VANISHING VISCOSITY FOR A 2×2 SYSTEM MODELING CONGESTED VEHICULAR TRAFFIC

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Abstract. We prove the convergence of the vanishing viscosity approximation for a class of 2×2 systems of conservation laws, which includes a model of traffic flow in congested regimes. The structure of the system allows us to avoid the typical constraints on the total variation and the L^1 norm of the initial data. The key tool is the compensated compactness technique, introduced by Murat and Tartar, used here in the framework developed by Panov. The structure of the Riemann invariants is used to obtain the compactness estimates.

1. Introduction

1.1. **Modeling traffic flow in the congested regime.** We consider the Cauchy problem associated to the following 2×2 system of conservation laws in one space dimension:

$$\begin{cases} \partial_t \rho + \partial_x (u \rho f(\rho)) = 0, & t > 0,\ x \in \mathbb{R}, \\ \partial_t u + \partial_x (u^2 f(\rho)) = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

The functions $\rho : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ represent respectively the vehicular density and the generalized momentum. The velocity law is given by $uf(\rho)$, where the function $f = f(\rho)$ describes the reaction of drivers to the different crowding level of the road.

System (1.1) describes the evolution of congested traffic in the second-order macroscopic traffic model, introduced in [13] as an extension of the classical first-order Lighthill-Whitham-Richards (LWR) model (see [31, 52]) for allowing different drivers to have different maximal speeds. According to the empirical evidence that vehicular traffic behaves differently in the situations of low and high densities, see [26], the model in [13] consists in two different regimes or phases: a free phase, described by a single transport equation, and a congested one, modeled by the 2×2 system (1.1).

We remark that the well-known second-order Aw-Rascle-Zhang (ARZ) model in its original form [1, Formula (2.10)], i.e.

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0, & t > 0,\ x \in \mathbb{R}, \\ \partial_t (\rho(v + p(\rho))) + \partial_x (\rho v (v + p(\rho))) = 0, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

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is obtained from (1.1) by formally setting $v = uf(\rho)$ and $p = \frac{u}{\rho} - uf(\rho)$.

The original ARZ model does not distinguish between a free and a congested phase, but it was extended in this direction in [20], where Goatin generalized the two-phase model proposed by Colombo in [12], coupling the LWR equation in the free phase with the ARZ model in the congested phase. A peculiar difference between the aformentioned models and the one formulated in [13], is that the two phases are here connected. For other second order macroscopic or two-phase models describing traffic evolution and for differences between models see [4, 17, 19, 21, 30, 58] and the references therein.

In the present paper, we do not consider phase transitions; we focus on the evolution of traffic in the congested regime given by system (1.1). Indeed, the more complex and richer dynamics happens in the congested phase; on the other hand, in the free phase the model reduces to a linearly degenerate 2×2 system, where each driver's speed is constantly equal to the maximal one. Our main contribution is a proof that the solutions of the viscous approximations of (1.1) converge to a weak solution of the hyperbolic system.

1.2. **Vanshing viscosity for systems of conservation laws.** The vanishing viscosity limit for the uniformly parabolic viscous regularizations of scalar conservation laws is a crucial point in Kružkov's well-posedness theory (see [29]; cf. [23, **?**] for a modern exposition). The developments concerning the vanishing viscosity approximation of systems of conservation laws are more recent. DiPerna proved convergence for certain classes of 2×2 genuinely nonlinear systems in [15, 28, 9]. His results were subsequently extended in many directions to more general systems describing gas dynamics or other physical phenomena (e.g. shallow waters, liquid chromatography, etc.) – see, e.g. [34, 25, 10, 27, 35, 44, 36, 24, 42, 43, 41, 54, 48, 40, 47, 59, 39, 22, 46, 45, 38, 37] and references therein. The proofs rely on a compensated compactness argument: the key idea, introduced by Tartar and Murat (see, e.g., [16, Chapter 5] for a survey), is as follows: the invariant region method provides uniform L^∞ bounds on the sequence of viscous approximation, but the weak-star convergence does not allow to pass to limit in the nonlinear terms of the equations; however, the weak limit can be represented in terms of Young measures, which reduce to a Dirac mass (hence giving strong convergence) due to the mechanism of entropy dissipation. In [53], Serre proved the global existence of weak solutions for a 2×2 Temple class systems, that is for systems with either linearly degenerate characteristic fields, or with straight characteristic curves (see also [57]). Coclite, Karlsen, Mishra, Risebro applied an improved compensated compactness result due to Panov (see [51, 50]) to prove convergence for 2×2 triangular systems in [11]. For strictly hyperbolic $n \times n$ systems with small initial total variation, in [3], Bianchini and Bressan managed to develop a theory of vanishing viscosity based a priori BV bounds on solutions. We remark that the general uniqueness results known for systems of conservation laws apply only to BV solutions (see [6, 32, 33, 5, 7, 8]); therefore, the uniqueness of the L^∞ solutions obtained by the compensated compactness method remains a long-standing open problem.

None of the previously known results can be directly applied to our problem: indeed, we do not assume any smallness condition on the initial data and system (1.1) is neither of Temple class nor genuinely nonlinear nor triangular.

1.3. **Outline of the paper.** The paper is organized as follows. In Section 2, we introduce the approximate viscous system and we state the main result together with the assumptions on the function f and on the initial data. Section 3 is dedicated to several a priori estimates for the solutions of the viscous system and to the compactness of the family of Riemann invariants, which is a preliminary step in the proof of the main result. Finally, in Section 4, we prove the existence of a solution to (1.1) by the vanishing viscosity approach. Here the main tool is the version of the compensated compactness proposed by Panov in [50, 51].

2. Main result

Before stating the main result of the paper, Theorem 2.1, we introduce the viscous approximation of (1.1) and all the required assumptions.

We consider a flux function f that satisfies the following hypothesis:

(F): $f \in C^2((0,1];\mathbb{R}^+) \cap L^1((0,1);\mathbb{R}^+)$ satisfies $f(1) = 0$ and

$$\mathcal{L}^1\left(\left\{\rho \in (0,1) : \partial^2_{\rho\rho}(\rho^2 f(\rho)) = 0\right\}\right) = 0,$$

where $\mathcal{L}^1$ denotes the Lebesgue measure in $\mathbb{R}$.

Assumption **(F)** guarantees that the function $g : (0,1] \to \mathbb{R}^+$, defined by

$$g(\rho) = \rho^2 f(\rho) \tag{2.1}$$

for every $\rho \in (0,1]$, is genuinely nonlinear.

Example 2.1. *The affine function $f(\rho) = 1 - \rho$ satisfies assumption* **(F)**. *Indeed $g''(\rho) = 2 - 6\rho$ is equal to 0 if and only if $\rho = \frac{1}{3}$.*

Example 2.2. *Choose $\delta \in (0,1)$ and define*

$$f(\rho) = \begin{cases} \frac{1}{\delta} - 1, & 0 < \rho \le \delta, \\ \frac{1}{\rho} - 1, & \delta \le \rho \le 1. \end{cases}$$

The function f satisfies **(F)**. *This is a typical choice in traffic flow modeling.*

On the initial data ρ_0 and u_0, we assume that there exist two constants $0 < \check{w} < \hat{w} < \infty$, such that

$$0 \le \rho_0 \le 1, \quad \check{w}\rho_0 \le u_0 \le \hat{w}\rho_0, \tag{2.2}$$

$$\ln(\rho_0) \in L^1(\mathbb{R}), \quad \frac{u_0}{\rho_0} \in BV(\mathbb{R}). \tag{2.3}$$

Remark 2.1. *Assumptions (2.2) and (2.3) on the function ρ_0 imply also that the function $\rho_0 - 1$ belongs to $L^1(\mathbb{R})$.*

We use the following definition of weak solution of problem (1.1).

Definition 2.1 (Weak solutions)**.** *Given $\rho_0 \in L^\infty(\mathbb{R};\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R};\mathbb{R})$, we say that the couple (ρ, u) is a weak solution to (1.1) if the following statements hold:*

(1) $\rho \in L^\infty((0,+\infty) \times \mathbb{R};\mathbb{R})$;

(2) $u \in L^\infty((0,+\infty) \times \mathbb{R};\mathbb{R})$;

(3) for every $\varphi \in C^\infty_c([0,+\infty) \times \mathbb{R};\mathbb{R})$,

$$\int_0^{+\infty}\int_{\mathbb{R}} [\rho(t,x)\partial_t\varphi(t,x) + u(t,x)\rho(t,x)f(\rho(t,x))\,\partial_x\varphi(t,x)]\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}} \rho_0(x)\varphi(0,x)\,\mathrm{d}x;$$

(4) for every $\varphi \in C_c^\infty([0,+\infty)\times\mathbb{R};\mathbb{R})$*,*

$$\int_0^{+\infty}\int_{\mathbb{R}}\left[u(t,x)\partial_t\varphi(t,x)+u^2(t,x)f\left(\rho(t,x)\right)\partial_x\varphi(t,x)\right]\mathrm{d}x\,\mathrm{d}t=\int_{\mathbb{R}}u_0(x)\varphi(0,x)\,\mathrm{d}x.$$

Let us consider the following viscous approximation of (1.1):

$$\begin{cases}\partial_t\rho_\varepsilon+\partial_x(u_\varepsilon\rho_\varepsilon f(\rho_\varepsilon))=\varepsilon\partial_{xx}^2\rho_\varepsilon, & t>0,\ x\in\mathbb{R},\\ \partial_t u_\varepsilon+\partial_x(u_\varepsilon^2 f(\rho_\varepsilon))=\varepsilon\partial_{xx}^2u_\varepsilon, & t>0,\ x\in\mathbb{R},\\ \rho_\varepsilon(0,x)=\rho_{0,\varepsilon}(x), & x\in\mathbb{R},\\ u_\varepsilon(0,x)=u_{0,\varepsilon}(x), & x\in\mathbb{R},\end{cases}\tag{2.4}$$

where $\varepsilon>0$ and the initial data $\rho_{0,\varepsilon}$ and $u_{0,\varepsilon}$ are smooth approximations of ρ_0 and u_0. More precisely we assume:

$$\rho_{0,\varepsilon},\ u_{0,\varepsilon}\in C^\infty(\mathbb{R};\mathbb{R})\text{ for every }\varepsilon>0,\tag{2.5}$$

$$\rho_{0,\varepsilon}\to\rho_0,\ u_{0,\varepsilon}\to u_0\text{ in }L^p_{loc}(\mathbb{R}),\ 1\le p<\infty,\text{ and a.e. as }\varepsilon\to 0,\tag{2.6}$$

$$\|\rho_{0,\varepsilon}-1\|_{L^1(\mathbb{R})}\le\|\rho_0-1\|_{L^1(\mathbb{R})}\text{ for every }\varepsilon>0,\tag{2.7}$$

$$\|u_{0,\varepsilon}\|_{L^2(\mathbb{R})}\le\|u_0\|_{L^2(\mathbb{R})},\tag{2.8}$$

$$\varepsilon\le\rho_{0,\varepsilon}\le 1,\ \check{w}\rho_{0,\varepsilon}\le u_{0,\varepsilon}\le\hat{w}\rho_{0,\varepsilon}\text{ for every }\varepsilon>0,\tag{2.9}$$

$$\|\ln(\rho_{0,\varepsilon})\|_{L^1(\mathbb{R})}\le\|\ln(\rho_0)\|_{L^1(\mathbb{R})},\ \left\|\left(\frac{u_{0,\varepsilon}}{\rho_{0,\varepsilon}}\right)'\right\|_{L^1(\mathbb{R})}\le TV\left(\frac{u_0}{\rho_0}\right)\text{ for all }\varepsilon>0.\tag{2.10}$$

The well-posedness of classical solutions to (2.4) is guaranteed for short time by the Cauchy-Kowaleskaya theorem (see [56]) and for large times by the classical parabolic theory (see [18]). Moreover, at least for short time we can assume $\rho_\varepsilon\ge\varepsilon/2$. A key ingredient for the proof is the analysis of the Riemann invariant

$$w_\varepsilon=\frac{u_\varepsilon}{\rho_\varepsilon}\tag{2.11}$$

(see [14, Section 7.3] for a definition of Riemann invariant). From (2.4), we easily deduce that w_ε satisfies the equation

$$\partial_t w_\varepsilon+\rho_\varepsilon f(\rho_\varepsilon)w_\varepsilon\partial_x w_\varepsilon=\varepsilon\partial_{xx}^2w_\varepsilon+2\varepsilon\frac{\partial_x\rho_\varepsilon\partial_x w_\varepsilon}{\rho_\varepsilon}.\tag{2.12}$$

By a L^2_{loc} estimate, Lemma 3.5, we then deduce that w_ε is well-defined for all $t>0$.

Our main result is the following convergence theorem.

Theorem 2.1 (Convergence of the vanishing viscosity approximation)**.** *Let us suppose that the assumptions* **(F)**, (2.7), (2.9)*, and* (2.10) *hold. Then, there exists a sequence* $\{\varepsilon_k\}_{k\in\mathbb{N}}\subset(0,\infty)$, $\varepsilon_k\to 0$*, and a weak solution* (ρ,u) *of problem* (1.1)*, in the sense of Definition 2.1, such that*

$$\begin{aligned}\rho_{\varepsilon_k}\to\rho,\ u_{\varepsilon_k}\to u\quad&\text{in }L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty,\\ &\text{and a.e. in }(0,\infty)\times\mathbb{R}\text{ as }k\to\infty,\end{aligned}\tag{2.13}$$

where $(\rho_{\varepsilon_k},u_{\varepsilon_k})$ *is a classical solution of the viscous problem* (2.4).

3. A priori estimates and compactness results

In this section, we obtain several a priori estimates on the functions ρ_ε, u_ε, solutions to (2.4), and on the function w_ε, defined in (2.11). For the sake of simplicity, throughout this section, we use c to denote various constants, which are independent from the parameter ε and from the time t.

Lemma 3.1 (L^∞ estimates on ρ_ε, u_ε, w_ε)**.** *Let us assume that* **(F)** *and* (2.9) *hold. For every $t > 0$ and $x \in \mathbb{R}$, we have that*

$$0 \le \rho_\varepsilon(t,x) \le 1, \quad \check{w}\rho_\varepsilon(t,x) \le u_\varepsilon(t,x) \le \hat{w}\rho_\varepsilon(t,x), \quad \check{w} \le w_\varepsilon(t,x) \le \hat{w}. \tag{3.1}$$

Proof. Due to **(F)** and (2.9), the functions $r = \rho_\varepsilon$, $r = 0$, and $r = 1$ are respectively a solution, a subsolution, and a supersolution of the Cauchy problem

$$\begin{cases} \partial_t r + \partial_x(u_\varepsilon r f(r)) = \varepsilon \partial_{xx}^2 r, & t > 0,\, x \in \mathbb{R}, \\ r(0,x) = \rho_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases}$$

Therefore, the first part of (3.1) follows from the comparison principle for parabolic equations (see [18]).

Due to (2.9), the functions $r = u_\varepsilon - \check{w}\rho_\varepsilon$ and $r = 0$ are respectively a solution and a subsolution of the Cauchy problem

$$\begin{cases} \partial_t r + \partial_x(r u_\varepsilon f(\rho_\varepsilon)) = \varepsilon \partial_{xx}^2 r, & t > 0,\, x \in \mathbb{R}, \\ r(0,x) = u_{0,\varepsilon}(x) - \check{w}\rho_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases}$$

Using the comparison principle for parabolic equations (see [18]), we gain $\check{w}\rho_\varepsilon \le u_\varepsilon$. An analogous argument proves that $u_\varepsilon \le \hat{w}\rho_\varepsilon$.

Finally, the third part of (3.1) follows from the second one, the definition of w_ε given in (2.11), and the positiveness of ρ_ε. □

Lemma 3.2 (L^1 estimates on $\rho_\varepsilon - 1$)**.** *Let us assume that* **(F)**, (2.7) *and* (2.9) *hold. For every $t \ge 0$, we have that*

$$\|\rho_\varepsilon(t,\cdot) - 1\|_{L^1(\mathbb{R})} \le \|\rho_0 - 1\|_{L^1(\mathbb{R})}. \tag{3.2}$$

Proof. Lemma 3.1 implies that $1 - \rho_\varepsilon$ is positive. Therefore, using (2.4) and observing

$$\lim_{x\to\pm\infty} \rho_\varepsilon(t,x) f(\rho_\varepsilon(t,x)) = f(1) = 0, \qquad \lim_{x\to\pm\infty} \partial_x \rho_\varepsilon(t,x) = 0,$$

due to (3.1), we deduce that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} |\rho_\varepsilon - 1| \,\mathrm{d}x =& \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (1 - \rho_\varepsilon) \,\mathrm{d}x = -\int_{\mathbb{R}} \partial_t \rho_\varepsilon \,\mathrm{d}x \\ &= -\int_{\mathbb{R}} \partial_x \left(\varepsilon \partial_x \rho_\varepsilon - u_\varepsilon \rho_\varepsilon f(\rho_\varepsilon)\right) \mathrm{d}x = 0. \end{aligned}$$

An integration over $(0,t)$ and assumption (2.7) give the claim. □

Lemma 3.3 (BV estimate on w_ε)**.** *Let us assume that* (2.10) *holds. We have that*

$$\|\partial_x w_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le TV\left(\frac{u_0}{\rho_0}\right) \tag{3.3}$$

for every $t \ge 0$.

Proof. Differentiating (2.12) with respect to x, we get

$$\partial_{tx}^2 w_\varepsilon + \partial_x(\rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon \partial_x w_\varepsilon) = \varepsilon \partial_{xxx}^3 w_\varepsilon + 2\varepsilon \partial_x \left(\frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} \right).$$

In light of [2, Lemma 2],

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} |\partial_x w_\varepsilon| \,\mathrm{d}x &= \int_{\mathbb{R}} \partial_{tx}^2 w_\varepsilon \operatorname{sign}(\partial_x w_\varepsilon) \,\mathrm{d}x \\
&= \varepsilon \int_{\mathbb{R}} \partial_{xxx}^3 w_\varepsilon \operatorname{sign}(\partial_x w_\varepsilon) \,\mathrm{d}x + 2\varepsilon \int_{\mathbb{R}} \partial_x \left(\frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} \right) \operatorname{sign}(\partial_x w_\varepsilon) \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \partial_x(\rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon \partial_x w_\varepsilon) \operatorname{sign}(\partial_x w_\varepsilon) \,\mathrm{d}x \\
&= \underbrace{-\varepsilon \int_{\mathbb{R}} (\partial_{xx}^2 w_\varepsilon)^2 \delta_{\{\partial_x w_\varepsilon = 0\}} \,\mathrm{d}x}_{\leq 0} \\
&\quad \underbrace{-2\varepsilon \int_{\mathbb{R}} \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} \partial_{xx}^2 w_\varepsilon \delta_{\{\partial_x w_\varepsilon = 0\}} \,\mathrm{d}x}_{=0} \\
&\quad \underbrace{+ \int_{\mathbb{R}} \rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon \partial_x w_\varepsilon \partial_{xx}^2 w_\varepsilon \delta_{\{\partial_x w_\varepsilon = 0\}} \,\mathrm{d}x}_{=0} \leq 0,
\end{aligned}$$

where $\delta_{\{\partial_x w_\varepsilon = 0\}}$ is the Dirac delta measure concentrated on the set $\{\partial_x w_\varepsilon = 0\}$. An integration over $(0, t)$ and assumption (2.10) give the claim. $\square$

Lemma 3.4 (L^1 estimate on $\ln(\rho_\varepsilon)$)**.** *Assume* **(F)**, (2.7), (2.9), *and* (2.10) *hold. We have that*

$$\begin{aligned}
&\|\ln(\rho_\varepsilon(t, \cdot))\|_{L^1(\mathbb{R})} + \varepsilon \int_0^t \left\| \frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s \\
&\qquad \leq \|\ln(\rho_0)\|_{L^1(\mathbb{R})} + t\, TV\left(\frac{u_0}{\rho_0} \right) \int_0^1 |f(\xi)| \,\mathrm{d}\xi,
\end{aligned} \tag{3.4}$$

for every $t \geq 0$.

Proof. Using the definition of w_ε (see (2.11)) in (2.4), we get

$$\partial_t \rho_\varepsilon + \partial_x(w_\varepsilon \rho_\varepsilon^2 f(\rho_\varepsilon)) = \varepsilon \partial_{xx}^2 \rho_\varepsilon. \tag{3.5}$$

Consider the function $F : (0, +\infty) \to \mathbb{R}$ defined, for every $\xi > 0$, by

$$F(\xi) = \int_1^\xi f(s) \,\mathrm{d}s.$$

Thanks to (3.1) and (3.3), we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} |\ln(\rho_\varepsilon)| \,\mathrm{d}x &= -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \ln(\rho_\varepsilon) \,\mathrm{d}x = -\int_{\mathbb{R}} \frac{\partial_t \rho_\varepsilon}{\rho_\varepsilon} \,\mathrm{d}x \\
&= -\varepsilon \int_{\mathbb{R}} \frac{\partial_{xx}^2 \rho_\varepsilon}{\rho_\varepsilon} \,\mathrm{d}x + \int_{\mathbb{R}} \frac{\partial_x(w_\varepsilon \rho_\varepsilon^2 f(\rho_\varepsilon))}{\rho_\varepsilon} \,\mathrm{d}x \\
&= -\varepsilon \int_{\mathbb{R}} \frac{(\partial_x \rho_\varepsilon)^2}{\rho_\varepsilon^2} \,\mathrm{d}x + \int_{\mathbb{R}} w_\varepsilon \underbrace{f(\rho_\varepsilon) \partial_x \rho_\varepsilon}_{\partial_x F(\rho_\varepsilon)} \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= -\varepsilon \int_{\mathbb{R}} \frac{(\partial_x \rho_\varepsilon)^2}{\rho_\varepsilon^2}\,\mathrm{d}x - \int_{\mathbb{R}} \partial_x w_\varepsilon F(\rho_\varepsilon)\,\mathrm{d}x \\
&\le -\varepsilon \int_{\mathbb{R}} \frac{(\partial_x \rho_\varepsilon)^2}{\rho_\varepsilon^2}\,\mathrm{d}x + \|F\|_{L^\infty(0,1)} \int_{\mathbb{R}} |\partial_x w_\varepsilon|\,\mathrm{d}x.
\end{aligned}
$$

An integration over $(0,t)$ and (3.3) give the claim. □

Lemma 3.5 (L^2_{loc} estimate on w_ε)**.** *Let us assume that the assumptions* **(F)**, (2.7), (2.9), *and* (2.10) *hold. Let $\chi \in C_c^\infty(\mathbb{R})$ be a non negative cut-off function with compact support. Then there exists a positive constant c, possibly depending on the function χ, such that*

$$
\|w_\varepsilon(t,\cdot)\sqrt{\chi}\|^2_{L^2(\mathbb{R})} + \varepsilon \int_0^t \|\partial_x w_\varepsilon(s,\cdot)\sqrt{\chi}\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \le c(t+1) \tag{3.6}
$$

for every $t \ge 0$.

Proof. Thanks to (2.12), (3.1), and (3.3), we have that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \frac{w_\varepsilon^2}{2} \chi(x)\,\mathrm{d}x =& \int_{\mathbb{R}} \partial_t w_\varepsilon w_\varepsilon \chi(x)\,\mathrm{d}x \\
=& \varepsilon \int_{\mathbb{R}} \partial^2_{xx} w_\varepsilon w_\varepsilon \chi(x)\,\mathrm{d}x + 2\varepsilon \int_{\mathbb{R}} \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} w_\varepsilon \chi(x)\,\mathrm{d}x \\
&- \int_{\mathbb{R}} \rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon^2 \partial_x w_\varepsilon \chi(x)\,\mathrm{d}x \\
=& -\varepsilon \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 \chi(x)\,\mathrm{d}x - \varepsilon \int_{\mathbb{R}} \partial_x w_\varepsilon w_\varepsilon \chi'(x)\,\mathrm{d}x \\
&+ 2\varepsilon \int_{\mathbb{R}} \frac{\partial_x \rho_\varepsilon \partial_x w_\varepsilon}{\rho_\varepsilon} w_\varepsilon \chi(x)\,\mathrm{d}x - \int_{\mathbb{R}} \rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon^2 \partial_x w_\varepsilon \chi(x)\,\mathrm{d}x \\
\le& -\frac{\varepsilon}{2} \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 \chi(x)\,\mathrm{d}x \\
&+ 4\varepsilon \int_{\mathbb{R}} \left(\frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon}\right)^2 w_\varepsilon^2 \chi(x)\,\mathrm{d}x + c \int_{\mathbb{R}} |\partial_x w_\varepsilon|\,\mathrm{d}x \\
\le& -\frac{\varepsilon}{2} \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 \chi(x)\,\mathrm{d}x + c\varepsilon \int_{\mathbb{R}} \left(\frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon}\right)^2 \mathrm{d}x + c.
\end{aligned}
$$

Integrating over $(0,t)$ and using (2.10) and (3.4), we deduce that

$$
\begin{aligned}
\|w_\varepsilon(t,\cdot)\sqrt{\chi}\|^2_{L^2(\mathbb{R})} &+ \varepsilon \int_0^t \|\partial_x w_\varepsilon(s,\cdot)\sqrt{\chi}\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\
&\le \left\| \frac{u_{0,\varepsilon}}{\rho_{0,\varepsilon}} \sqrt{\chi} \right\|^2_{L^2(\mathbb{R})} + \varepsilon c \int_0^t \left\| \frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon}(s,\cdot) \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + ct \\
&\le c(t+1),
\end{aligned}
$$

where we used assumption (2.8)-(2.9) in the last line. This concludes the proof. □

3.1. **Compactness of w_ε.** This subsection deals with the compactness of $\{w_\varepsilon\}_{\varepsilon>0}$, which is a preliminary step for the proof of Theorem 2.1. We use the following result, due to Murat (see [49]).

Theorem 3.1 (Murat's compact embedding)**.** *Let Ω be a bounded and open subset of $\mathbb{R}^N$ with $N \geq 2$. Assume $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ is a bounded sequence of distributions in $W^{-1,\infty}(\Omega)$. Suppose also that, for every $n \in \mathbb{N}$, there exists a decomposition*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ belongs to a compact subset of $H^{-1}_{loc}(\Omega)$.

The following result about the compactness of w_ε holds.

Lemma 3.6 (Compactness of $\{w_\varepsilon\}_{\varepsilon>0}$)**.** *Let us assume that the assumptions* **(F)**, (2.7), (2.9), *and* (2.10) *hold. Then, there exist a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$, $\varepsilon_k \to 0$, and a function*

$$w \in L^\infty((0,\infty)\times\mathbb{R}) \cap L^\infty(0,\infty; BV(\mathbb{R}))$$

such that

$$\begin{aligned} w_{\varepsilon_k} \to w \quad & \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \leq p < \infty, \\ & \text{and a.e. in } (0,\infty)\times\mathbb{R} \end{aligned} \tag{3.7}$$

as $k \to +\infty$.

Proof. Note that the equation (2.12) for w_ε can be rewritten in the form

$$\partial_t w_\varepsilon = \partial_x(\sqrt{\varepsilon}(\sqrt{\varepsilon}\partial_x w_\varepsilon)) + 2\varepsilon\frac{\partial_x\rho_\varepsilon\partial_x w_\varepsilon}{\rho_\varepsilon} - \rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon \partial_x w_\varepsilon. \tag{3.8}$$

Thanks to Lemma 3.1,

$$\{\partial_t w_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } W^{-1,\infty}((0,\infty)\times\mathbb{R}). \tag{3.9}$$

Observing that $\{\sqrt{\varepsilon}\partial_x w_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2_{loc}((0,\infty)\times\mathbb{R})$ (see Lemma 3.5) we gain

$$\{\partial_x(\sqrt{\varepsilon}(\sqrt{\varepsilon}\partial_x w_\varepsilon))\}_{\varepsilon>0} \quad \text{compact in } H^{-1}_{loc}((0,\infty)\times\mathbb{R}). \tag{3.10}$$

Using Lemmas 3.4 and 3.5

$$\left\{\varepsilon\frac{\partial_x\rho_\varepsilon\partial_x w_\varepsilon}{\rho_\varepsilon}\right\}_{\varepsilon>0} \quad \text{bounded in } L^1_{loc}((0,\infty)\times\mathbb{R}). \tag{3.11}$$

Finally, Lemmas 3.1 and 3.3 guarantee that

$$\{-\rho_\varepsilon f(\rho_\varepsilon) w_\varepsilon \partial_x w_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in } L^1_{loc}((0,\infty)\times\mathbb{R}). \tag{3.12}$$

Therefore, in light of Theorem 3.1, we deduce that

$$\{\partial_t w_\varepsilon\}_{\varepsilon>0} \quad \text{is compact in } H^{-1}_{loc}((0,\infty)\times\mathbb{R}). \tag{3.13}$$

This concludes the proof. □

4. Proof of the main theorem

In this section, we prove Theorem 2.1. To do that, first we state – in our setting – a result due to Panov (see [51, Theorem 5],[50]), which improves the classical compensated compactness theorem by Tartar (see [55]).

Theorem 4.1 (Panov's compensated compactness). *Let $\{v_\nu\}_{\nu>0}$ be a family of functions defined on $(0,\infty)\times\mathbb{R}$ and w the limit function introduced in Lemma 3.6. If $\{v_\nu\}_{\nu\in\mathbb{N}}$ lies in a bounded set of $L^\infty_{loc}((0,\infty)\times\mathbb{R})$ and if, for every constant $c\in\mathbb{R}$, the family*

$$\{\partial_t\,|v_\nu - c| + \partial_x(\text{sign}\,(v_\nu - c)\,(g(v_\nu) - g(c))w)\}_{\nu>0}\,,$$

where g is a genuinely nonlinear function, lies in a compact set of $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$, then there exist a sequence $\{\nu_k\}_{k\in\mathbb{N}}\subset(0,\infty)$, $\nu_k\to 0$, and a map $v\in L^\infty((0,\infty)\times\mathbb{R})$ such that

$$\begin{aligned} v_{\nu_k}\to v \quad & \textit{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty,\\ & \textit{and a.e. in } (0,\infty)\times\mathbb{R} \end{aligned}$$

1 *as $k\to\infty$.*

Proof of Theorem 2.1. We begin by proving the compactness of $\{\rho_\varepsilon\}_{\varepsilon>0}$. Let $c\in\mathbb{R}$ be fixed. We claim that the family

$$\{\partial_t\,|\rho_{\varepsilon_k} - c| + \partial_x\,[\text{sign}\,(\rho_{\varepsilon_k} - c)\,(g(\rho_{\varepsilon_k}) - g(c))w]\}_{k\in\mathbb{N}}$$

is compact in $H^{-1}_{loc}\,((0,+\infty)\times\mathbb{R})$, where g is the function defined in (2.1), which is genuinely nonlinear due to assumption **(F)**. For simplicity we introduce the following notations:

$$\begin{aligned} \eta_0(\xi) &= |\xi - c| - |c|\,,\\ q_0(\xi) &= \text{sign}\,(\xi - c)\,\big(g(\xi) - g(c)\big) + \text{sign}\,(-c)\,g(c). \end{aligned}$$

2 Let us remark that

$$\begin{aligned} &\eta_0(0) = q_0(0) = 0,\\ &\partial_t\,|\rho_{\varepsilon_k} - c| + \partial_x\,\big[\text{sign}\,(\rho_{\varepsilon_k} - c)\,\big(g(\rho_{\varepsilon_k}) - g(c)\big)w\big]\\ &\qquad\qquad = \partial_t\eta_0(\rho_{\varepsilon_k}) + \partial_x(q_0(\rho_{\varepsilon_k})w) - \text{sign}\,(-c)\,g(c)\partial_x w. \end{aligned}\tag{4.1}$$

3 Let $\{(\eta_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ be a family of maps such that

$$\begin{aligned} \eta_\varepsilon\in C^2(\mathbb{R}),&\quad q_\varepsilon\in C^2(\mathbb{R}),\\ q'_\varepsilon = g'\eta'_\varepsilon,&\quad \eta''_\varepsilon\ge 0\\ \|\eta_\varepsilon - \eta_0\|_{L^\infty(0,1)}\le\varepsilon,&\quad \|\eta'_\varepsilon - \eta'_0\|_{L^1(0,1)}\le\varepsilon,\\ \|\eta'_\varepsilon\|_{L^\infty(0,1)}\le 1,&\quad \eta_\varepsilon(0) = q_\varepsilon(0) = 0, \end{aligned}\tag{4.2}$$

4 for every $\varepsilon>0$.

Using (2.1), (2.4), (2.11), and (4.2), we deduce that

$$\begin{aligned} &\partial_t\eta_0(\rho_{\varepsilon_k}) + \partial_x(q_0(\rho_{\varepsilon_k})w)\\ &= \partial_t\eta_{\varepsilon_k}(\rho_{\varepsilon_k}) + \partial_x(q_{\varepsilon_k}(\rho_{\varepsilon_k})w_{\varepsilon_k}) + \underbrace{\partial_t\left(\eta_0(\rho_{\varepsilon_k}) - \eta_{\varepsilon_k}(\rho_{\varepsilon_k})\right)}_{I_{4,k}}\\ &\quad + \underbrace{\partial_x((q_0(\rho_{\varepsilon_k}) - q_{\varepsilon_k}(\rho_{\varepsilon_k}))w)}_{I_{5,k}} + \underbrace{\partial_x(q_{\varepsilon_k}(\rho_{\varepsilon_k})(w - w_{\varepsilon_k}))}_{I_{6,k}}\\ &= \eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\partial_t\rho_{\varepsilon_k} + q'_{\varepsilon_k}(\rho_{\varepsilon_k})w_{\varepsilon_k}\partial_x\rho_{\varepsilon_k} + q_{\varepsilon_k}(\rho_{\varepsilon_k})\partial_x w_{\varepsilon_k} + I_{4,k} + I_{5,k} + I_{6,k}\\ &= \varepsilon_k\eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\partial^2_{xx}\rho_{\varepsilon_k} - \eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\partial_x\left(w_{\varepsilon_k}g\left(\rho_{\varepsilon_k}\right)\right) + g'(\rho_{\varepsilon_k})\eta'_{\varepsilon_k}(\rho_{\varepsilon_k})w_{\varepsilon_k}\partial_x\rho_{\varepsilon_k}\\ &\quad + q_{\varepsilon_k}(\rho_{\varepsilon_k})\partial_x w_{\varepsilon_k} + I_{4,k} + I_{5,k} + I_{6,k} \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\varepsilon_k \partial^2_{xx} \eta_{\varepsilon_k}(\rho_{\varepsilon_k})}_{I_{2,k}} \underbrace{- \varepsilon_k \eta''_{\varepsilon_k}(\rho_{\varepsilon_k})(\partial_x \rho_{\varepsilon_k})^2}_{I_{3,k}} - \eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\, g(\rho_{\varepsilon_k})\, \partial_x w_{\varepsilon_k} \\
&\quad - \eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\, g'(\rho_{\varepsilon_k})\, w_{\varepsilon_k} \partial_x \rho_{\varepsilon_k} + \eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\, g'(\rho_{\varepsilon_k})\, w_{\varepsilon_k} \partial_x \rho_{\varepsilon_k} \\
&\quad + q_{\varepsilon_k}(\rho_{\varepsilon_k}) \partial_x w_{\varepsilon_k} + I_{4,k} + I_{5,k} + I_{6,k} \\
&= \underbrace{- \left(\eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\, g(\rho_{\varepsilon_k}) - q_{\varepsilon_k}(\rho_{\varepsilon_k})\right) \partial_x w_{\varepsilon_k}}_{I_{1,k}} + I_{2,k} + I_{3,k} + I_{4,k} + I_{5,k} + I_{6,k}.
\end{aligned}
$$

By Lemma 3.1, Lemma 3.3, and (4.2), there exist $c_1 > 0$ and $c_2 > 0$ such that

$$
\|I_{1,k}\|_{L^1((0,T)\times\mathbb{R})} \le c_1 \int_0^T \|\partial_x w_{\varepsilon_k}(s)\|_{L^1(\mathbb{R})}\, \mathrm{d}s \le c_2 T,
$$

1 proving that $I_{1,k}$ is bounded in $L^1((0,T)\times\mathbb{R})$ for every $T > 0$.

By Lemma 3.1, Lemma 3.4, and (4.2), we deduce that there exist $c_1 > 0$ and $c_2 > 0$ such that, for every $T > 0$,

$$
\begin{aligned}
\varepsilon_k^2 \int_0^T \int_{\mathbb{R}} |\partial_x \eta_{\varepsilon_k}(\rho_{\varepsilon_k})|^2\, \mathrm{d}x\, \mathrm{d}t &= \varepsilon_k^2 \int_0^T \int_{\mathbb{R}} \left|\rho^2_{\varepsilon_k} \eta'_{\varepsilon_k}(\rho_{\varepsilon_k})\right|^2 \left|\frac{\partial_x \rho_{\varepsilon_k}}{\rho_{\varepsilon_k}}\right|^2 \mathrm{d}x\, \mathrm{d}t \\
&\le c_1\, \varepsilon_k^2 \int_0^T \left\|\frac{\partial_x \rho_{\varepsilon_k}}{\rho_{\varepsilon_k}}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}t \\
&\le \varepsilon_k\, c_1\, c_1\, (1+T),
\end{aligned}
$$

2 proving that $I_{2,k} \to 0$ as $k \to +\infty$ in $H^{-1}((0,T)\times\mathbb{R})$.

By Lemma 3.1 and Lemma 3.4, there exists $c > 0$ such that, for every $T > 0$,

$$
\begin{aligned}
\varepsilon_k \int_0^T \int_{\mathbb{R}} |\eta''_{\varepsilon_k}(\rho_{\varepsilon_k})|\, |\partial_x \rho_{\varepsilon_k}|^2\, \mathrm{d}x\, \mathrm{d}t &= \varepsilon_k \int_0^T \int_{\mathbb{R}} |\rho^2_{\varepsilon_k} \eta''_{\varepsilon_k}(\rho_{\varepsilon_k})| \left|\frac{\partial_x \rho_{\varepsilon_k}}{\rho_{\varepsilon_k}}\right|^2 \mathrm{d}x\, \mathrm{d}t \\
&\le c\,(1+T),
\end{aligned}
$$

3 proving that $I_{3,k}$ is bounded in $L^1_{loc}((0,\infty)\times\mathbb{R})$.

By Lemma 3.1 and (4.2), there exists $c > 0$ such that

$$
\begin{aligned}
\|\eta_0(\rho_{\varepsilon_k}) - \eta_{\varepsilon_k}(\rho_{\varepsilon_k})\|_{L^\infty((0,\infty)\times\mathbb{R})} &\le \|\eta_0 - \eta_{\varepsilon_k}\|_{L^\infty(0,1)} \le \varepsilon_k, \\
\|(q_0(\rho_{\varepsilon_k}) - q_{\varepsilon_k}(\rho_{\varepsilon_k}))w\|_{L^\infty((0,\infty)\times\mathbb{R})} &\le \|q_0 - q_{\varepsilon_k}\|_{L^\infty(0,1)}\, \hat{w} \\
&\le \hat{w}\, \|g'\|_{L^\infty(0,1)} \left\|\eta'_{\varepsilon_k} - \eta'_0\right\|_{L^1(0,1)} \le c\, \varepsilon_k,
\end{aligned}
$$

4 proving that both $I_{4,k} \to 0$ and $I_{5,k} \to 0$ as $k \to +\infty$ in $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$.

Finally, (4.2) implies that, for every $\xi \in (0,1)$,

$$
|q_{\varepsilon_k}(\xi)| \le \int_0^1 |g'(s)| \left|\eta'_{\varepsilon_k}(s)\right| \mathrm{d}s \le \int_0^1 |g'(s)|\, \mathrm{d}s \le c
$$

for a suitable constant $c > 0$. By Lemma 3.1 and Lemma 3.6, for every set K which is compactly embedded in $(0,\infty)\times\mathbb{R}$, we get

$$
\begin{aligned}
\|q_{\varepsilon_k}(\rho_{\varepsilon_k})(w - w_{\varepsilon_k})\|_{L^2(K)} &\le \|q_{\varepsilon_k}(\rho_{\varepsilon_k})\|_{L^\infty(K)}\, \|w - w_{\varepsilon_k}\|_{L^2(K)} \\
&\le c\, \|w - w_{\varepsilon_k}\|_{L^2(K)},
\end{aligned}
$$

and so

$$
I_{6,k} \to 0 \qquad \text{in} \qquad H^{-1}_{loc}((0,\infty)\times\mathbb{R}).
$$

Having proved that the family

$$\{\partial_t |\rho_{\varepsilon_k} - c| + \partial_x [\text{sign}(\rho_{\varepsilon_k} - c)(g(\rho_{\varepsilon_k}) - g(c))w]\}_{k\in\mathbb{N}}$$

is compact in $H^{-1}_{loc}((0,+\infty)\times\mathbb{R})$, the compactness of $\{\rho_\varepsilon\}_{\varepsilon>0}$ follows from Theorem 4.1. This, together with the compactness of $\{w_\varepsilon\}_{\varepsilon>0}$ established in Lemma 3.6, yields the compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ since $u_\varepsilon = w_\varepsilon\rho_\varepsilon$ (see (2.11)).

In conclusion, we have proved that there exists $(u,\rho)\in L^\infty((0,\infty)\times\mathbb{R};\mathbb{R})$ such that

$$\rho_{\varepsilon_k}\to\rho,\ u_{\varepsilon_k}\to u \quad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty,$$
$$\text{and a.e. in } (0,\infty)\times\mathbb{R} \text{ as } k\to\infty.$$

By Lebesgue's dominated convergence theorem, we conclude that (ρ,u) is a weak solution of (1.1) in the sense of Definition 2.1.

□

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References

[1] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938 (electronic), 2000.
[2] C. Bardos, A. Y. le Roux, and J.-C. Nédélec. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4(9):1017–1034, 1979.
[3] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math. (2)*, 161(1):223–342, 2005.
[4] S. Blandin, D. Work, P. Goatin, B. Piccoli, and A. Bayen. A general phase transition model for vehicular traffic. *SIAM J. Appl. Math.*, 71(1):107–127, 2011.
[5] A. Bressan. *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000.
[6] A. Bressan and R. M. Colombo. The semigroup generated by 2×2 conservation laws. *Arch. Rational Mech. Anal.*, 133(1):1–75, 1995.
[7] A. Bressan, G. Crasta, and B. Piccoli. Well-posedness of the Cauchy problem for $n\times n$ systems of conservation laws. *Mem. Amer. Math. Soc.*, 146(694):viii+134, 2000.
[8] A. Bressan, T.-P. Liu, and T. Yang. L^1 stability estimates for $n\times n$ conservation laws. *Arch. Ration. Mech. Anal.*, 149(1):1–22, 1999.
[9] G.-Q. Chen. Remarks on R. J. DiPerna's paper: "Convergence of the viscosity method for isentropic gas dynamics" [Comm. Math. Phys. **91** (1983), no. 1, 1–30; MR0719807 (85i:35118)]. *Proc. Amer. Math. Soc.*, 125(10):2981–2986, 1997.

[10] G.-Q. Chen and H. Frid. Vanishing viscosity limit for initial-boundary value problems for conservation laws. In *Nonlinear partial differential equations (Evanston, IL, 1998)*, volume 238 of *Contemp. Math.*, pages 35–51. Amer. Math. Soc., Providence, RI, 1999.

[11] G. M. Coclite, K. H. Karlsen, S. Mishra, and N. H. Risebro. Convergence of vanishing viscosity approximations of 2×2 triangular systems of multi-dimensional conservation laws. *Boll. Unione Mat. Ital. (9)*, 2(1):275–284, 2009.

[12] R. M. Colombo. Hyperbolic phase transitions in traffic flow. *SIAM J. Appl. Math.*, 63(2):708–721, 2002.

[13] R. M. Colombo, F. Marcellini, and M. Rascle. A 2-phase traffic model based on a speed bound. *SIAM J. Appl. Math.*, 70(7):2652–2666, 2010.

[14] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, fourth edition, 2016.

[15] R. J. DiPerna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, 91(1):1–30, 1983.

[16] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.

[17] S. Fan, M. Herty, and B. Seibold. Comparative model accuracy of a data-fitted generalized Aw-Rascle-Zhang model. *Netw. Heterog. Media*, 9(2):239–268, 2014.

[18] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[19] M. Garavello and F. Marcellini. The Riemann problem at a junction for a phase transition traffic model. *Discrete Contin. Dyn. Syst.*, 37(10):5191–5209, 2017.

[20] P. Goatin. The Aw-Rascle vehicular traffic flow model with phase transitions. *Math. Comput. Modelling*, 44(3-4):287–303, 2006.

[21] J. M. Greenberg, A. Klar, and M. Rascle. Congestion on multilane highways. *SIAM J. Appl. Math.*, 63(3):818–833 (electronic), 2003.

[22] F. Gu, Y.-g. Lu, and Q. Zhang. Global solutions to one-dimensional shallow water magneto-hydrodynamic equations. *J. Math. Anal. Appl.*, 401(2):714–723, 2013.

[23] H. Holden and N. H. Risebro. *Front tracking for hyperbolic conservation laws*, volume 152 of *Applied Mathematical Sciences*. Springer, Heidelberg, second edition, 2015.

[24] Y.-b. Hu, Y.-g. Lu, and N. Tsuge. Global existence and stability to the polytropic gas dynamics with an outer force. *Appl. Math. Lett.*, 95:36–40, 2019.

[25] F. Huang and Z. Wang. Convergence of viscosity solutions for isothermal gas dynamics. *SIAM J. Math. Anal.*, 34(3):595–610, 2002.

[26] B. S. Kerner. *The Physics of Traffic: Empirical Freeway Pattern Features, Engineering Applications, and Theory*. Springer, Berlin, New York, 2004.

[27] C. Klingenberg and Y.-g. Lu. Existence of solutions to hyperbolic conservation laws with a source. *Comm. Math. Phys.*, 187(2):327–340, 1997.

[28] C. Klingenberg and Y.-g. Lu. The vacuum case in Diperna's paper. *J. Math. Anal. Appl.*, 225(2):679–684, 1998.

[29] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[30] J. P. Lebacque, X. Louis, S. Mammar, B. Schnetzlera, and H. Haj-Salem. Modélisation du trafic autoroutier au second ordre. *Comptes Rendus Mathematique*, 346(21–22):1203–1206, November 2008.

[31] M. J. Lighthill and G. B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.*, 229:317–345, 1955.

[32] T.-P. Liu and T. Yang. L_1 stability for 2×2 systems of hyperbolic conservation laws. *J. Amer. Math. Soc.*, 12(3):729–774, 1999.

[33] T.-P. Liu and T. Yang. L_1 stability of conservation laws with coinciding Hugoniot and characteristic curves. *Indiana Univ. Math. J.*, 48(1):237–247, 1999.

[34] Y. Lu. *Hyperbolic conservation laws and the compensated compactness method*, volume 128 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[35] Y. G. Lu. Convergence of the viscosity method for a non-strictly hyperbolic conservation law. *Comm. Math. Phys.*, 150(1):59–64, 1992.
[36] Y.-G. Lu. Existence of global entropy solutions of a nonstrictly hyperbolic system. *Arch. Ration. Mech. Anal.*, 178(2):287–299, 2005.
[37] Y.-g. Lu. Existence of global bounded weak solutions to nonsymmetric systems of Keyfitz-Kranzer type. *J. Funct. Anal.*, 261(10):2797–2815, 2011.
[38] Y.-g. Lu. Existence of global bounded weak solutions to a symmetric system of Keyfitz-Kranzer type. *Nonlinear Anal. Real World Appl.*, 13(1):235–240, 2012.
[39] Y.-g. Lu. Existence of global entropy solutions to general system of Keyfitz-Kranzer type. *J. Funct. Anal.*, 264(10):2457–2468, 2013.
[40] Y.-g. Lu. Global entropy solutions of Cauchy problem for the Le Roux system. *Appl. Math. Lett.*, 60:61–66, 2016.
[41] Y.-g. Lu. Global existence of solutions to system of polytropic gas dynamics with friction. *Nonlinear Anal. Real World Appl.*, 39:418–423, 2018.
[42] Y.-g. Lu. Global solutions to isothermal system in a divergent nozzle with friction. *Appl. Math. Lett.*, 84:176–180, 2018.
[43] Y.-g. Lu. Global weak solutions for the chromatography system. *Israel J. Math.*, 225(2):721–741, 2018.
[44] Y.-g. Lu. Existence of global solutions for isentropic gas flow with friction. *Nonlinearity*, 33(8):3940–3969, 2020.
[45] Y.-G. Lu and F. Gu. Existence of global bounded weak solutions to a Keyfitz-Kranzer system. *Commun. Math. Sci.*, 10(4):1133–1142, 2012.
[46] Y.-g. Lu and F. Gu. Existence of global entropy solutions to the isentropic Euler equations with geometric effects. *Nonlinear Anal. Real World Appl.*, 14(2):990–996, 2013.
[47] Y.-g. Lu, X.-z. Lu, and C. Klingenberg. The Cauchy problem for multiphase first-contact miscible models with viscous fingering. *Nonlinear Anal. Real World Appl.*, 27:43–54, 2016.
[48] Y.-g. Lu, E. Villamizar Roa, and J. Xie. Global existence of weak solutions for $n \times n$ system of chromatography. *Nonlinear Anal. Real World Appl.*, 37:309–316, 2017.
[49] F. Murat. L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$. *J. Math. Pures Appl. (9)*, 60(3):309–322, 1981.
[50] E. Y. Panov. Erratum to: Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux [mr2592291]. *Arch. Ration. Mech. Anal.*, 196(3):1077–1078, 2010.
[51] E. Y. Panov. Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux. *Arch. Ration. Mech. Anal.*, 195(2):643–673, 2010.
[52] P. I. Richards. Shock waves on the highway. *Operations Res.*, 4:42–51, 1956.
[53] D. Serre. Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservation. *J. Differential Equations*, 68(2):137–168, 1987.
[54] Q. Sun, Y. Lu, and C. Klingenberg. Global L∞ Solutions to System of Isentropic Gas Dynamics in a Divergent Nozzle with Friction. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 39(5):1213–1218, 2019.
[55] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, Mass.-London, 1979.
[56] M. E. Taylor. *Partial differential equations I. Basic theory*, volume 115 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011.
[57] B. Temple. Systems of conservation laws with invariant submanifolds. *Trans. Amer. Math. Soc.*, 280(2):781–795, 1983.
[58] G. Wong and S. Wong. A multi-class traffic flow model - an extension of lwr model with heterogeneous drivers. *Transportation Research Part A: Policy and Practice*, 36(9):827–841, 2002. cited By 191.
[59] D.-y. Zheng, Y.-g. Lu, G.-q. Song, and X.-z. Lu. Global existence of solutions for a nonstrictly hyperbolic system. *Abstr. Appl. Anal.*, pages Art. ID 691429, 7, 2014.

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