mu_1,\varepsilon) \big) \dot z_{\mu_1} \right|$$
$$\leq \text{const} \int_{\mathbb{R}^N} \big(|w^{k_1+k_2}(\mu_1,\varepsilon)|^{2^*-1} + |w^{k_1+k_2}(\mu_1,\varepsilon)|^2 z_{\mu_1}^{2^*-3} \big) z_{\mu_1}$$
$$\leq \text{const} \big(\|w^{k_1+k_2}(\mu_1,\varepsilon)\|^{2^*-1} + \|w^{k_1+k_2}(\mu_1,\varepsilon)\|^2 \big) \leq \text{const}\, |\varepsilon|^2 = o(\varepsilon) \quad (6.16)$$

if $N < 6$. On the other hand, (A.11), (2.13), and (6.1) imply that

$$\left| \varepsilon \int_{\mathbb{R}^N} k_2 z_{\mu_1}^{2^*-1} \dot z_{\mu_1} \right| \leq \text{const}\, |\varepsilon| \int_{\mathbb{R}^N} |k_2| z_{\mu_1}^{2^*} \leq \text{const}\, |\varepsilon| g(\nu)^{\frac{2N}{N+2}} = o(\varepsilon). \quad (6.17)$$

Collecting (6.13)–(6.17), we have

$$(\alpha^{k_1+k_2}(\mu_1,\varepsilon) \dot\xi_{\mu_1}, \dot z_{\mu_1}) = -\varepsilon \int_{\mathbb{R}^N} k_1 z_{\mu_1}^{2^*-1} \dot z_{\mu_1} + o(\varepsilon). \quad (6.18)$$

From (6.5), (6.6), (6.7), (6.8), (6.10), (6.18), we finally get

$$\partial_{\mu_1} \Phi_\varepsilon(\mu_1,\mu_2) = -\varepsilon \int_{\mathbb{R}^N} k_1 z_{\mu_1}^{2^*-1} \dot z_{\mu_1} + o(\varepsilon)$$

uniformly with respect to $(\mu_1,\mu_2) \in (a_1,b_1) \times (\nu a_2, \nu b_2)$, namely

$$\partial_{\mu_1} \Phi_\varepsilon(\mu_1,\mu_2) = -\varepsilon (\Gamma^{k_1})'(\mu_1) + o(\varepsilon) = -\varepsilon \partial_{\mu_1} \widetilde\Gamma(\mu_1,\mu_2) + o(\varepsilon).$$

Expansion (6.2) is thereby proved. The proof of (6.3) is analogous. □

7 Study of $\widetilde\Gamma$ and Proof of Theorem 2.6

Stability properties of the topological degree allow to reduce the computation of the topological degree of the jacobian map of Φ_ε to the computation of the topological degree of the jacobian map of $\widetilde\Gamma$, as the following lemma states.

Lemma 7.1. *Under the same assumptions of Proposition 6.1, there exists $\bar\varepsilon$ such that for all $|\varepsilon| \leq \bar\varepsilon$*

$$\deg\big(-\varepsilon^{-1} \mathrm{Jac}\, \Phi_\varepsilon, Q_\nu, 0 \big) = \deg\big(\mathrm{Jac}\, \widetilde\Gamma, Q_\nu, 0 \big), \quad (7.1)$$

where $Q_\nu := (a_1,b_1) \times (\nu\, a_2, \nu\, b_2)$.

Proof. From (6.2) and (6.3) we have that

$$\deg\big(-\varepsilon^{-1}\mathrm{Jac}\,\Phi_\varepsilon, Q_\nu, 0\big) = \deg\Big(\mathrm{Jac}\,\widetilde{\Gamma} + \mathcal{E}_{\nu,\varepsilon}, Q_\nu, 0\Big)$$

where $\mathcal{E}_{\nu,\varepsilon} = o(1)\binom{1}{1/\nu}$. We can choose $\delta > 0$ such that $\big((-\delta,\delta)\times\nu^{-1}(-\delta,\delta)\big)\cap \mathrm{Jac}\,\widetilde{\Gamma}(\partial Q_\nu) = \emptyset$. Let $\bar\varepsilon$ be such that for all $|\varepsilon| \le \bar\varepsilon$ we have $\mathcal{E}_{\nu,\varepsilon} \in (-\delta,\delta)\times\nu^{-1}(-\delta,\delta)$. From well-known properties of the topological degree it follows that

$$\deg\Big(\mathrm{Jac}\,\widetilde{\Gamma} + \mathcal{E}_{\nu,\varepsilon}, Q_\nu, 0\Big) = \deg\Big(\mathrm{Jac}\,\widetilde{\Gamma}, Q_\nu, 0\Big).$$

The lemma is thereby proved. □

In order to prove that $\deg\Big(\mathrm{Jac}\,\widetilde{\Gamma}, Q_\nu, 0\Big) \neq 0$ we use the theorem below, which is due to Miranda, see [23].

Theorem 7.2. **[Miranda's Theorem]** *Let* $Q = [\alpha_1^-, \alpha_1^+] \times [\alpha_2^-, \alpha_2^+] \times \cdots \times [\alpha_k^-, \alpha_k^+]$ *and let* $f = (f_1, f_2, \dots, f_k) : Q \to \mathbb{R}^k$ *be a continuous function. Set* $\gamma_i^\pm = \partial Q \cap \{x = (x_1, \dots, x_k) : \; x_i = \alpha_i^\pm\}$. *Assume that* $f_i\big|_{\gamma_i^\pm}$ *never vanish, have a fixed sign, and*

$$\mathrm{sign}\,\big(f_i\big|_{\gamma_i^+}\big)\cdot \mathrm{sign}\,\big(f_i\big|_{\gamma_i^-}\big) < 0. \tag{7.2}$$

Then there exists $\bar x \in Q$ *such that* $f_i(\bar x) = 0$ *for all* $i = 1, 2, \dots, k$. *Moreover if we set*

$$\sigma(i) = \begin{cases} +1 & \text{if } f_i\big|_{\gamma_i^-} < 0 < f_i\big|_{\gamma_i^+}, \\ -1 & \text{if } f_i\big|_{\gamma_i^+} < 0 < f_i\big|_{\gamma_i^-}, \end{cases}$$

then

$$\deg(f, Q, 0) = \prod_{i=1}^{k} \sigma(i).$$

Proposition 7.3. *Let* $\lambda < (N-2)^2/4$ *satisfying* (1.4) *and assume* (1.6) *and* (2.11) *hold. Suppose that* $K_1 \in L^\infty(\mathbb{R}^N)\cap C^1(\mathbb{R}^N)$ *satisfies either* $(1.8)_i$ *or* $(1.9)_i$, *and* $K_2 \in L^\infty(\mathbb{R}^N)\cap C^1(\mathbb{R}^N)$ *satisfies either* $(1.8)_i$ *or* $(1.9)_i$. *Then*

$$\deg\Big(\mathrm{Jac}\,\widetilde{\Gamma}, Q_\nu, 0\Big) \neq 0.$$

Proof. We use Miranda's Theorem with $k = 2$, $\alpha_1^- = a_1$, $\alpha_1^+ = b_1$, $\alpha_2^- = \nu a_2$, $\alpha_2^+ = \nu b_2$, and $f = \mathrm{Jac}\,\widetilde{\Gamma}$. Corollary 2.5 ensures that (7.2) is satisfied. The conclusion follows from Theorem 7.2. □

Proof of Theorem 2.6. From Lemma 7.1 and Proposition 7.3, it follows that

$$\deg\big(\mathrm{Jac}\,\Phi_\varepsilon, Q_\nu, 0\big) \neq 0,$$

provided $|\varepsilon| \le \bar\varepsilon$ and (6.1) holds. From the solution property of the topological degree it follows that Φ_ε has a critical point $(\mu_1, \mu_2) \in Q_\nu$. From Proposition 5.3 we deduce that $z_{\mu_1,\varepsilon}^{k_1+k_2} + z_{\mu_2,\varepsilon}^{k_1+k_2} + w(\mu_1, \mu_2, \varepsilon)$ is a critical point of $f_\varepsilon^{k_1+k_2}$, and hence a nonnegative solution to equation $(\mathcal{P}_{\lambda,K}^\varepsilon)$. Positivity of solution outside 0 follows from the Maximum Principle □

Appendix

In this appendix we collect some technical lemmas. The first result provides some elementary inequalities, the proof of which is omitted since it is quite standard.

Lemma A.1. *The following inequalities*

$$\Big|(a+b)_+^s - a_+^s - s a_+^{s-1} b\Big| \leq \begin{cases} C(a_+^{s-2}|b|^2 + |b|^s) & \text{if } s \geq 2 \\ C\ |b|^s & \text{if } 1 < s < 2, \end{cases} \tag{$A.1$}_s$$

$$\Big|(a+b)_+^s - a_+^s\Big| \leq C|b|^s \quad \text{if } 0 < s \leq 1 \tag{$A.2$}_s$$

$$\Big|(a+b)_+^s - a_+^s\Big| \leq C(|a|^{s-1}|b| + |b|^s) \quad \text{if } s \geq 1 \tag{$A.3$}_s$$

hold for some $C = C(s) > 0$ and for any $a, b \in \mathbb{R}$. Moreover

$$\Big|(a+b)_+^s - a_+^s - b_+^s\Big| \leq \begin{cases} C(|a|^{s-1}|b| + |a||b|^{s-1}) & \text{if } s \geq 1 \\ C\ |a|^r|b|^q & \text{if } s \leq 1 \end{cases} \tag{$A.4$}_s$$

for any $a, b \in \mathbb{R}$, $r > 0$, $q > 0$, $r + q = s$ and for some $C = C(s, r, q) > 0$.

Corollary A.2. *If $s \geq 1$ there exists a constant $C = C(s) > 0$ such that for any $a, b \in \mathbb{R}$ there holds*

$$\Big|(a+b+w)_+^s - a_+^s - b_+^s\Big| \leq C(|w|^s + |a+b|^{s-1}|w| + |a|^{s-1}|b| + |a||b|^{s-1}). \tag{$A.5$}_s$$

If $s \leq 1$ and $r > 0$, $q > 0$, $r + q = s$ there exists a constant $C = C(s, r, q) > 0$ such that for any $a, b \in \mathbb{R}$ there holds

$$\Big|(a+b+w)_+^s - a_+^s - b_+^s\Big| \leq C(|w|^s + |a|^r|b|^q). \tag{$A.6$}_s$$

Proof. Since

$$\big|(a+b+w)_+^s - a_+^s - b_+^s\big| \leq \big|(a+b+w)_+^s - (a+b)_+^s\big| + \big|(a+b)_+^s - a_+^s - b_+^s\big|,$$

$(A.5)_s$ follows from $(A.3)_s$ and $(A.4)_s$, while $(A.6)_s$ comes from $(A.2)_s$ and $(A.4)_s$. □

Lemma A.3. *For any $k \in L^\infty(\mathbb{R}^N)$ there exist $C = C(N, \|k\|_{L^\infty(\mathbb{R}^N)}) > 0$ such that for any $|\varepsilon| \leq 1$ and $u, w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$*

$$\|Df_\varepsilon^k(u+w) - Df_\varepsilon^k(u)\| \leq C\|w\|\Big(\|u\|^{\frac{4}{N-2}} + \|w\|^{\frac{4}{N-2}}\Big) \tag{A.7}$$

$$\|D^2 f_\varepsilon^k(u+w) - D^2 f_\varepsilon^k(u)\| \leq \begin{cases} C\|w\|\Big(\|u\|^{\frac{6-N}{N-2}} + \|w\|^{\frac{6-N}{N-2}}\Big) & \text{if } 3 \leq N < 6 \\ C\|w\|^{\frac{4}{N-2}} & \text{if } N \geq 6, \end{cases} \tag{A.8}$$

and

$$\|D^2 f_\varepsilon^k(u) - D^2 f_0(u)\| \leq C|\varepsilon|\|u\|^{\frac{4}{N-2}}. \tag{A.9}$$

Proof. From Hölder and Sobolev inequality we have that for any $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\begin{aligned}
\Big|\Big(Df_\varepsilon^k(u+w) - Df_\varepsilon^k(u), v\Big)\Big| &= \Big|(w,v) - \int_{\mathbb{R}^N}(1+\varepsilon k(x))\big((u+w)_+^{2^*-1} - u_+^{2^*-1}\big)v\,dx\Big| \\
&\le \|w\|\|v\| + (1+\|k\|_{L^\infty(\mathbb{R}^N)})\|v\|_{L^{2^*}(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N}\big|(u+w)_+^{2^*-1} - u_+^{2^*-1}\big|^{\frac{2N}{N+2}}\Big)^{\frac{N+2}{2N}} \\
&\le C\|v\|\Big[\|w\| + \Big(\int_{\mathbb{R}^N}\big|(u+w)_+^{2^*-1} - u_+^{2^*-1}\big|^{\frac{2N}{N+2}}\Big)^{\frac{N+2}{2N}}\Big].
\end{aligned}$$

Estimate (A.7) follows from $(A.3)_s$ with $s = 2^* - 1$. Using again Hölder and Sobolev inequality, we have that for any v_1 and $v_2 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$\begin{aligned}
&\Big|\Big(\Big[D^2 f_\varepsilon^k(u+w) - D^2 f_\varepsilon^k(u)\Big]v_1, v_2\Big)\Big| \\
&= (2^*-1)\Big|\int_{\mathbb{R}^N}(1+\varepsilon k(x))\big((u+w)_+^{2^*-2} - u_+^{2^*-2}\big)v_1 v_2\,dx\Big| \\
&\le (2^*-1)(1+\|k\|_{L^\infty(\mathbb{R}^N)})\|v_1\|_{L^{2^*}(\mathbb{R}^N)}\|v_2\|_{L^{2^*}(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N}\big|(u+w)_+^{2^*-2} - u_+^{2^*-2}\big|^{\frac{N}{2}}\Big)^{\frac{2}{N}} \\
&\le C\|v_1\|\|v_2\|\Big(\int_{\mathbb{R}^N}\big|(u+w)_+^{2^*-2} - u_+^{2^*-2}\big|^{\frac{N}{2}}\Big)^{\frac{2}{N}}.
\end{aligned}$$

Using $(A.3)_s$ if $N < 6$ and $(A.2)_s$ if $N \ge 6$ with $s = 2^* - 2$ we get estimate (A.8). Estimate (A.9) follows easily from Hölder inequality. □

Lemma A.4. *There holds*

$$\|\dot z_\mu\| = \frac{1}{\mu}\|\dot z_1\| \quad \textit{and} \quad \|\ddot z_\mu\| = \frac{1}{\mu^2}\|\ddot z_1\|. \tag{A.10}$$

Moreover there exists a positive constant C depending only on λ and N such that

$$|\dot z_\mu| \le c\,\mu^{-1} z_\mu, \tag{A.11}$$
$$|\ddot z_\mu| \le c\,\mu^{-2} z_\mu. \tag{A.12}$$

Proof. Let $U_\mu : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathcal{D}^{1,2}(\mathbb{R}^N)$ de defined in (5.8). Differentiating the identity $z_\sigma = U_\mu z_{\sigma/\mu}$ with respect to σ we obtain

$$\dot z_\mu = \frac{1}{\mu}U_\mu \dot z_1 \quad \text{and} \quad \ddot z_\mu = \frac{1}{\mu^2}U_\mu \ddot z_1. \tag{A.13}$$

Since U_μ conserves the norm, we obtain

$$\|\dot z_\mu\| = \frac{1}{\mu}\|\dot z_1\| \quad \text{and} \quad \|\ddot z_\mu\| = \frac{1}{\mu^2}\|\ddot z_1\|$$

thus proving (A.10). An explicit calculation shows that

$$\dot z_1(x) = A(N,\lambda)\frac{N-2}{2}\Big(1-\frac{2a_\lambda}{N-2}\Big)\frac{|x|^{\frac{2a_\lambda}{N-2}}-|x|^{\frac{2(N-2-a_\lambda)}{N-2}}}{\Big(|x|^{\frac{2a_\lambda}{N-2}}+|x|^{\frac{2(N-2-a_\lambda)}{N-2}}\Big)^{\frac{N}{2}}} \tag{A.14}$$

hence for some positive constant c depending only on N and λ

$$|\dot z_1(x)| \le c\, z_1(x). \tag{A.15}$$

From (A.13) and (A.15), it follows that

$$|\dot z_\mu(x)| = \mu^{-1}\mu^{-\frac{N-2}{2}}|\dot z_1(x/\mu)| \le c\mu^{-1}\mu^{-\frac{N-2}{2}} z_1(x/\mu) = c\mu^{-1} z_\mu(x)$$

thus proving (A.11). The proof of (A.12) is similar. □

Lemma A.5. *Let Ω be an open bounded subset of $\mathbb{R}^N$. Then there exists a positive constant $C = C(\lambda, N)$ such that for any $\mu > 0$ and $s \ge 1$*

$$\int_\Omega \big|(z_\mu^\star)^{2^*-2}(\dot\xi_\mu)^\star\big|^s |x|^{-a_\lambda(2^*-2^*s+s)}\,dx \le C\mu^{\frac{N+2}{2}s\left(1-\frac{2a_\lambda}{N-2}\right)} \tag{A.16}$$

and

$$\int_\Omega \big|z_\mu^{2^*-2}\dot\xi_\mu\big|^s |x|^{-a_\lambda(2^*-2^*s+s)}\,dx \le C\mu^{-\frac{N+2}{2}s\left(1-\frac{2a_\lambda}{N-2}\right)} \tag{A.17}$$

where z_μ, respectively $\dot\xi_\mu$, are defined in (1.1), *respectively* (2.6), *and $\star$ denotes the Kelvin transform defined in* (3.4).

Proof. A direct calculation shows that

$$\dot\xi_\mu(x) = \frac{\mu^{-\frac{N-2}{2}}\dot z_1(x/\mu)}{\|\dot z_1\|} \tag{A.18}$$

where the explicit expression of $\dot z_1$ is given in (A.14). Hence from (A.15)

$$|\dot\xi_\mu(x)| \le c(\lambda, N) z_\mu(x) \tag{A.19}$$

and

$$\begin{aligned}|(\dot\xi_\mu)^\star(x)| &= \left|\frac{\mu^{-\frac{N-2}{2}}|x|^{-(N-2)}}{\|\dot z_1\|}\dot z_1\Big(\frac{x}{\mu|x|^2}\Big)\right| \\ &\le c(\lambda,N)\mu^{-\frac{N-2}{2}}|x|^{-(N-2)} z_1\Big(\frac{x}{\mu|x|^2}\Big) = c(\lambda,N) z_\mu^\star(x).\end{aligned} \tag{A.20}$$

A direct calculation shows that

$$z_\mu^\star(x) = A(N,\lambda)\mu^{\frac{N-2}{2}-a_\lambda}|x|^{-a_\lambda}\Big(1+\mu^{2-\frac{4a_\lambda}{N-2}}|x|^{2-\frac{4a_\lambda}{N-2}}\Big)^{-\frac{N-2}{2}}. \tag{A.21}$$

From (A.20) and (A.21) it follows that for some positive constant c depending only on N and λ

$$\big|(z^\star_\mu(x))^{2^*-2}(\dot\xi_\mu)^\star(x)\big| \le c\mu^{\frac{N+2}{2}-a_\lambda\frac{N+2}{N-2}}|x|^{-a_\lambda\frac{N+2}{N-2}}$$

hence for some positive constant c depending only on N and λ

$$\int_\Omega \big|(z^\star_\mu)^{2^*-2}(\dot\xi_\mu)^\star\big|^s|x|^{-a_\lambda(2^*-2^*s+s)}\,dx \le c\mu^{\frac{N+2}{2}s\left(1-\frac{2a_\lambda}{N-2}\right)}\int_\Omega |x|^{-2^*a_\lambda}\,dx = C(\lambda,N)\mu^{\frac{N+2}{2}s\left(1-\frac{2a_\lambda}{N-2}\right)}.$$

Estimate (A.16) is thereby proved. From (A.19), (1.1), and (1.2) we get

$$\int_\Omega \big|z_\mu^{2^*-2}\dot\xi_\mu\big|^s|x|^{-a_\lambda(2^*-2^*s+s)}\,dx \le c\mu^{-\frac{N+2}{2}s\left(1-\frac{2a_\lambda}{N-2}\right)}\int_\Omega |x|^{-2^*a_\lambda}\,dx = C(\lambda,N)\mu^{-\frac{N+2}{2}s\left(1-\frac{2a_\lambda}{N-2}\right)}$$

thus proving (A.17). □

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