Profinite groups
with a cyclotomic $p$-orientation

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To the memory of Vladimir Voevodsky

Abstract. Let $p$ be a prime. A continuous representation
\[ \theta : G \to \text{GL}_1(\mathbb{Z}_p) \]
of a profinite group $G$ is called a cyclotomic $p$-orientation if for all open subgroups $U \subseteq G$ and for all $k, n \geq 1$ the natural maps
\[ H^k(U, \mathbb{Z}_p(k)/p^n) \to H^k(U, \mathbb{Z}_p(k)/p) \]
are surjective. Here $\mathbb{Z}_p(k)$ denotes the $\mathbb{Z}_p$-module of rank 1 with $U$-action induced by $\theta|_U$. By the Rost-Voevodsky theorem, the cyclotomic character of the absolute Galois group $G_K$ of a field $K$ is, indeed, a cyclotomic $p$-orientation of $G_K$. We study profinite groups with a cyclotomic $p$-orientation. In particular, we show that cyclotomicity is preserved by several operations on profinite groups, and that Bloch-Kato pro-$p$ groups with a cyclotomic $p$-orientation satisfy a strong form of Tits’ alternative and decompose as semi-direct product over a canonical abelian closed normal subgroup.

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1. Introduction

For a prime $p$ let $\mathbb{Z}_p$ denote the ring of $p$-adic integers. For a profinite group $G$, we call a continuous representation $\theta : G \to \mathbb{Z}_p^\times = \text{GL}_1(\mathbb{Z}_p)$ a $p$-orientation of $G$ and call the couple $(G, \theta)$ a $p$-oriented profinite group. Given a $p$-oriented profinite group $(G, \theta)$, for $k \in \mathbb{Z}$ let $\mathbb{Z}_p(k)$ denote the left $\mathbb{Z}_p[G]$-module

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induced by $\theta^k$, namely, $\mathbb{Z}_p(k)$ is equal to the additive group $\mathbb{Z}_p$ and the left $G$-action is given by
$$g \cdot z = \theta(g)^k \cdot z, \quad g \in G, \quad z \in \mathbb{Z}_p(k). \quad (1.1)$$
Vice-versa, if $M$ is a topological left $\mathbb{Z}_p[G]$-module which as an abelian pro-$p$ group is isomorphic to $\mathbb{Z}_p$, then there exists a unique $p$-orientation $\theta: G \to \mathbb{Z}_p^\times$ such that $M \cong \mathbb{Z}_p(1)$.

The $\mathbb{Z}_p[G]$-module $\mathbb{Z}_p(1)$ and the representation $\theta: G \to \mathbb{Z}_p^\times$ are said to be $k$-cyclotomic, for $k \geq 1$, if for every open subgroup $U$ of $G$ and every $n \geq 1$ the natural maps
$$H^k(U, \mathbb{Z}_p(k)/p^n) \longrightarrow H^k(U, \mathbb{Z}_p(k)/p),$$
induced by the epimorphism of $\mathbb{Z}_p[U]$-modules $\mathbb{Z}_p(k)/p^n \to \mathbb{Z}_p(k)/p$, are surjective. If $\mathbb{Z}_p(1)$ (respectively $\theta$) is $k$-cyclotomic for every $k \geq 1$, then it is called simply a cyclotomic $\mathbb{Z}_p[G]$-module (resp., cyclotomic $p$-orientation).

Note that $\mathbb{Z}_p(1)$ is $k$-cyclotomic if, and only if, $H^{k+1}(U, \mathbb{Z}_p(k))$ is a torsion free $\mathbb{Z}_p$-module for every open subgroup $U \subseteq G$ — here $H^\text{cts}_{k}$ denotes continuous cochain cohomology as introduced by J. Tate in [34] (see § 2.1).

Cyclotomic modules of profinite groups have been introduced and studied by C. De Clercq and M. Florence in [5]. Previously J.P. Labute, in [16], considered surjectivity of (1.2) in the case $k = 1$ and $U = G$ — note that demanding surjectivity for $U = G$ only is much weaker than demanding it for every open subgroup $U \subseteq G$, and this is what makes the definition of cyclotomic modules truly new.

Let $K$ be a field, and let $\bar{\mathbb{K}} / K$ be a separable closure of $K$. If $\text{char}(K) \neq p$, the absolute Galois group $G_K = \text{Gal}(\bar{\mathbb{K}} / K)$ of $K$ comes equipped with a canonical $p$-orientation
$$\theta_{K,p}: G_K \longrightarrow \text{Aut}(\mu_p(\bar{\mathbb{K}})) \cong \mathbb{Z}_p^\times, \quad (1.3)$$
where $\mu_p(\bar{\mathbb{K}}) \subseteq \bar{\mathbb{K}}^\times$ denotes the subgroup of roots of unity of $\bar{\mathbb{K}}$ of $p$-power order. If $p = \text{char}(K)$, we put $\theta_{K,p} = 1_{G_K}$, the function which is constantly 1 on $G_K$. The following result (cf. [5, Prop. 14.19]) is a consequence of the positive solution of the Bloch-Kato Conjecture given by M. Rost and V. Voevodsky with the “C. Weibel patch” (cf. [29, 36, 40]), which from now on we will refer to as the Rost-Voevodsky Theorem.

**Theorem 1.1.** Let $K$ be a field, and let $p$ be prime number. The canonical $p$-orientation $\theta_{K,p}: G_K \to \mathbb{Z}_p^\times$ is cyclotomic.

**Theorem 1.1** provides a fundamental class of examples of profinite groups endowed with a cyclotomic $p$-orientation. Bearing in mind the exotic character of absolute Galois groups, it also provides a strong motivation to the study of cyclotomically $p$-oriented profinite groups — which is the main purpose of this manuscript. In fact, one may recover several Galois-theoretic statements already for profinite groups with a 1-cyclotomic $p$-orientation — e.g., the only finite group endowed with a 1-cyclotomic $p$-orientation is the finite group $C_2$ of order 2, with non-constant 2-orientation $\theta: C_2 \to \{\pm 1\}$ (cf. [11]
and this implies the Artin-Schreier obstruction for absolute Galois groups. In their paper, De Clercq and Florence formulated the “Smoothness Conjecture”, which can be restated in this context as follows: for a $p$-oriented profinite group, 1-cyclotomicity implies $k$-cyclotomicity for all $k \geq 1$ (cf. [5, Conj. 14.25]).

A $p$-oriented profinite group $(G, \theta)$ is said to be Bloch-Kato if the $\mathbb{F}_p$-algebra

$$H^\bullet(U, \hat{\theta}|_U) = \prod_{k \geq 0} H^k(U, \mathbb{F}_p(k)),$$

where $\mathbb{F}_p(k) = \mathbb{Z}_p(k)/p$, with product given by cup-product, is quadratic for every open subgroup $U$ of $G$. Note that if $\text{im}(\theta) \subseteq 1 + p\mathbb{Z}_p$ and $p \neq 2$ then $G$ acts trivially on $\mathbb{Z}_p(k)/p$. By the Rost-Voevodsky Theorem $(G, \theta)$ is, indeed, Bloch-Kato.

For a profinite group $G$, let $O_p(G)$ denote the maximal closed normal pro-$p$ subgroup of $G$. A $p$-oriented profinite group $(G, \theta)$ has two particular closed normal subgroups: the kernel $\ker(\theta)$ of $\theta$, and the $\theta$-center of $(G, \theta)$, given by

$$Z_{\theta}(G) = \{ x \in O_p(\ker(\theta)) \mid gxg^{-1} = x^{\theta(g)} \text{ for all } g \in G \}.$$  \hspace{1cm} (1.5)

As $Z_{\theta}(G)$ is contained in the center $Z(\ker(\theta))$ of $\ker(\theta)$, it is abelian. The $p$-oriented profinite group $(G, \theta)$ will be said to be $\theta$-abelian, if $\ker(\theta) = Z_{\theta}(G)$ and if $Z_{\theta}(G)$ is torsion free. In particular, for such a $p$-oriented profinite group $(G, \theta)$, $G$ is a virtual pro-$p$ group (i.e., $G$ contains an open subgroup which is a pro-$p$ group). Moreover, a $\theta$-abelian pro-$p$ group $(G, \theta)$ will be said to be split if $G \cong Z_{\theta}(G) \times \text{im}(\theta)$.

As $Z_{\theta}(G)$ is contained in $\ker(\theta)$, by definition, the canonical quotient $\bar{G} = G/Z_{\theta}(G)$ carries naturally a $p$-orientation $\bar{\theta}: \bar{G} \to \mathbb{Z}_p^\times$, and one has the following short exact sequence of $p$-oriented profinite groups.

$$\{1\} \to Z_{\theta}(G) \to G \xrightarrow{\pi} \bar{G} \to \{1\}.$$ \hspace{1cm} (1.6)

The following result can be seen as an analogue of the equal characteristic transition theorem (cf. [31, §II.4, Exercise 1(b), p. 86]) for cyclotomically $p$-oriented Bloch-Kato profinite groups.

**Theorem 1.2.** Let $(G, \theta)$ be a cyclotomically $p$-oriented Bloch-Kato profinite group. Then \hspace{1cm} (1.6) \hspace{1cm} splits, provided that $\text{cd}_p(G) < \infty$, and one of the following conditions hold:

(i) $G$ is a pro-$p$ group,
(ii) $(G, \theta)$ is an oriented virtual pro-$p$ group (see §4),
(iii) $(\bar{G}, \bar{\theta})$ is cyclotomically $p$-oriented and Bloch-Kato.

In the case that $(G, \theta)$ is the maximal pro-$p$ Galois group of a field $\mathbb{K}$ containing a primitive $p^th$-root of unity endowed with the $p$-orientation induced by $\theta_{\mathbb{K},p}$, $Z_{\theta}(G)$ is the inertia group of the maximal $p$-henselian valuation of $\mathbb{K}$ (cf. Remark 7.8).
Note that the 2-oriented pro-2 group \((C_2 \times \mathbb{Z}_2, \theta)\) may be \(\theta\)-abelian, but \(\theta\) is never \(1\)-cyclotomic (cf. Proposition 6.5). As a consequence, in a cyclotomically 2-oriented pro-2 group every element of order 2 is self-centralizing.

For \(p\) odd it was shown in [25] that a Bloch-Kato pro-\(p\) group \(G\) satisfies a strong form of Tits alternative, i.e., either \(G\) contains a closed non-abelian free pro-\(p\) subgroup, or there exists a \(p\)-orientation \(\theta : G \to \mathbb{Z}_p^\times\) such that \(G\) is \(\theta\)-abelian. In Subsection 7.1 we extend this result to pro-2 groups with a cyclotomic orientation, i.e., one has the following analogue of R. Ware’s theorem (cf. [38]) for cyclotomically oriented Bloch-Kato pro-\(p\) groups (cf. Fact 7.4).

**Theorem 1.3.** Let \((G, \theta)\) be a cyclotomically \(p\)-oriented Bloch-Kato pro-\(p\) group. If \(p = 2\) assume further that \(\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2\). Then one — and only one — of the following cases hold:

(i) \(G\) contains a closed non-abelian free pro-\(p\) subgroup; or 
(ii) \(G\) is \(\theta\)-abelian.

It should be mentioned that for \(p = 2\) the additional hypothesis is indeed necessary (cf. Remark 5.8). The class of cyclotomically \(p\)-oriented Bloch-Kato profinite groups is closed with respect to several constructions.

**Theorem 1.4.** (a) The inverse limit of an inverse system of cyclotomically \(p\)-oriented Bloch-Kato profinite groups with surjective structure maps is a cyclotomically \(p\)-oriented Bloch-Kato profinite group (cf. Corollary 3.3 and Corollary 3.6).

(b) The free profinite (resp. pro-\(p\)) product of two cyclotomically \(p\)-oriented Bloch-Kato profinite (resp. pro-\(p\)) groups is a cyclotomically \(p\)-oriented Bloch-Kato profinite group (resp. pro-\(p\)) group (cf. Theorem 3.14).

(c) The fibre product of a cyclotomically \(p\)-oriented Bloch-Kato profinite group \((G_1, \theta_1)\) with a split \(\theta_2\)-abelian profinite group \((G_2, \theta_2)\) is a cyclotomically \(p\)-oriented Bloch-Kato profinite group (cf. Theorem 3.11 and Theorem 3.13).

(d) The quotient of a cyclotomically \(p\)-oriented Bloch-Kato profinite group \((G, \theta)\) with respect to a closed normal subgroup \(N \subseteq G\) satisfying \(N \subseteq \ker(\theta)\) and \(N\) a \(p\)-perfect group is a cyclotomically \(p\)-oriented Bloch-Kato profinite group (cf. Proposition 4.6).

Some time ago I. Efrat (cf. [7, 8, 9]) has formulated the so-called elementary type conjecture concerning the structure of finitely generated pro-\(p\) groups occurring as maximal pro-\(p\) quotients of an absolute Galois group. His conjecture can be reformulated in the class of cyclotomically \(p\)-oriented Bloch-Kato pro-\(p\) groups. Such a \(p\)-oriented pro-\(p\) group \((G, \theta)\) is said to be indecomposable if \(Z_\theta(G) = \{1\}\) and if \(G\) is not a proper free pro-\(p\) product. A positive answer to the following question would settle the elementary type conjecture affirmatively.

**Question 1.5.** Let \((G, \theta)\) be a finitely generated, torsion free, indecomposable, cyclotomically oriented Bloch-Kato pro-\(p\) group. Does this imply that \(G\) is a Poincaré duality pro-\(p\) group of dimension \(\text{cd}_p(G) \leq 2\)?
The paper is organized as follows. In §2 we give some equivalent definitions for cyclotomic $p$-orientations. In §3 we study some operations of profinite groups (inverse limits, free products and fibre products) in relation with the properties of cyclotomicity and Bloch-Kato-ness, and we prove Theorem 1.4(a)-(b)-(c). In §4 we study the quotients of cyclotomically $p$-oriented profinite groups over closed normal $p$-perfect subgroups — in particular, we introduce oriented virtual pro-$p$ groups and we prove Theorem 1.4(d). In §5 we study $p$-oriented profinite Poincaré duality groups. In §6 we focus on the presence of torsion in cyclotomically 2-oriented pro-2 groups, and we prove that in a 1-cyclotomically 2-oriented pro-2 group every element of order 2 is self-centralizing (see Proposition 6.5). In §7 we focus on the structure of cyclotomically $p$-oriented Bloch-Kato pro-$p$ groups: we prove Theorems 1.2 and 1.3 and show that in many cases the $\theta$-center is the maximal abelian closed normal subgroup (cf. Theorem 7.7).

2. Absolute Galois groups and cyclotomic $p$-orientations

Throughout the paper, we study profinite groups with a cyclotomic module $\mathbb{Z}_p(1)$. In contrast to [5, §14], we refer to the associated representation $\theta: G \to \mathbb{Z}_p^\times$, rather than to the module itself. As we study several subgroups of $G$ associated to this cyclotomic module $\mathbb{Z}_p(1)$, like $\ker(\theta)$ and $\mathbb{Z}_p(G)$, this choice of notation turns out to be convenient for our purposes. We follow the convention as established in [25, 26] and call such representations “$p$-orientations”. In the case that $G$ is a pro-$p$ group, the couple $(G, \theta)$ was called a cyclotomic pro-$p$ pair, in [9, §3].

2.1. The connecting homomorphism $\delta^k$

Let $G$ be a profinite group, and let $\theta: G \to \mathbb{Z}_p^\times$ be a $p$-orientation of $G$. For every $k \geq 0$ one has the short exact sequence of left $\mathbb{Z}_p[G]$-modules

$$0 \to \mathbb{Z}_p(k) \to \mathbb{Z}_p(k) \to \mathbb{F}_p(k) \to 0,$$

which induces the long exact sequence in cohomology

$$\cdots \to H_{\text{cts}}^k(G, \mathbb{Z}_p(k)) \xrightarrow{p} H_{\text{cts}}^k(G, \mathbb{Z}_p(k)) \xrightarrow{\pi^k} H^k(G, \mathbb{F}_p(k)) \xrightarrow{\delta^k} H_{\text{cts}}^{k+1}(G, \mathbb{Z}_p(k)) \to \cdots$$

(2.2)

with connecting homomorphism $\delta^k$ (cf. [34 §2]). In particular, $\delta^k$ is trivial if, and only if, multiplication by $p$ on $H_{\text{cts}}^{k+1}(G, \mathbb{Z}_p(k))$ is a monomorphism. This is equivalent to $H_{\text{cts}}^{k+1}(G, \mathbb{Z}_p(k))$ being torsion free. Therefore, one concludes the following:

1. For a Poincaré duality group $G$ the representation associated to the dualizing module — which coincides with the cyclotomic module in the case of a Poincaré duality pro-$p$ group of dimension 2 (cf. Theorem 5.7) — is sometimes also called the “orientation” of $G$. 
Proposition 2.1. Let \((G, \theta)\) be a \(p\)-oriented profinite group. For \(k \geq 1\) and \(U \subseteq G\) an open subgroup the following are equivalent.

(i) The map \(\pi^k \circ C. Quadrelli and Thomas S. Weigel\)

(ii) The map \(\pi^k: H^k_{cts}(U, \mathbb{Z}_p(k)) \to H^k(U, \mathbb{F}_p(k))\) is surjective.

(iii) The connecting homomorphism \(\delta^k: H^k(U, \mathbb{F}_p(k)) \to H^{k+1}_{cts}(U, \mathbb{Z}_p(k))\) is trivial.

(iv) The \(\mathbb{Z}_p\)-module \((U, \mathbb{Z}_p(k))\) is torsion free.

Proof. By the long exact sequence (2.2), the equivalences between (ii), (iii) and (iv) are straightforward. For \(m \geq n \geq 1\) let \(\pi_{m,n}^k\) denote the natural maps

\[
\pi_{m,n}^k: H^k(U, \mathbb{Z}_p(k)/p^m) \to H^k(U, \mathbb{Z}_p(k)/p^n)
\]

(if \(m = \infty\) we set \(p^\infty = 0\)). If condition (i) holds then the system

\[
(H^k(U, \mathbb{Z}_p/p^n), \pi_{m,n}^k)
\]

satisfies the Mittag-Leffler property. In particular,

\[
H^k(U, \mathbb{Z}_p(k)) \simeq \lim_{n \geq 1} H^k(U, \mathbb{Z}_p(k)/p^n)
\]

(cf. [28 and 23 Thm. 2.7.5]). Thus \(\pi^k = \pi_{n,1}^k \circ \pi_{\infty,n}^k\) is surjective if, and only if, \(\pi_{n,1}^k\) is surjective for every \(n \geq 1\). Conversely, if \(\pi^k\) is surjective then \(\pi^k = \pi_{n,1}^k \circ \pi_{\infty,n}^k\) yields the surjectivity of \(\pi_{n,1}^k\) for every \(n\). \(\square\)

2.2. Profinite groups of cohomological \(p\)-dimension at most 1

Let \(G\) be a profinite group, and let \(\theta: G \to \mathbb{Z}_p^\times\) be a \(p\)-orientation of \(G\). Then

\[
H^1_{cts}(G, \mathbb{Z}_p(0)) = \text{Hom}_{\text{grp}}(G, \mathbb{Z}_p)
\]

(2.3)

is a torsion free abelian group for every profinite group \(G\), i.e., \(\theta\) is 0-cyclotomic. If \(G\) is of cohomological \(p\)-dimension less or equal to 1, then \(H^1_{cts}(G, \mathbb{Z}_p(m)) = 0\) for all \(m \geq 1\) showing that \(\theta\) is cyclotomic. Moreover, \(H^*(G, \theta)\) is a quadratic \(\mathbb{F}_p\)-algebra for every profinite group with \(\text{cd}_p(G) \leq 1\) and for any \(p\)-orientation \(\theta: G \to \mathbb{Z}_p^\times\). If \(G\) is of cohomological \(p\)-dimension less or equal to 1, one has \(\text{cd}_p(C) \leq 1\) for every closed subgroup \(C\) of \(G\) (cf. [31 §1.3.3, Proposition 14]). Thus one has the following.

Fact 2.2. Let \(G\) be a profinite group with \(\text{cd}_p(G) \leq 1\), and let \(\theta: G \to \mathbb{Z}_p^\times\) be a \(p\)-orientation for \(G\). Then \((G, \theta)\) is Bloch-Kato and \(\theta\) is cyclotomic.

2.3. The \(m\)th-norm residue symbol

Throughout this subsection we fix a field \(\mathbb{K}\) and a separable closure \(\overline{\mathbb{K}}\) of \(\mathbb{K}\). For \(p \neq \text{char}(\mathbb{K})\), \(\mu_{p^\infty}(\overline{\mathbb{K}})\) is a divisible abelian group. By construction, one has a canonical isomorphism

\[
\lim_{\leftarrow k \geq 0} (\mu_{p^\infty}(\overline{\mathbb{K}}), p^k) \simeq \mathbb{Z}_p(1) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p = \mathbb{Q}_p(1)
\]

(2.4)

and a short exact sequence \(0 \to \mathbb{Z}_p(1) \to \mathbb{Q}_p(1) \to \mu_{p^\infty}(\overline{\mathbb{K}}) \to 0\) of topological left \(\mathbb{Z}_p[G_\mathbb{K}]-\)modules, where \(G_\mathbb{K} = \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})\) is the absolute Galois group of \(\mathbb{K}\).
Let $K_m^M(\mathbb{K})$, $m \geq 0$, denote the $m^{th}$-Milnor $K$-group of $\mathbb{K}$ (cf. [10 §24.3]). For $p \neq \text{char}(\mathbb{K})$, J. Tate constructed in [31] a homomorphism of abelian groups

$$h_m(\mathbb{K}): K_m^M(\mathbb{K}) \longrightarrow H_{cts}^m(G_{\mathbb{K}}, \mathbb{Z}_p(m)),$$  

the so-called $m^{th}$-norm residue symbol. Let $K_m^M(\mathbb{K})/p = K_m^M(\mathbb{K})/pK_m^M(\mathbb{K})$. Around ten years later S. Bloch and K. Kato conjectured in [1] that the conjecture has been proved by V. Voevodsky and M. Rost with a “patch” of C. Weibel (cf. [29, 36, 40]). In particular, since $K_m^M(\mathbb{K})/p$ is a quadratic $\mathbb{F}_p$-algebra and as $h_m(\mathbb{K})/p$ is a homomorphism of algebras, this implies that the absolute Galois group of a field $\mathbb{K}$ is Bloch-Kato (cf. [10 §23.4]). The Rost-Voevodsky Theorem has also the following consequence.

**Proposition 2.3.** Let $\mathbb{K}$ be a field, let $G_{\mathbb{K}}$ denote its absolute Galois group, and let $\theta_{\mathbb{K}, p}: G_{\mathbb{K}} \rightarrow \mathbb{Z}_p^\times$ denote its canonical $p$-orientation. Then $\theta_{\mathbb{K}, p}$ is cyclotomic.

Although Proposition 2.3 might be well known to specialists, we add a short proof of it. By Proposition 2.1, Proposition 2.3 in combination with Theorem 1.4-(d) is equivalent to [5 Prop. 14.19].

**Proof of Proposition 2.3** If char($\mathbb{K}$) = $p$, then $\text{cd}_p(G_{\mathbb{K}}) \leq 1$ (cf. [31 II.2.2, Proposition 3]), and the $p$-orientation $\theta_{\mathbb{K}, p}$ is cyclotomic by Fact 2.2. So we may assume that char($\mathbb{K}$) $\neq p$. In the commutative diagram

$$
\begin{array}{ccc}
K_k^M(\mathbb{K}) & \longrightarrow & K_k^M(\mathbb{K}) \\
\downarrow h_k & & \downarrow h_k \\
H_{cts}^k(G_{\mathbb{K}}, \mathbb{Z}_p(k)) & \longrightarrow & H_{cts}^k(G_{\mathbb{K}}, \mathbb{Z}_p(k))
\end{array}
\quad
\begin{array}{ccc}
K_k^M(\mathbb{K})/p & \longrightarrow & K_k^M(\mathbb{K})/p \\
\downarrow (h_k)/p & & \downarrow (h_k)/p \\
H_{cts}^k(G_{\mathbb{K}}, \mathbb{Z}_p(k))/p & \longrightarrow & H_{cts}^k(G_{\mathbb{K}}, \mathbb{Z}_p(k))/p
\end{array}
\longrightarrow 0
$$

the map $\pi$ is surjective, and $(h_k)/p$ is an isomorphism. Hence $\alpha$ must be surjective, and thus $\beta = 0$, i.e.,

$$p: H_{cts}^{k+1}(G_{\mathbb{K}}, \mathbb{Z}_p(k)) \longrightarrow H_{cts}^{k+1}(G_{\mathbb{K}}, \mathbb{Z}_p(k))$$

is an injective homomorphism of $\mathbb{Z}_p$-modules. Thus $H_{cts}^{k+1}(G_{\mathbb{K}}, \mathbb{Z}_p(k))$ must be $p$-torsion free. Any open subgroup $U$ of $G_{\mathbb{K}}$ is the absolute Galois group of $\mathbb{K}^U$. Hence $\theta_{\mathbb{K}, p}$ is cyclotomic, and this yields the claim. \hfill \square

**Remark 2.4.** Let $\mathbb{K}$ be a number field, let $S$ be a set of places containing all finite places of $\mathbb{K}$ and all places lying above $p$, and let $G_{\mathbb{K}}^S$ be the Galois group of $\mathbb{K}^S/\mathbb{K}$, where $\mathbb{K}^S/\mathbb{K}$ is the maximal extension of $\mathbb{K}/\mathbb{K}$ which is unramified outside $S$. Then $\theta_{\mathbb{K}, p}: G_{\mathbb{K}} \rightarrow \mathbb{Z}_p^\times$ induces a $p$-orientation $\theta_{\mathbb{K}, p}^S: G_{\mathbb{K}}^S \rightarrow \mathbb{Z}_p^\times$. However, it is well known (cf. [23 Prop. 8.3.11(ii)]) that,

$$H^1(G_{\mathbb{K}}^S, {\mathcal{O}^S}_{\mathbb{K}}(1)) \simeq H^1(G_{\mathbb{K}}^S, {\mathcal{O}^S}_{\mathbb{K}}(p)) \simeq \text{cl}(O_{\mathbb{K}}^S)(p)$$  

(2.8)
(for the definition of $I_p(1)$ see [3], where $\text{cl}(O_{\mathbb{K}}^S)$ denotes the ideal class group of the Dedekind domain $O_{\mathbb{K}}^S$, and $\text{cl}(p)$ denotes the $p$-primary component. Hence $(G_{\mathbb{K}}^S, \theta_{\mathbb{K}, p})$ is in general not cyclotomic (cf. Proposition 3.1).

3. Cohomology of $p$-oriented profinite groups

A homomorphism $\phi: (G_1, \theta_1) \to (G_2, \theta_2)$ of two $p$-oriented profinite groups $(G_1, \theta_1)$ and $(G_2, \theta_2)$ is a continuous group homomorphism $\phi: G_1 \to G_2$ satisfying $\theta_1 = \theta_2 \circ \phi$.

Let $(G, \theta)$ be a $p$-oriented profinite group. For $k \in \mathbb{Z}$, put $\mathbb{Q}_p(k) = \mathbb{Z}_p(k) \otimes \mathbb{Z}_p \mathbb{Q}_p$, and also $\mathbb{I}_p(k) = \mathbb{Q}_p(k)/\mathbb{Z}_p(k)$, i.e., $\mathbb{I}_p(k)$ is a discrete left $G$-module and — as an abelian group — a divisible $p$-torsion module.

Let $\mathbb{I}_p = \mathbb{Q}_p/\mathbb{Z}_p$, and let $\mathbb{I}_p^* = \text{Hom}_{\mathbb{Z}_p}(\mathbb{I}_p, \mathbb{I}_p)$ denote the Pontryagin duality functor. Then $\mathbb{I}_p(k)^*$ is a profinite left $\mathbb{Z}_p[G]$-module which is isomorphic to $\mathbb{Z}_p(-k)$.

3.1. Criteria for cyclotomicity

The following proposition relates the continuous co-chain cohomology groups, Galois cohomology and the Galois homology groups as defined by A. Brumer in [3].

Proposition 3.1. Let $(G, \theta)$ be a $p$-oriented profinite group, let $k$ be an integer, and let $m$ be a non-negative integer. Then the following are equivalent:

(i) $H^{m+1}_{\text{cts}}(G, \mathbb{Z}_p(k))$ is torsion free;
(ii) $H^m(G, \mathbb{I}_p(k))$ is divisible;
(iii) $H_m(G, \mathbb{Z}_p(-k))$ is torsion free.

Proof. The equivalence (i)$\iff$(ii) is a direct consequence of [34, Prop. 2.3], and (ii)$\iff$(iii) follows from [33, (3.4.5)].

The direct limit of divisible $p$-torsion modules is a divisible $p$-torsion module. From this fact — and Proposition 3.1 — one concludes the following.

Corollary 3.2. Let $(G, \theta)$ be a cyclotomically $p$-oriented profinite group. Then $H^m(C, \mathbb{I}_p(m))$ is divisible for all $m \geq 0$ and all $C$ closed in $G$.

Proof. It suffices to show (ii)$\Rightarrow$(i). Let $C$ be a closed subgroup of $G$. Then

$$H^m(C, \mathbb{I}_p(m)) \simeq \lim_{\mathcal{U} \in \mathcal{B}_C} H^m(U, \mathbb{I}_p(m)),$$

where $\mathcal{B}_C$ denotes the set of all open subgroups of $G$ containing $C$ (cf. [31, §I.2.2, Proposition 8]). Hence Proposition 3.1 yields the claim.

In combination with [3 Corollary 4.3(ii)], Proposition 3.1 implies the following.
Corollary 3.3. Let \((I, \leq)\) be a directed set, let \((G, \theta)\) be a p-oriented profinite group, and let \((N_i)_{i \in I}\) be a family of closed normal subgroups of \(G\) satisfying \(N_j \subseteq N_i \subseteq \ker(\theta)\) for \(i \leq j\) such that \(\bigcap_{i \in I} N_i = \{1\}\) and the induced p-orientation \(\theta_i: G/N_i \to \mathbb{Z}_p^\times\) is cyclotomic for all \(i \in I\). Then \(\theta: G \to \mathbb{Z}_p^\times\) is cyclotomic.

Proof. Let \(U \subseteq G\) be a open subgroup of \(G\). Hypothesis (iii) implies that the group \(H_m(U N_i/N_i, \mathbb{Z}_p(-m))\) is torsion free for all \(i \in I\) (cf. Proposition [3.1]). Thus, by [3, Corollary 4.3(ii)], \(H_m(U, \mathbb{Z}_p(-m))\) is torsion free, and hence, by Proposition [3.1] \(\theta: G \to \mathbb{Z}_p^\times\) is a cyclotomic p-orientation. 

3.2. The mod-p cohomology ring

An \(N_0\)-graded \(\mathbb{F}_p\)-algebra \(A = \coprod_{k \geq 0} A_k\) is said to be anti-commutative if for \(x \in A_s\) and \(y \in A_t\) one has \(y \cdot x = (-1)^{st} \cdot x \cdot y\). E.g., if \(V\) is an \(\mathbb{F}_p\)-vector space, the exterior algebra \(\Lambda(V)\) (cf. [18, Chapter 4]) is an \(N_0\)-graded anti-commutative \(\mathbb{F}_p\)-algebra. Moreover, if \(G\) is a profinite group, then its cohomology ring \(H^*(G, \mathbb{F}_p)\) is an \(N_0\)-graded anti-commutative \(\mathbb{F}_p\)-algebra (cf. [23, Prop. 1.4.4]).

Let \(T(V) = \coprod_{k \geq 0} V^\otimes k\) denote the tensor algebra generated by the \(\mathbb{F}_p\)-vector space \(V\). A \(N_0\)-graded associative \(\mathbb{F}_p\)-algebra \(A\) is said to be quadratic if the canonical homomorphism \(\eta^A: T(A_1) \to A\) is surjective, and

\[
\ker(\eta^A) = T(A_1) \otimes \ker(\eta^A_2) \otimes T(A_1) \tag{3.1}
\]

(cf. [24, §1.2]). E.g., \(A = \Lambda(V)\) is quadratic.

If \(A\) and \(B\) are anti-commutative \(N_0\)-graded \(\mathbb{F}_p\)-algebras, then \(A \otimes B\) is again an anti-commutative \(N_0\)-graded \(\mathbb{F}_p\)-algebra, where

\[
(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (-1)^{s_2 t_1} \cdot (x_1 \cdot x_2) \otimes (y_1 \cdot y_2), \tag{3.2}
\]

for \(x_1 \in A_{s_1}\), \(x_2 \in A_{s_2}\), \(y_1 \in B_{t_1}\), \(y_2 \in B_{t_2}\). In particular, if \(A\) and \(B\) are quadratic, then \(A \otimes B\) is quadratic as well.

A direct set \((I, \leq)\) maybe considered as a small category with objects given by the set \(I\) and precisely one morphism \(\iota_{i,j}\) for all \(i \leq j\), \(i, j \in I\), i.e., \(\iota_{i,i} = id_i\). One has the following.

Fact 3.4. Let \(\mathbb{F}\) be a field, let \((I, \leq)\) be a direct system, and let \(A: (I, \leq) \to \mathbb{F}\text{-}\text{qalg}\) be a covariant functor with values in the category of quadratic \(\mathbb{F}\)-algebras. Then \(B = \varprojlim_{i \in I} A(i)\) is a quadratic \(\mathbb{F}\)-algebra.

Let \((G, \theta)\) be a p-oriented profinite group, and let \(\hat{\theta}: G \to \mathbb{F}_p^\times\) be the map induced by \(\theta\). If \(\hat{\theta} = 1_G\), then the mod-p cohomology ring of \(H^*(G, \hat{\theta})\) coincides with \(H^*(G, \mathbb{F}_p)\) (see (1.4)), and hence it is anti-commutative. Furthermore, if \(\hat{\theta} \neq 1_G\) and \(G^\circ = \ker(\theta)\), restriction

\[
\text{res}_{G, G^\circ}^*: H^*(G, \hat{\theta}) \to H^*(G^\circ, \mathbb{F}_p) \tag{3.3}
\]

is an injective homomorphism of \(N_0\)-graded algebras. Hence the mod-p cohomology ring \(H^*(G, \theta)\) is anti-commutative. In particular, if \(M_{(k)}\) denotes the homogeneous component of the left \(\mathbb{F}_p[G/G^\circ]\)-module \(M\), on which \(G/G^\circ\) acts
by $\hat{\theta}^k$, the Hochschild-Serre spectral sequence (cf. [23, § II.4, Exercise 4(ii)]) shows that
\[ H^k(G, \hat{\theta}) = H^k(G^\circ, \mathbb{F}_p)(-k). \] (3.4)

From [31, § I.2.2, Prop. 8] and Fact 3.4 one concludes the following.

**Corollary 3.5.** Let $(G, \theta)$ be a $p$-oriented profinite group which is Bloch-Kato. Then $H^\bullet(C, \hat{\theta}|_C)$ is quadratic for all $C$ closed in $G$.

**Corollary 3.6.** Let $(I, \preceq)$ be a directed set, let $(G, \theta)$ be a $p$-oriented profinite group, and let $(N_i)_{i \in I}$ be a family of closed normal subgroups of $G$, $N_j \subseteq N_i$ for $i \preceq j$, such that $\bigcap_{i \in I} N_i = \{1\}$ and $(G/N_i, \hat{\theta}_{N_i})$ is Bloch-Kato. Then $(G, \theta)$ is Bloch-Kato.

**Remark 3.7.** Let $G$ be a pro-$p$ group with minimal presentation
\[ G = \langle x_1, \ldots, x_d \mid [x_1, x_2][x_3, x_4], x_5 \rangle, \]
with $d \geq 5$. In [22, Ex. 7.3] and [21, § 4.3] it is shown that $G$ does not occur as maximal pro-$p$ Galois group of a field containing a primitive $p^{th}$-root of unity, relying on the properties of Massey products. It would be interesting to know whether $G$ admits a cyclotomic $p$-orientation $\theta: G \to \mathbb{Z}_p^\times$ such that $(G, \theta)$ is Bloch-Kato. By Theorem [1.1] a negative answer would provide a “Massey-free” proof of the aforementioned fact.

### 3.3. Fibre products

Let $(G_1, \theta_1), (G_2, \theta_2)$ be $p$-oriented profinite groups. The fibre product $(G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)$ denotes the pull-back of the diagram
\[ G_1 \xrightarrow{\theta_1} \mathbb{Z}_p^\times \quad \xleftarrow{\theta_2} \quad G_2 \]
\[ \downarrow \quad \searrow \quad \downarrow \theta \]
\[ G \quad \xrightarrow{\theta} \quad G \]
\[ \theta \]

**Remark 3.8.** By restricting to the suitable subgroups if necessary, for the analysis of a fibre product $(G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)$ one may assume that $\text{im}(\theta_1) = \text{im}(\theta_2)$. In particular, if $(G_2, \theta_2)$ is split $\theta_2$-abelian and $G_2 \simeq A \times \text{im}(\theta_2)$ for some free abelian pro-$p$ group $A$, then $G \simeq A \times G_1$ with $gag^{-1} = a^{\theta_1(g)}$ for all $a \in A$ and $g \in G_1$.

**Fact 3.9.** Let $(G, \theta)$ be a $p$-oriented profinite group, and let $N$ be a finitely generated non-trivial free closed subgroup of $\mathbb{Z}_p(G)$, i.e., $N \simeq \mathbb{Z}_p(1)^r$ as left $\mathbb{Z}_p[G]$-modules for some $r \geq 1$. Then for $k \geq 0$ one has
\[ H^1(N, \mathbb{L}_p(k)) \simeq \mathbb{L}_p(k-1)^r \] (3.6)
as left $\mathbb{Z}_p[G]$-module.

The following property will be useful for the analysis of fibre products.
Lemma 3.10. Let \((G_1, \theta)\) be a cyclotomically \(p\)-oriented profinite group, and set \((G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)\), where \((G_2, \theta_2)\) is split \(\theta_2\)-abelian with \(Z = Z_{\theta_2}(G_2)\). Let \(\pi : G \to G_1\) be the canonical projection, and let \(U \subseteq G\) be an open subgroup. Then \(U \simeq (Z \cap U) \times \pi(U)\).

Proof. Without loss of generality we may assume that \(Z \simeq \mathbb{Z}_p\), so that \(Z \cap U = \mathbb{Z}_p^k\) for some \(k \geq 0\). It suffices to show that there exists an open subgroup \(U_1\) of \(U\) satisfying \(Z \cap U_1 = \{1\}\) and \(\pi(U_1) = \pi(U)\).

By choosing a section \(\sigma : G_1 \to G\) (see Remark 3.8), one has a continuous homomorphism \(\tau = \sigma \circ \pi : G \to G_1\) and a continuous function \(\eta : G \to Z\) such that each \(g \in G\) can be uniquely written as \(g = \eta(g) \cdot \tau(g)\). In particular, for \(h, h_1, h_2 \in U\) and \(z \in Z \cap U = \mathbb{Z}_p^k\) one has

\[
\eta(z \cdot h) = z \cdot \eta(h) \quad \text{and} \quad \eta(h_1 \cdot h_2) = \eta(h_1) \cdot h_1 \eta(h_2). \tag{3.7}
\]

Let \(\eta_U = \chi \circ \eta|_U\), where \(\chi : Z \to \mathbb{Z}/\mathbb{Z}_p^k\) is the canonical projection. By (3.7), \(\eta_U\) defines a crossed-homomorphism \(\tilde{\eta}_U : \bar{U} \to \mathbb{Z}/\mathbb{Z}_p^k\), where \(\bar{U} = U/\mathbb{Z}_p^k\).

As \(\bar{U}\) is canonically isomorphic to an open subgroup of \(G_1\), \((\bar{U}, \theta_1|_\bar{U})\) is cyclotomically \(p\)-oriented. (Note that \(Z \simeq \mathbb{Z}_p(1)\) as \(\mathbb{Z}_p[U]\)-modules.) Hence, \(H^1_{\text{cts}}(U, \mathbb{Z}_p(1)) \to H^1(\bar{U}, \mathbb{Z}_p(1)/p^k)\) is surjective by Proposition 2.1 and the snake lemma applied to the commutative diagram

\[
\begin{array}{c}
0 \longrightarrow B^1(\bar{U}, Z) \longrightarrow Z^1(\bar{U}, Z) \longrightarrow H^1(\bar{U}, \mathbb{Z}_p(1)) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow B^1(\bar{U}, \mathbb{Z}/\mathbb{Z}_p^k) \longrightarrow Z^1(\bar{U}, \mathbb{Z}/\mathbb{Z}_p^k) \longrightarrow H^1(\bar{U}, \mathbb{Z}_p(1)/p^k) \longrightarrow 0
\end{array} \tag{3.8}
\]

where the left-side and right-side vertical arrows are surjective, shows that \(Z^1(\bar{U}, Z) \to Z^1(\bar{U}, \mathbb{Z}/\mathbb{Z}_p^k)\) is surjective. Thus there exists \(\eta_0 \in Z^1(\bar{U}, Z)\) such that \(\tilde{\eta}_U = \chi \circ \eta_0\). It is straightforward to verify that \(U_1 = \{\eta_0(\bar{h}) \cdot \sigma(\bar{h}) \mid \bar{h} \in \bar{U}\}\) is an open subgroup of \(G_1\) satisfying the requirements. \(\square\)

Theorem 3.11. Let \((G_1, \theta_1)\) be a cyclotomically \(p\)-oriented profinite group, and let \((G_2, \theta_2)\) be split \(\theta_2\)-abelian. Then \((G_1, \theta_1) \boxtimes (G_2, \theta_2)\) is cyclotomically \(p\)-oriented.

Remark 3.12. (a) If \(p\) is odd, then every \(\theta\)-abelian profinite group \((G, \theta)\) is split. However, a 2-oriented \(\theta\)-abelian profinite group \((G, \theta)\) is split if, and only if, it is cyclotomically 2-oriented (cf. Proposition 6.7).

(b) If \((G, \theta)\) is \(\theta\)-abelian and \(H \subseteq G\) is a closed subgroup, then \((H, \theta|_H)\) is also \(\theta\)-abelian.

Proof of Theorem 3.11. Put \((G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)\) and \(Z = Z_{\theta_2}(G_2)\). We may also assume that \(\text{im}(\theta_1) = \text{im}(\theta_2)\). As \((G_2, \theta_2)\) is split \(\theta_2\)-abelian, one has \(G = Z \times G_1\).

We first show the claim for \(Z \simeq \mathbb{Z}_p\). Let \(U\) be an open subgroup of \(G\). By Lemma 3.10, \((U, \theta|_U) \simeq (U_1, \theta_1) \boxtimes (U_2, \theta_2)\) where \(U_1\) is isomorphic to an open subgroup of \(G_1\) and \((U_2, \theta_2)\) is split \(\theta_2\)-abelian with \(N = \ker(\theta_2)\) open in \(Z\). As \(\text{cd}_p(N) = 1\), one has \(H^m(N, \mathbb{Z}_p(k)) = 0\) for \(m \geq 2\) and \(k \geq 0\).
Therefore, the $E_2$-term of the Hochschild-Serre spectral sequence associated to the short exact sequence of profinite groups

$$
\{1\} \longrightarrow N \longrightarrow U \longrightarrow U_1 \longrightarrow \{1\} \quad (3.9)
$$

and evaluated on the discrete $\mathbb{Z}_p[U]$-module $\mathbb{I}_p(k)$, is concentrated on the first and the second row. In particular, $d_{s,t}^{n,0} = 0$ for $r \geq 3$. As $\mathbb{I}_p(k)$ is inflated from $U_1$, one has $E_{2}^{s,0}(\mathbb{I}_p(k)) = E_{\infty}^{s,0}(\mathbb{I}_p(k))$ for $s \geq 0$ (cf. [23 Prop. 2.4.5]). Hence $d_{s,t}^{s,0} = 0$ for all $s, t \geq 0$, i.e., $E_{2}^{s,t}(\mathbb{I}_p(k)) = E_{\infty}^{s,t}(\mathbb{I}_p(k))$, and the spectral sequence collapses. Thus, using the isomorphism (3.6), for every $k \geq 1$ one has a short exact sequence

$$
0 \longrightarrow H^k(U_1, \mathbb{I}_p(k)) \xrightarrow{\text{inf}} H^k(U, \mathbb{I}_p(k)) \longrightarrow H^{k-1}(U_1, \mathbb{I}_p(k-1)) \longrightarrow 0,
$$

(3.10)

where the right- and left-hand side are divisible $p$-torsion modules. As such $\mathbb{Z}_p$-modules are injective, (3.10) splits showing that $H^k(U, \mathbb{I}_p(k))$ is $p$-divisible. Therefore, by Proposition 3.1, $(G, \theta)$ is cyclotomic.

Thus, by induction the claim holds for all split $\theta_2$-abelian groups $(G_2, \theta_2)$ satisfying $\text{rk}(Z_{\theta_2}(G_2)) < \infty$. In general, as $Z$ is a torsion free abelian pro-$p$ group, there exists an inverse system $(Z_i)_{i \in I}$ of closed subgroups of $Z$ such that $Z/Z_i$ is torsion free, of finite rank, and $Z = \varprojlim_{i \in I} Z/Z_i$. Since $Z_i$ is normal in $G$ and

$$(G/Z_i, \overline{\theta}) \simeq (G_1, \theta_1) \otimes (G_2/Z_i, \theta_2)$$

is cyclotomically $p$-oriented, Corollary 3.3 yields the claim. □

The following theorem can be seen as a generalization of a result of A. Wadsworth [37 Thm. 3.6].

**Theorem 3.13.** Let $(G_i, \theta_i)$, $i = 1, 2$, be $p$-oriented profinite groups satisfying $\text{im}(\theta_1) = \text{im}(\theta_2)$. Assume further that $(G_2, \theta_2)$ is split $\theta_2$-abelian. Then for $(G, \theta) = (G_1, \theta_1) \otimes (G_2, \theta_2)$ one has that

$$H^\bullet(G, \hat{\theta}) \simeq H^\bullet(G_1, \hat{\theta}_1) \otimes \Lambda^\bullet((\ker(\theta_2)/\ker(\theta_2)^p)^*) \cdot (3.11)$$

Moreover, if $(G_1, \theta_1)$ is Bloch-Kato, then $(G, \theta)$ is Bloch-Kato.

**Proof.** Assume first that $d(Z_{\theta_2}(G_2))$ is finite. If $d(Z_{\theta_2}(G_2)) = 1$ then one obtains the isomorphism (3.11) from [37 Thm. 3.1], which uses the Hochschild-Serre spectral sequence associated to the short exact sequence of profinite groups

$$
\{1\} \longrightarrow Z_{\theta_2}(G_2) \longrightarrow G \longrightarrow G/Z_{\theta_2}(G_2) \longrightarrow \{1\}
$$

and evaluated on the discrete $\mathbb{Z}_p[G]$-module $\mathbb{F}_p(k)$, to compute $H^\bullet(G, \hat{\theta})$. If $d(Z_{\theta_2}(G_2)) > 1$, then applying induction on $d(Z_{\theta_2}(G_2))$ yields the isomorphism (3.11). Finally, if $Z_{\theta_2}(G_2)$ is not finitely generated, then a limit argument similar to the one used in the proof Theorem 3.11 and Corollary 3.6 yield the claim. □
3.4. Coproducts

For two profinite groups $G_1$ and $G_2$ let $G = G_1 \amalg G_2$ denote the coproduct (or free product) in the category of profinite groups (cf. [27, § 9.1]). In particular, if $(G_1, \theta_1)$ and $(G_2, \theta_2)$ are two $p$-oriented profinite groups, the $p$-orientations $\theta_1$ and $\theta_2$ induce a $p$-orientation $\theta : G \to \mathbb{Z}_p^\times$ via the universal property of of the free product. Thus, we may interpret $\amalg$ as the coproduct in the category of $p$-oriented profinite groups (cf. [21, §3]). The same applies to $\amalg^p$ — the coproduct in the category of pro-$p$ groups.

Theorem 3.14. Let $(G_1, \theta_1)$ and $(G_2, \theta_2)$ be two cyclotomically $p$-oriented profinite groups. Then their coproduct $(G, \theta) = (G_1, \theta_1) \amalg (G_2, \theta_2)$ is cyclotomically oriented. Moreover, if $(G_1, \theta_1)$ and $(G_2, \theta_2)$ are Bloch-Kato, then $(G, \theta)$ is Bloch-Kato.

Proof. Let $(U, \theta|_U)$ be an open subgroup of $(G, \theta)$. Then, by the Kurosh subgroup theorem (cf. [27, Thm. 9.1.9]),

\[ U \simeq \coprod_{s \in \mathcal{S}_1} (sG_1 \cap U) \amalg \coprod_{t \in \mathcal{S}_2} (tG_2 \cap U) \amalg F, \tag{3.12} \]

where $yG_i = yG_i y^{-1}$ for $y \in G$. The sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are sets of representatives of the double cosets $U \setminus G/G_1$ and $U \setminus G/G_2$, respectively. In particular, the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are finite, and $F$ is a free profinite subgroup of finite rank.

Put $U_s = sG_1 \cap U$ for all $s \in \mathcal{S}_1$, and $V_t = tG_2 \cap U$ for all $t \in \mathcal{S}_2$. By [23, Thm. 4.1.4], one has an isomorphism

\[ H^k(U, \mathbb{I}_p(k)) \simeq \bigoplus_{s \in \mathcal{S}_1} H^k(U_s, \mathbb{I}_p(k)) \oplus \bigoplus_{t \in \mathcal{S}_2} H^k(V_t, \mathbb{I}_p(k)), \tag{3.13} \]

for $k \geq 2$, and an exact sequence

\[ M \xrightarrow{\alpha} H^1(U, \mathbb{I}_p(1)) \xrightarrow{} M' \xrightarrow{} 0. \tag{3.14} \]

If $(G_1, \theta_1)$ and $(G_2, \theta_2)$ are cyclotomically $p$-oriented, then, by hypothesis and [3.13], $H^k(U, \mathbb{I}_p(k))$ is a divisible $p$-torsion module for $k \geq 2$. In [3.14], the module $M$ is a homomorphic image of a $p$-divisible $p$-torsion module, and the module $M'$ is the direct sum of $p$-divisible $p$-torsion modules, showing that $H^1(U, \mathbb{I}_p(1))$ is divisible. Hence, by Proposition 3.1 and Corollary 3.3, $(G, \theta)$ is cyclotomically $p$-oriented.

Assume that $(G_1, \theta_1)$ and $(G_2, \theta_2)$ are Bloch-Kato. Then — for $U$ as in (3.12) — one has by (3.13) and (3.14) that

\[ H^\bullet(U, \tilde{\theta}|_U) \simeq \mathbb{A} \oplus \bigoplus_{s \in \mathcal{S}_1} H^\bullet(U_s, \tilde{\theta}|_{U_s}) \oplus \bigoplus_{t \in \mathcal{S}_2} H^\bullet(V_t, \tilde{\theta}|_{V_t}) \oplus H^\bullet(F, \tilde{\theta}|_F) \tag{3.15} \]

where $\mathbb{A}$ is a quadratic algebra, and $\oplus$ denotes the direct sum in the category of quadratic algebras (cf. [23, p. 55]). In particular, $H^\bullet(U, \tilde{\theta}|_U)$ is quadratic.

For pro-$p$ groups one has also the following.
Lemma 22.10.1) implies that the short exact sequence of profinite groups provided for all \( c \in \text{im}(\hat{\theta}) \) and for all \( g \in G^\circ \) one has

\[ \theta^\circ(\sigma(c) \cdot g \cdot \sigma(c)^{-1}) = \theta^\circ(g) \quad (4.2) \]

Proof. By (4.1), one has \( G = G^\circ \times_{\beta} \hat{\Sigma} \), where \( \hat{\Sigma} = \text{im}(\hat{\theta}) \), \( \beta: \hat{\Sigma} \to \text{Aut}(G^\circ) \) and \( \beta(c) \) is the 

4. Oriented virtual pro-\( p \) groups

We say that a \( p \)-oriented profinite group \((G, \theta)\) is an oriented virtual pro-\( p \) group if \( \ker(\theta) \) is a pro-\( p \) group. In particular, \( G \) is a virtual pro-\( p \) group. Since \( \mathbb{Z}^\times_p \) is a pro-2 group, every oriented virtual pro-2 group is in fact a pro-2 group. For \( p \neq 2 \) let \( \hat{\theta}: G \to \mathbb{F}_p^\times \) be the homomorphism induced by \( \theta \), and put \( G^\circ = \ker(\hat{\theta}) \). Then \( G/G^\circ \simeq \text{im}(\hat{\theta}) \) is a finite cyclic group of order co-prime to \( p \). The profinite version of the Schur-Zassenhaus theorem (cf. [14, Lemma 22.10.1]) implies that the short exact sequence of profinite groups

\[
\begin{array}{cccccc}
\{1\} & \longrightarrow & G^\circ & \longrightarrow & G & \longrightarrow \text{im}(\hat{\theta}) & \longrightarrow \{1\} \\
& & \sigma & \downarrow & \theta & \downarrow & \\
& & \{1\} & & \{1\} & & \\
\end{array}
\]

splits. Indeed, if \( C \subseteq G \) is a \( p' \)-Hall subgroup of \( G \), then \( \pi|_C: C \to \text{im}(\hat{\theta}) \) is an isomorphism, and \( \sigma = (\pi|_C)^{-1} \) is a canonical section for \( \hat{\theta} \).

Note that \( \mathbb{Z}_p^\times = \mathbb{F}_p^\times \times \Xi_p \), where \( \Xi_p = O_p(\mathbb{Z}_p^\times) \) is the pro-\( p \) Sylow subgroup of \( \mathbb{Z}_p^\times \), and where we denoted by \( \mathbb{F}_p^\times \) also the image of the Teichmüller section \( \tau: \mathbb{F}_p^\times \to \mathbb{Z}_p^\times \). Hence a \( p \)-orientation \( \theta: G \to \mathbb{Z}_p^\times \) on \( G \) defines a homomorphism \( \hat{\theta}: G \to \mathbb{F}_p^\times \) and also a homomorphism \( \theta^\vee: G \to \Xi_p \). On the contrary a pair of continuous homomorphisms \((\hat{\theta}, \theta^\vee)\), where \( \hat{\theta}: G \to \mathbb{F}_p^\times \) and \( \theta^\vee: G \to \Xi_p \), defines a \( p \)-orientation \( \theta: G \to \mathbb{Z}_p^\times \) given by \( \theta(g) = \hat{\theta}(g) \cdot \theta^\vee(g) \) for \( g \in G \).

Fact 4.1. Let \( \hat{\theta}: G \to \mathbb{F}_p^\times \), \( \sigma: \text{im}(\hat{\theta}) \to G \) be homomorphisms of groups satisfying (4.1). A homomorphism \( \theta^\circ: G^\circ \to \Xi_p \) defines a \( p \)-orientation \( \theta: G \to \mathbb{Z}_p^\times \), provided for all \( c \in \text{im}(\hat{\theta}) \) and for all \( g \in G^\circ \) one has

Proof. By (4.1), one has \( G = G^\circ \times_{\beta} \hat{\Sigma} \), where \( \hat{\Sigma} = \text{im}(\hat{\theta}) \), \( \beta: \hat{\Sigma} \to \text{Aut}(G^\circ) \) and \( \beta(c) \) is left conjugation by \( \sigma(c) \) for \( c \in \hat{\Sigma} \). Thus, by (4.2), the map \( \theta^\vee: G \to \Xi_p \) given by \( \theta^\vee(g, c) = \theta^\circ(g) \) is a continuous homomorphism of groups, and \((\iota, \theta^\vee)\), where \( \iota: \hat{\Sigma} \to \mathbb{F}_p^\times \) is the canonical inclusion, defines a \( p \)-orientation of \( G \).
Let \((G, \theta)\) be an oriented virtual pro-\(p\) group satisfying \((4.1)\). As \(\theta: G \to \mathbb{Z}_p^\times\) is a homomorphism onto an abelian group one has
\[
\theta(c \cdot g \cdot c^{-1}) = \theta(g)
\] (4.3)
for all \(c \in C = \text{im}(\sigma)\) and \(g \in G\). Thus, if \(i_c \in \text{Aut}(G)\) denotes left conjugation by \(c \in C\), one has
\[
\theta = \theta \circ i_c
\] (4.4)
for all \(c \in C\).

### 4.1. Oriented \(\Sigma\)-virtual pro-\(p\) groups

From now on let \(p\) be odd, and fix a subgroup \(\Sigma\) of \(\mathbb{F}_p^\times\). An oriented virtual pro-\(p\) group \((G, \theta)\) is said to be an oriented \(\Sigma\)-virtual pro-\(p\) group, if \(\text{im}(\hat{\theta}) = \Sigma\). Hence, by the previous subsection, for such a group one has a split short exact sequence
\[
\{1\} \longrightarrow G^\circ \longrightarrow G \xrightarrow{\sigma} \Sigma \longrightarrow \{1\}. \tag{4.5}
\]
By abuse of notation, we consider from now on \((G, \theta, \sigma)\) as an oriented \(\Sigma\)-virtual pro-\(p\) group. As the following fact shows there is also an alternative form of a \(\Sigma\)-virtual pro-\(p\) group.

**Fact 4.2.** Let \(\Sigma\) be a subgroup of \(\mathbb{F}_p^\times\). Let \(Q\) be a pro-\(p\) group, let \(\theta^\circ: Q \to \Xi_p\) be a continuous homomorphism, and let \(\gamma_Q: \Sigma \to \text{Aut}_c(Q)\) be a homomorphism of groups, where \(\text{Aut}_c(\_\) is the group of continuous automorphisms, satisfying
\[
\theta^\circ(\gamma_Q(c)(q)) = \theta^\circ(q), \tag{4.6}
\]
for all \(q \in Q\) and \(c \in \Sigma\), then \((Q \rtimes_{\gamma_Q} \Sigma, \theta, \iota)\) is an oriented \(\Sigma\)-virtual pro-\(p\) group, where \(\iota: \Sigma \to Q \rtimes_{\gamma_Q} \Sigma\) is the isomorphism map, and \(\theta: Q \rtimes_{\gamma_Q} \Sigma \to \mathbb{Z}_p^\times\) is the homomorphism induced by \(\theta^\circ\) (cf. Fact [4.1]).

If \((G_1, \theta_1, \sigma_1)\) and \((G_2, \theta_2, \sigma_2)\) are oriented \(\Sigma\)-virtual pro-\(p\) groups, a continuous group homomorphism \(\phi: G_1 \to G_2\) is said to be a morphism of \(\Sigma\)-virtual pro-\(p\) groups, if \(\sigma_2 = \phi \circ \sigma_1\) and \(\theta_1 = \theta_2 \circ \phi\). Similarly, if \((Q, \theta^\circ_Q, \gamma_Q)\) and \((R, \theta^\circ_R, \gamma_R)\) are \(\Sigma\)-virtual pro-\(p\) groups in alternative form (cf. Fact [4.2]), the continuous group homomorphism \(\phi: Q \to R\) is a homomorphism of \(\Sigma\)-virtual pro-\(p\) groups provided \(\theta_R \circ \phi = \theta_Q\) and if for all \(c \in \Sigma\) and for all \(q \in Q\) one has that
\[
\gamma_R(c)(\phi(q)) = \phi(\gamma_Q(c)(q)). \tag{4.7}
\]

With this slightly more sophisticated set-up the category of \(\Sigma\)-virtual pro-\(p\) groups admits coproducts. In more detail, let \((Q, \theta^\circ_Q, \gamma_Q)\) and \((R, \theta^\circ_R, \gamma_R)\) be \(\Sigma\)-virtual pro-\(p\) groups in alternative form. Put \(X = Q \amalg^p R\). Then for every element \(c \in \Sigma\) there exists an element \(\delta(c) \in \text{Aut}(X)\) making the
diagram

\[
\begin{array}{c}
Q \xrightarrow{\iota_1} X \xleftarrow{\iota_2} R \\
\gamma_Q(c) \downarrow \delta(c) \Downarrow \gamma_R(c) \\
Q \xrightarrow{\iota_1} X \xleftarrow{\iota_2} R
\end{array}
\]

(4.8)

commute. Since \(\Xi_p\) is a pro-\(p\) group, there exists a continuous group homomorphism \(\theta^\circ : X \to \Xi_p\) making the lower two rows of the diagram

\[
\begin{array}{c}
Q \xrightarrow{j_Q} X \xleftarrow{j_R} R \\
\gamma_Q(c) \downarrow \delta(c) \Downarrow \gamma_R(c) \\
Q \xrightarrow{j_Q} X \xleftarrow{j_R} R
\end{array}
\]

(4.9)

commute. Since \(\theta^\circ_{Q/R} = \theta^\circ_{Q/R} \circ \gamma_{Q/R}(c)\) for all \(c \in \bar{\Sigma}\), one has \(\theta^\circ = \theta^\circ \circ \delta(c)\) for all \(c \in \bar{\Sigma}\). The commutativity of the diagram (4.9) yields that the group homomorphisms \(j_Q : (Q, \theta^\circ_Q, \gamma_Q) \to (X, \theta^\circ, \delta)\) and \(j_R : (R, \theta^\circ_R, \gamma_R) \to (X, \theta^\circ, \delta)\) are homomorphisms of oriented \(\Sigma\)-virtual pro-\(p\) groups in alternative form. Moreover, one has the following.

**Proposition 4.3.** The oriented \(\Sigma\)-virtual pro-\(p\) group \((X, \theta^\circ, \delta)\) together with the homomorphisms \(j_Q : Q \to X\), and \(j_R : R \to X\) is a coproduct in the category of oriented \(\Sigma\)-virtual pro-\(p\) groups.

**Proof.** Let \((H, \theta_H, \gamma_H)\) be an oriented \(\Sigma\)-virtual pro-\(p\) group in alternative form, and let \(\phi_Q : Q \to H\) and \(\phi_R : R \to H\) be homomorphisms of oriented \(\Sigma\)-virtual pro-\(p\) groups in alternative form. Then there exists a unique homomorphism of pro-\(p\) groups \(\phi : X \to H\) making the diagram concentrated on the second and third row of

\[
\begin{array}{c}
Q \xrightarrow{j_Q} X \xleftarrow{j_R} R \\
\gamma_Q(c) \downarrow \delta(c) \Downarrow \gamma_R(c) \\
Q \xrightarrow{j_Q} X \xleftarrow{j_R} R
\end{array}
\]

(4.10)

commute. Since \(\phi_{Q/R} \circ \gamma_{Q/R}(c) = \gamma_H(c) \circ \phi_{Q/R}\) for all \(c \in \bar{\Sigma}\), the uniqueness of \(\phi\) implies that \(\phi \circ \delta(c) = \gamma_H(c) \circ \phi\) for all \(c \in \bar{\Sigma}\). As \(\phi_Q : Q \to H\) and \(\phi_R : R \to H\) are homomorphisms of \(\bar{\Sigma}\)-virtual pro-\(p\) groups, one has that
\( \theta_{Q/R}^\circ = \theta_H^\circ \circ \phi_{Q/R} \). This implies that \((\theta_H^\circ \circ \phi) \circ j_{Q/R} = \theta_{Q/R}^\circ \), and from the construction of \( \theta^\circ \): \( X \to \Xi_p \) one concludes that \( \theta^\circ = \theta_H^\circ \circ \phi \). This implies that \( \phi \) is a homomorphism of oriented \( \Sigma \)-virtual pro-\( p \) groups. \( \square \)

**Example 4.4.** For \( p = 3 \) set \( \Sigma = \mathbb{F}_3^\times = \{1, s\} \). Then the free product \((\mathbb{Z}_3^\times, \text{id}) \amalg (\mathbb{Z}_3^\times, \text{id})\) is isomorphic to \( F \times \Sigma \), where \( F = \langle x, y \rangle \) is a free pro-3 group of rank 2 and the induced isomorphism \( s: F \to F \) satisfies \( s(x) = x^{-1}, s(y) = y^{-1} \).

**Proposition 4.5.** Let \((Q, \theta_Q, \gamma_Q)\) be an oriented \( \Sigma \)-virtual pro-\( p \) group, and let \( Z \) be a normal \( \Sigma \)-invariant subgroup of \( Q \) isomorphic to \( \mathbb{Z}_p \), which is not contained in the Frattini subgroup \( \Phi(Q) = \text{cl}([Q, Q]_{\mathbb{Q}^p}) \) of \( Q \). Then there exists a maximal closed subgroup \( M \) of \( Q \) which is \( \Sigma \)-invariant, such that \( M \cdot Z = Q \) and \( M \cap Z = Z_p \).

**Proof.** Let \( \bar{Q} = Q/\Phi(Q) \). Then \( \gamma_Q \) induces a homomorphism \( \bar{\gamma}_Q: \bar{\Sigma} \to \text{Aut}_c(\bar{Q}) \) making \( \bar{Q} \) a compact \( \mathbb{F}_p[\Sigma] \)-module. Let \( \Omega = \text{Hom}_c(\bar{Q}, \mathbb{F}_p) \), where \( \mathbb{F}_p \) denotes the finite field \( \mathbb{F}_p \) with canonical left \( \Sigma \)-action. By Pontryagin duality, one has \( \bigcap_{\omega \in \bar{\Omega}} \ker(\omega) = \{0\} \). Thus, by hypothesis, there exists \( \psi \in \bar{\Omega} \) such that \( \psi|_Z \neq 0 \). Hence \( M = \ker(\psi) \) has the desired properties. \( \square \)

### 4.2. The maximal oriented virtual pro-\( p \) quotient

For a prime \( p \) and a profinite group \( G \) we denote by \( O^p(G) \) the closed subgroup of \( G \) generated by all Sylow pro-\( \ell \) subgroups of \( G \), \( \ell \neq p \). In particular, \( O^p(G) \) is \( p \)-perfect, i.e., \( H^1(O^p(G), \mathbb{F}_p) = 0 \), and one has the short exact sequence

\[
\begin{align*}
\{1\} &\longrightarrow O^p(G) \longrightarrow G \longrightarrow G(p) \longrightarrow \{1\},
\end{align*}
\]

where \( G(p) \) denotes the maximal pro-\( p \) quotient of \( G \).

For a \( p \)-oriented profinite group \((G, \theta)\), we denote by

\[
G(\theta) = G/O^p(G^\circ)
\]

the maximal \( p \)-oriented virtual pro-\( p \) quotient of \( G \) (for the definition of \( G^\circ \) see the beginning of §[3]. By construction, it carries naturally a \( p \)-orientation \( \theta: G(\theta) \to \mathbb{Z}_p^\times \) inherited by \( G \).

Note that if \( \text{im}(\theta) \) is a pro-\( p \) group, then \( G^\circ = G \), and \( G(\theta) = G(p) \).

**Proposition 4.6.** Let \((G, \theta)\) be a \( p \)-oriented Bloch-Kato profinite group, and let \( O \subseteq G \) be a \( p \)-perfect subgroup such that \( O \subseteq \ker(\theta) \). Then the inflation map

\[
\text{inf}^k(M): H^k_{\text{cts}}(G/O, M) \longrightarrow H^k_{\text{cts}}(G, M),
\]

is an isomorphism for all \( k \geq 0 \) and all \( M \in \text{ob}(\mathbb{Z}_p[G/O]_{\text{prf}}) \), where \( \mathbb{Z}_p[G/O]_{\text{prf}} \) denotes the abelian category of profinite left \( \mathbb{Z}_p[G/O] \)-modules.

**Proof.** As \( O \subseteq \ker(\theta), Z_p(k) \) is a trivial \( \mathbb{Z}_p[O] \)-module for every \( k \in \mathbb{Z} \). Since \( O \) is \( p \)-perfect, and as the \( \mathbb{F}_p \)-algebra \( H^\bullet(O, \mathbb{F}_p) \) is quadratic, \( H^\bullet(O, \mathbb{F}_p) \) is 1-dimensional concentrated in degree 0. By Pontryagin duality, this is equivalent to \( H_k(O, \mathbb{F}_p) = 0 \) for all \( k > 0 \), where \( H_k(O, \mathbb{F}_p) \) denotes Galois homology
as defined by A. Brumer in [3]. Thus, the long exact sequence in Galois homology implies that \( H_k(O, \mathbb{Z}_p) = 0 \) for all \( k > 0 \).

Let \((P_\bullet, \partial_\bullet, \varepsilon)\) be a projective resolution of the trivial left \( \mathbb{Z}_p\llbracket G \rrbracket \)-module in the category \( \mathbb{Z}_p\llbracket G \rrbracket \text{prf} \). For a projective left \( \mathbb{Z}_p\llbracket G \rrbracket \)-module \( P \in \text{ob}(\mathbb{Z}_p\llbracket G \rrbracket \text{prf}) \) define

\[
def(P) = \text{def}_G^\mathbb{Z}/O(P) = \mathbb{Z}_p\llbracket G/O \rrbracket \hat{\otimes} P,
\]

where \( \hat{\otimes} \) denotes the completed tensor product as defined in [3]. Then, by the Eckmann-Shapiro lemma in homology, one has that

\[
H_k(\text{def}(P_\bullet), \text{def}(\partial_\bullet)) \simeq H_k(O, \mathbb{Z}_p).
\]

Hence, by the previously mentioned remark, \((\text{def}(P_\bullet), \text{def}(\partial_\bullet))\) is a projective resolution of \( \mathbb{Z}_p \) in the category \( \mathbb{Z}_p\llbracket G/O \rrbracket \text{prf} \).

Let \( M \in \text{ob}(\mathbb{Z}_p\llbracket G/O \rrbracket \text{prf}) \). Then for every projective profinite left \( \mathbb{Z}_p\llbracket G \rrbracket \)-module \( P \), one has a natural isomorphism

\[
\text{Hom}_{G/O}(\text{def}(P), M) \simeq \text{Hom}_G(P, M).
\]

Hence \( \text{Hom}_{G/O}(\text{def}(P_\bullet), M) \) and \( \text{Hom}_G(P_\bullet, M) \) are isomorphic co-chain complexes, and the induced maps in cohomology — which coincide with inf\(^*\)(M) — are isomorphisms.

**Corollary 4.7.** Let \((G, \theta)\) be a \( p \)-oriented profinite group which is Bloch-Kato, respectively cyclotomically oriented. Then the maximal oriented virtual pro-\( p \) quotient \((G(\theta), \theta)\) is Bloch-Kato, respectively cyclotomically oriented.

5. Profinite Poincaré duality groups and \( p \)-orientations

5.1. Profinite Poincaré duality groups

Let \( G \) be a profinite group, and let \( p \) be a prime number. Then \( G \) is called a \( p \)-Poincaré duality group of dimension \( d \), if

\[
(\text{PD}_1) \quad \text{cd}_p(G) = d;
\]

\[
(\text{PD}_2) \quad |H^k_{\text{cts}}(G, A)| < \infty \text{ for every finite discrete left } G\text{-module } A \text{ of } p\text{-power order;}
\]

\[
(\text{PD}_3) \quad H^k_{\text{cts}}(G, \mathbb{Z}_p\llbracket G \rrbracket) = 0 \text{ for } k \neq d, \text{ and } H^d_{\text{cts}}(G, \mathbb{Z}_p\llbracket G \rrbracket) \simeq \mathbb{Z}_p.
\]

Although quite different at first glance, for a pro-\( p \) group our definition of \( p \)-Poincaré duality coincides with the definition given by J-P. Serre in [31 §I.4.5]. However, some authors prefer to omit the condition (PD\(_2\)) in the definition of a \( p \)-Poincaré duality group (cf. [23 Chap. III, §7, Definition 3.7.1]).

For a profinite \( p \)-Poincaré duality group \( G \) of dimension \( d \) the profinite right \( \mathbb{Z}_p\llbracket G \rrbracket \)-module \( D_G = H^d_{\text{cts}}(G, \mathbb{Z}_p\llbracket G \rrbracket) \) is called the dualizing module. Since \( D_G \) is isomorphic to \( \mathbb{Z}_p \times \) as a pro-\( p \) group, there exists a unique \( p \)-orientation \( \partial_G : G \rightarrow \mathbb{Z}_p \times \) such that for \( g \in G \) and \( z \in D_G \) one has

\[
z \cdot g = z \cdot \partial_G(g) = \partial_G(g) \cdot z.
\]

We call \( \partial_G \) the dualizing \( p \)-orientation.
Let $^{\times}D_G$ denote the associated profinite left $\mathbb{Z}_p[G]$-module, i.e., setwise $^{\times}D_G$ coincides with $D_G$ and for $g \in G$ and $z \in ^{\times}D_G$ one has
\[ g \cdot z = z \cdot g^{-1} = \partial_G(g^{-1}) \cdot z. \]
For a profinite $p$-Poincaré duality group of dimension $d$ the usual standard arguments (cf. [2] §VIII.10 for the discrete case) provide natural isomorphisms
\begin{align*}
\text{Tor}^G_k(D_G, -) &\simeq H_{cts}^{d-k}(G, -), \\
\text{Ext}^k_G(^{\times}D_G, -) &\simeq H_{d-k}(G, -),
\end{align*}
\tag{5.1}
where $\text{Tor}^G(-, -)$ denotes the left derived functor of $\_\otimes_G\_\_\_\$, and $\text{Ext}^G(-, -)$ denotes the right derived functors of $\text{Hom}_G(-, -)$ in the category $\mathbb{Z}_p[G]_{\text{Prof}}$ (cf. [3]).

If $A$ is a discrete left $G$-module which is also a $p$-torsion module, then $A^*$ carries naturally the structure of a left (profinite) $\mathbb{Z}_p[G]$-module (cf. [27] p. 171]). Then, by [31, §I.3.5, Proposition 17], Pontryagin duality and [33 (3.4.5)], one obtains for every finite discrete left $\mathbb{Z}_p[G]$-module $A$ of $p$-power order that
\[ H_{cts}^d(G, A) \simeq \text{Hom}_G(A, I_G)^* \simeq \text{Hom}_G(I_G^*, A^*) \simeq (I_G^*)^\times \hat{\otimes}_G A, \tag{5.2} \]
where $I_G$ denotes the discrete left dualizing module of $G$ (cf. [31] §I.3.5]). In particular, by (5.1), $D_G \simeq (I_G^*)^\times$.

**Example 5.1.** Let $G_\mathbb{K}$ be the absolute Galois group of an $\ell$-adic field $\mathbb{K}$. Then $G_\mathbb{K}$ satisfies $p$-Poincaré duality of dimension $2$ for all prime numbers $p$. One has $I_G \simeq \mu_{p^\infty}(\mathbb{K})$ (cf. [31] II.5.2, Theorem 1]). Hence $^{\times}D_{G_\mathbb{K}} \simeq \mathbb{Z}_p(-1)$ with respect to the cyclotomic $p$-orientation $\theta_{\mathbb{K}, p}: G_\mathbb{K} \to \mathbb{Z}_p^\times$, i.e., $\partial_{G_\mathbb{K}} = \theta_{\mathbb{K}, p}$.

As we will see in the next proposition, the final conclusion in Example 5.1 is a consequence of a general property of Poincaré duality groups.

**Proposition 5.2.** Let $G$ be a $p$-Poincaré duality group of dimension $d$, and let $\theta: G \to \mathbb{Z}_p^\times$ be a cyclotomic $p$-orientation of $G$. Then $\theta^{d-1} = \partial_G$ and $^{\times}D_G \simeq \mathbb{Z}_p(1-d)$.

**Proof.** By (5.1) and the hypothesis, $H_{cts}^d(G, \mathbb{Z}_p(d-1)) \simeq D_G \hat{\otimes}_G \mathbb{Z}_p(d-1)$ is torsion free, and hence isomorphic to $\mathbb{Z}_p$. This implies $\partial_G = \theta^{d-1}$. \qed

### 5.2. Finitely generated $\theta$-abelian pro-$p$ groups

Recall that $(G, \theta)$ is said to be $\theta$-abelian if $\ker(\theta) = Z_\theta(G)$ and $Z_\theta(G)$ is $p$-torsion free — in particular $\ker(\theta)$ is an abelian pro-$p$ group. If $G$ is finitely generated then one has an isomorphism of left $\mathbb{Z}_p[G]$-modules $N \simeq \mathbb{Z}_p(1)^r$ for some non-negative integer $r$, and either $\Gamma = \text{im}(\theta)$ is a finite group of order coprime to $p$, or $\Gamma$ is a $p$-Poincaré duality group of dimension 1 satisfying $\partial_\Gamma = 1_\Gamma$ (cf. [23] Prop. 3.7.6]). Moreover, one has isomorphisms of left $\mathbb{Z}_p[G]$-modules
\[ H_k(N, \mathbb{Z}_p) \simeq \Lambda_k(N) \simeq \mathbb{Z}_p(k)^{(1)}, \tag{5.3} \]
where \( \Lambda_s(\_\_\_) \) denotes the exterior algebra over the ring \( \mathbb{Z}_p \). Since \( \text{cd}_p(\Gamma) \leq 1 \),
the Hochschild-Serre spectral sequence for homology (cf. [39, § 6.8])
\[
E^2_{s,t} = H_s(\Gamma, H_t(\mathbb{Z}_p(m))) \implies H_{s+t}(\mathbb{Z}_p(m))
\]
(5.4)
is concentrated in the first two columns. Hence, the spectral sequence collapses at the \( E^2 \)-term, i.e., \( E^2_{s,t} = E^\infty_{s,t} \). Thus, for \( n \geq 1 \) one has a short exact sequence
\[
0 \longrightarrow H_{n-1}(\mathbb{Z}_p(m))^{\Gamma} \longrightarrow H_n(\mathbb{Z}_p(m)) \longrightarrow H_n(\mathbb{Z}_p(m))^{\Gamma} \longrightarrow 0
\]
(5.5)
if \( \text{cd}_p(\Gamma) = 1 \), and isomorphisms
\[
H^n(\mathbb{Z}_p(m)) \cong H_n(\mathbb{Z}_p(m))^{\Gamma}
\]
(5.6)
if \( \Gamma \) is a finite group of order coprime \( p \). Here we used the fact that \( H_0(\Gamma, \_\_\_) = \_\_\_ \) coincides with the coinvariants of \( \Gamma \), and that \( H_1(\Gamma, \_\_\_) = \_\_\^\Gamma \) coincides with the invariants of \( \Gamma \) if \( \Gamma \) is a \( p \)-Poincaré duality group of dimension 1 with \( \delta_\Gamma = 1_\Gamma \). Since \( H_{n-1}(\mathbb{Z}_p(m))^{\Gamma} \) is a torsion free abelian pro-\( p \) group, and as
\[
H_n(\mathbb{Z}_p(m))^{\Gamma} = (H_n(\mathbb{Z}_p(m)) \otimes \mathbb{Z}_p(m))^{\Gamma} \cong \Lambda_m(\_\_\_)
\]
(5.7)
by (5.3), one concludes from (5.5) and (5.6) that \( H_m(\mathbb{Z}_p(m))^{\Gamma} \) is torsion free.

**Proposition 5.3.** Let \( (G, \theta) \) be a \( \theta \)-abelian \( p \)-oriented virtual pro-\( p \) group such that \( N = \ker(\theta) \) is a finitely generated torsion free abelian pro-\( p \) group, and
that \( \Gamma = \text{im}(\theta) \) is \( p \)-torsion free. Then \( G \) is a \( p \)-Poincaré duality group of dimension \( d = \text{cd}(G) \), and \( \theta \) is cyclotomic.

**Proof.** By hypothesis, \( G \) is a \( p \)-torsion free \( p \)-adic analytic group. Hence the former assertion is a direct consequence of M. Lazard’s theorem (cf. [33, Thm. 5.1.5]). The latter follows from Proposition 3.1. \( \square \)

From Proposition 5.2 one concludes the following:

**Corollary 5.4.** Let \( (G, \theta) \) be a \( \theta \)-abelian pro-\( p \) group. If \( p = 2 \) assume further that \( \text{im}(\theta) \) is torsion free.

(a) The orientation \( \theta \) is cyclotomic.

(b) Suppose that \( G \) is finitely generated with minimum number of generators \( d = d(G) < \infty \). If \( p = 2 \) assume further that \( \text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2 \). Then \( G \) is a Poincaré duality pro-\( p \) group of dimension \( d \). Moreover, \( \delta_G = \theta^{d-1} \).

(c) If \( G \) satisfies the hypothesis of (b) and \( d(G) \geq 2 \), then for \( p \) odd, any cyclotomic orientation \( \theta' : G \to \mathbb{Z}_p^\times \) of \( G \) must coincide with \( \theta \), i.e.,
\[
\theta' = \theta.
\]
For \( p = 2 \) any cyclotomic orientation \( \theta' : G \to \mathbb{Z}_2^\times \) satisfying \( \text{im}(\theta') \subseteq 1 + 4\mathbb{Z}_2 \) must coincide with \( \theta \).

**Proof.** (a) follows from Proposition 5.3.
(b) By hypothesis, \( G \) is uniformly powerful (cf. [6, Ch. 4]), or equi-\( p \)-value, as it is called in [17]. Hence the claim follows from Proposition 5.3. By Proposition 5.2, \( \delta_G = \theta^{d-1} \).
(c) An element $\phi \in \text{Hom}_{\text{grp}}(G, \mathbb{Z}_p^\times)$ has finite order if, and only if, $\text{im}(\phi)$ is finite. Proposition 5.2 and part (b) imply that
\[
\theta^{d-1} = \partial_G = (\theta')^{d-1}.
\]
Hence $(\theta^{-1}\theta')^{d-1} = 1_G$. For $p$ odd, $\text{Hom}_{\text{grp}}(G, \mathbb{Z}_p^\times)$ does not contain non-trivial elements of finite order. Hence $\theta' = \theta$. For $p = 2$ the hypothesis implies that $\text{im}(\theta^{-1}\theta') \subseteq 1 + 4\mathbb{Z}_2$. Hence $(\theta^{-1}\theta')^{d-1} = 1_G$ implies that $\theta' = \theta$. \qed

Note that, by Fact 2.2, Corollary 5.4(c) cannot hold if $d(G) = 1$.

5.3. Profinite $p$-Poincaré duality groups of dimension 2

As the following theorem shows, for a profinite $p$-Poincaré duality group $G$ of dimension 2, the dualizing $p$-orientation $\partial_G: G \to \mathbb{Z}_p^\times$ is always cyclotomic.

\begin{theorem}
Let $G$ be a profinite $p$-Poincaré duality group of dimension 2. Then $\partial_G: G \to \mathbb{Z}_p^\times$ is a cyclotomic $p$-orientation.
\end{theorem}

\begin{proof}
As every $p$-oriented profinite group is 0-cyclotomic, it suffices to show that $H^2_{\text{cts}}(U, \mathbb{Z}_p(1))$ is torsion free for every open subgroup $U \subseteq G$. By Proposition 5.2, $\mathbb{Z}_p(-1) \cong^\times D_G$. Hence, from the Eckmann-Shapiro lemma in homology and (5.1), one concludes that
\[
H_1(U, \mathbb{Z}_p(-1)) = \text{Tor}^U_1(\mathbb{Z}_p, \mathbb{Z}_p(-1)) \cong \text{Tor}^U_1(\mathbb{Z}_p(-1)^\times, \mathbb{Z}_p) \\
\cong \text{Tor}^U_1(D_G, \mathbb{Z}_p[G/U]) \cong H^1_{\text{cts}}(G, \mathbb{Z}_p[G/U]) \equiv \text{Hom}_{\text{grp}}(U, \mathbb{Z}_p).
\]
Hence $H_1(U, \mathbb{Z}_p(-1))$ is a torsion free $\mathbb{Z}_p$-module, and, by Proposition 3.1, $H^2_{\text{cts}}(U, \mathbb{Z}_p(1))$ is torsion free as well. \qed
\end{proof}

\begin{remark}
Let $G$ be a profinite $p$-Poincaré duality group of dimension 2, and let $\partial_G: G \to \mathbb{Z}_p^\times$ be the dualizing $p$-orientation. Then $(G, \partial_G)$ is not necessarily Bloch-Kato, as the following example shows.

Let $p = 2$ and let $A = \text{PSL}_2(q)$ where $q \equiv 3 \mod 4$. Then there exists a $p$-Frattini extension $\pi: G \to A$ of $A$ such that $G$ is a 2-Poincaré duality group of dimension 2, i.e., ker$(\pi)$ is a pro-2 group contained in the Frattini subgroup of $G$ (cf. [41]). In particular, $G$ is perfect, and thus $\partial_G = 1_G$. Hence $\mathbb{F}_2(1) = \mathbb{F}_2(0)$ is the trivial $\mathbb{F}_2[G]$-module, and — as $G$ is perfect — $H^1(G, \mathbb{F}_2(1)) = 0$. Moreover, $H^2(G, \mathbb{F}_2(2)) \cong \mathbb{F}_2$, as $G$ is a profinite 2-Poincaré duality group of dimension 2 with $\partial_G = 1_G$. Therefore, $H^*(G, 1_G)$ is not quadratic.

A pro-$p$ group $G$ which satisfies $p$-Poincaré duality in dimension 2 is also called a Demuškin group (cf. [23] Def. 3.9.9)). For this class of groups one has the following.

\begin{corollary}
Let $G$ be a Demuškin pro-$p$ group. Then $G$ is a Bloch-Kato pro-$p$ group, and $\partial_G: G \to \mathbb{Z}_p^\times$ is a cyclotomic $p$-orientation.
\end{corollary}
Proof. By Theorem 5.5, it suffices to show that \((G, \delta_G)\) is Bloch-Kato. It is well known that \(H^*(G, \hat{\delta}_G)\) is quadratic (cf. [31 §I.4.5]). Moreover, every open subgroup \(U\) of \(G\) is again a Demuškin group, with \(\hat{\delta}_U = \delta_G|_U\) (cf. [23 Thm. 3.9.15]). Hence \((G, \delta_G)\) is Bloch-Kato. □

Remark 5.8. [The Klein bottle pro-2 group] Let \(G\) be the pro-2 group given by the presentation
\[
G = \langle x, y \mid xyx^{-1}y = 1 \rangle \tag{5.9}
\]
Then \(G\) is a Demuškin pro-2 group containing the free abelian pro-2 group \(H = \langle x^2, y \rangle\) of rank 2. Thus, by Corollary 5.7 \((G, \delta_G)\) is cyclotomic. Since \(H^1(G, \mathbb{I}_2(0)) \simeq \mathbb{I}_2 \oplus \mathbb{Z}/2\mathbb{Z}\), Proposition 3.1 implies that \(\delta_G \neq 1_G\) is non-trivial. In particular, since \(\delta_G|_H = 1_H\), this implies that \(\text{im}(\delta_G) = \{ \pm 1 \}\).

Note that \(H = \ker(\delta_G)\) and that one has a canonical isomorphism
\[
H = \langle x^2 \rangle \oplus \langle y \rangle \simeq \mathbb{Z}_2(0) \oplus \mathbb{Z}_2(1). \tag{5.10}
\]
In particular, \((G, \delta_G)\) is not \(\delta_G\)-abelian.

Example 5.9. Let \(G\) be the pro-\(p\) group with presentation
\[
G = \langle x, y, z \mid [x, y] = z^{-p} \rangle.
\]
If \(p = 2\) then \(G\) is a Demuškin group, and \(\bar{\delta}_G : G \to \mathbb{Z}_2^\times\) is given by \(\bar{\delta}_G(x) = \bar{\delta}_G(y) = 1, \bar{\delta}_G(z) = -1\). On the other hand, if \(p \neq 2\) then \(G\) is not a Demuškin group, and any \(p\)-orientations \(\theta : G \to \mathbb{Z}_p^\times\) is not 1-cyclotomic (cf. [11 Thm. 8.1]). However, \(H^*(G, \hat{\theta})\) is still quadratic.

6. Torsion

It is well known that a Bloch-Kato pro-\(p\) group may have non-trivial torsion only if, \(p = 2\). More precisely, a Bloch-Kato pro-2 group \(G\) is torsion if, and only if, \(G\) is abelian and of exponent 2. Moreover, any such group is a Bloch-Kato pro-2 group (cf. [25 §2]). The following result — which appeared first in [26 Prop. 2.13] — holds for 1-cyclotomically oriented pro-\(p\) groups (see also [11 Ex. 3.5] and [5 Ex. 14.27]).

Proposition 6.1. Let \((G, \theta)\) be a 1-cyclotomically oriented pro-\(p\) group.

(a) If \(\text{im}(\theta)\) is torsion free, then \(G\) is torsion free.
(b) If \(G\) is non-trivial and torsion, then \(p = 2\), \(G \simeq C_2\) and \(\theta\) is injective.

Remark 6.2. Let \(\theta : C_2 \to \mathbb{Z}_2^\times\) be an injective homomorphism of groups. Then \(\mathbb{Z}_2(1) \simeq \omega_{C_2}\) is isomorphic to the augmentation ideal
\[
\omega_{C_2} = \ker(\mathbb{Z}_2[C_2] \to \mathbb{Z}_2).
\]
Hence — by dimension shifting —
\[
H^2(C_2, \mathbb{Z}_2(1)) = H^1(C_2, \mathbb{Z}_2(0)) = 0.
\]
Thus — as \(C_2\) has periodic cohomology of period 2 — one concludes that \(H^s(C_2, \mathbb{Z}_2(t)) = 0\) for \(s\) odd and \(t\) even, and also for \(s\) even and \(t\) odd. Hence \((C_2, \theta)\) is cyclotomic.
From Proposition 6.1 and the profinite version of Sylow’s theorem one concludes the following corollary, which can be seen as a version of the Artin-Schreier theorem for 1-cyclotomically \( p \)-oriented profinite groups.

**Corollary 6.3.** Let \( p \) be a prime number, and let \( (G, \theta) \) be a profinite group with a 1-cyclotomic \( p \)-orientation.

(a) If \( p \) is odd, then \( G \) has no \( p \)-torsion.

(b) If \( p = 2 \), then every non-trivial 2-torsion subgroup is isomorphic to \( C_2 \). Moreover, if \( \text{im}(\theta) \) has no 2-torsion, then \( G \) has no 2-torsion.

**Remark 6.4.** Let \( \theta : \mathbb{Z}_2 \to \mathbb{Z}_2^\times \) be the homomorphism of groups given by \( \theta(1 + \lambda) = -1 \) and \( \theta(\lambda) = 1 \) for all \( \lambda \in 2\mathbb{Z}_2 \). Then \( \theta \) is a 2-orientation of \( G = \mathbb{Z}_2 \) satisfying \( \text{im}(\theta) = \{\pm 1\} \). As \( cd_2(\mathbb{Z}_2) = 1 \), Fact 2.2 implies that \( (\mathbb{Z}_2, \theta) \) is Bloch-Kato and cyclotomically 2-oriented. However, \( \text{im}(\theta) \) is not torsion free.

### 6.1. Orientations on \( C_2 \times \mathbb{Z}_2 \)

As we have seen in Proposition 5.3 for \( p \) odd, every \( \theta \)-abelian oriented pro-\( p \) group is cyclotomically \( p \)-oriented. For \( p = 2 \), this is not true. Indeed, one has the following.

**Proposition 6.5.** Any 2-orientation \( \theta : G \to \mathbb{Z}_2^\times \) on \( G \cong C_2 \times \mathbb{Z}_2 \) is not 1-cyclotomic.

**Proof.** Suppose that \((G, \theta)\) is 1-cyclotomically 2-oriented. Let \( x, y \) be elements of \( G \) such that \( x^2 = 1 \) and \( \text{ord}(y) = 2^\infty \), and that \( x, y \) generate \( G \). Proposition 6.1 applied to the cyclic pro-2 group generated by \( x \) yields \( \theta(x) = -1 \). Put \( \theta(y) = 1 + 2\lambda \) for some \( \lambda \in \mathbb{Z}_2 \). By [16, Prop. 6], if \( \theta \) is 1-cyclotomic then for any pair of elements \( c_x, c_y \in \mathbb{Z}_2(1) \) there exists a continuous crossed-homomorphism \( c : G \to \mathbb{Z}_2(1) \) (i.e., a map satisfying \( c(g_1g_2) = c(g_1) + \theta(g_1)c(g_2) \), cf. [23, p. 15]) such that \( c(x) = c_x \), \( c(y) = c_y \). Set \( c_x = c_y = 1 \). Then one computes

\[
\begin{align*}
    c(xy) &= c_x + \theta(x)c_y = 1 - 1 = 0, \\
    c(yx) &= c_y + \theta(y)c_x = 1 + 1 + 2\lambda,
\end{align*}
\]

which yields \( \lambda = -1 \). The element \( xy \) has the same properties as \( y \). Hence the previously mentioned argument applied to the element \( xy \) yields \( \theta(xy) = 1 - 2 = -1 \), whereas \( \theta(xy) = \theta(x)\theta(y) = 1 \), a contradiction. \( \square \)

**Remark 6.6.** From Proposition 6.1 and Proposition 6.5 one deduces that in a 1-cyclotomically 2-oriented pro-2 group, every element of order 2 is self-centralizing, which is a remarkable property of absolute Galois groups (cf. [4, Prop. 2.3] and [19, Cor. 2.3]).

**Proposition 6.7.** Let \((G, \theta)\) be a \( \theta \)-abelian oriented pro-2 group. Then \( \theta \) is cyclotomic if, and only if, either

(a) \( \text{im}(\theta) \) is torsion free; or

(b) \( \text{im}(\theta) \) has order 2.

In both these cases \((G, \theta)\) is split \( \theta \)-abelian.
Proof. Assume first that \(\text{im}(\theta)\) is torsion free. Then the short exact sequence \(\{1\} \to \ker(\theta) \to G \to \text{im}(\theta) \to \{1\}\) splits, as \(\text{im}(\theta) \simeq \mathbb{Z}_2\) is a projective pro-2 group. Moreover, \((G, \theta)\) is cyclotomic by Proposition 5.3.

Second assume that \(\theta\) is cyclotomic, \(p = 2\) and that \(\text{im}(\theta) \supseteq \{\pm 1\}\). If \(g \in G\) satisfies \(\theta(g) = -1\), then \(g^2 \in \ker(\theta) = \mathbb{Z}_2(G)\), and consequently

\[
g^2 = g \cdot g^2 \cdot g^{-1} = (g^2)^{\theta(g)} = g^{-2},
\]
i.e., \(g^4 = 1\). Since \((\ker(\theta), 1)\) is cyclotomically 2-oriented, \(\ker(\theta)\) is torsion free, and one deduces that \(g^2 = 1\). Therefore, the short exact sequence

\[
\{1\} \longrightarrow H \longrightarrow G \longrightarrow C_2 \longrightarrow \{1\}
\]
splits (here \(H = \ker(\pi \circ \theta)\), where \(\pi\) is the canonical epimorphism \(\mathbb{Z}_2^\times \to \{\pm 1\}\)). Since \((H, \theta|_H)\) is again cyclotomically 2-oriented and as \(\text{im}(\theta|_H)\) is torsion free, \((H, \theta|_H)\) is split \(\theta|_H\)-abelian by the previously mentioned argument. We claim that \(H = \ker(\theta)\). Indeed, suppose there exists \(h \in H\) such that \(\theta(h) \neq 1\). Put \(\lambda = (1 + \theta(h))/2\) and let \(z = ghgh^{-1} = [g, h^{-1}] \in \ker(\theta)\). Then — as \(g = g^{-1}\) and \(\theta(g) = -1\) — one has

\[
g(z^\lambda h^2)g^{-1} = (gzg)^\lambda \cdot gh^2 g = z^{-\lambda} \cdot (ghg)^2 = z^{-\lambda} \cdot (ghgh^{-1} \cdot h)^2
\]

\[
= z^{-\lambda} \cdot (zhzh^{-1} \cdot h^2) = z^{-\lambda + 1 + \theta(h)}h^2 = z^\lambda h^2,
\]
i.e., \(g\) and \(z^\lambda h^2\) commute which implies that \(\langle g, z^\lambda h^2 \rangle \simeq C_2 \times \mathbb{Z}_p\) contradicting Proposition 6.5. Therefore, \(H = \ker(\theta)\) is a free abelian pro-2 group, and \(G \simeq H \times C_2\).

Finally, let \(p = 2\) and assume that \(\text{im}(\theta) = \{\pm 1\}\). By Remark 6.2 we may also assume that \(\ker(\theta)\) is non-trivial. Then, either Case I: \(\theta^{-1}(\{-1\})\) contains an element of order 2 and \((G, \theta)\) is split \(\theta\)-abelian, i.e., \(G \simeq \ker(\theta) \times C_2\) with \(\ker(\theta)\) a free abelian pro-2 group, or Case II: all elements in \(x \in \theta^{-1}(\{-1\})\) are of infinite order. Then for \(y \in \ker(\theta)\), the group \(K = \langle x, y \rangle\) must be isomorphic to the Klein bottle pro-2 group which is impossible as \(G\) is \(\theta\)-abelian and thus contains only \(\theta\)-abelian closed subgroups (cf. Remark 3.12 b)). Hence Case II is impossible.

By Lemma 3.10 if \(U \subseteq G\) is an open subgroup, then either \(U \subseteq \ker(\theta)\), or \(U \simeq V \times C_2\) for some open subgroup \(V\) of \(\ker(\theta)\). In the first case, \((U, 1)\) is cyclotomically 2-oriented by Proposition 5.3. For the second case, we claim that \(H^k(U, \mathbb{I}_2(k))\) is 2-divisible for all \(k \geq 1\).

Recall that \(\mathbb{Z}_2[\mathbb{I}_2]\) has periodic cohomology (of period 2), and that one has the equalities of \(\mathbb{Z}_2[\mathbb{I}]/\mathbb{I}\)-modules \(\mathbb{I}_2(k) = \mathbb{I}_2(0)\) for \(k\) even and \(\mathbb{I}_2(k) = \mathbb{I}_2(-1)\) for \(k\) odd. Moreover,

\[
\hat{H}^0(C_2, \mathbb{I}_2(0)) = \mathbb{I}_2(0)/C_2/\mathbb{I}_2(0) = \mathbb{I}_2(0)/2 \cdot \mathbb{I}_2(0) = 0,
\]

\[
\hat{H}^{-1}(C_2, \mathbb{I}_2(-1)) = \ker(N_{C_2})/\omega_{C_2} \mathbb{I}_2(-1) = \mathbb{I}_2(-1)/2 \cdot \mathbb{I}_2(-1) = 0,
\]

(6.1)
where $\hat{H}^k$ denotes Tate cohomology, $N_{C_2} = \sum_{x \in C_2} x \in \mathbb{Z}_2[C_2]$ is the norm element, and $\omega_{C_2}$ is the augmentation ideal of the group algebra $\mathbb{Z}_2[C_2]$ (cf. [23 § I.2]). Thus, by (6.1), one has
\begin{equation}
H^m(C_2, \mathbb{I}_2(m)) = \hat{H}^m(C_2, \mathbb{I}_2(m)) \simeq \hat{H}^k(C_2, \mathbb{I}_2(k)) = 0, \quad (6.2)
\end{equation}
for all positive integers $m > 0$ and $m \equiv k \pmod{2}$.

Suppose first that $V \simeq \mathbb{Z}_2$. As in the proof of Theorem 3.11 the $E_2$-term of the Hochschild-Serre spectral sequence associated to the short exact sequence $\{1\} \to V \to U \to C_2 \to \{1\}$ evaluated on $\mathbb{I}_2(k)$ is concentrated in the first and the second row. In particular, $d_2^{\bullet, \bullet} = 0$ and thus $E^{s,t}_2(\mathbb{I}_2(k)) = E^{\infty, t}_{(2)}(\mathbb{I}_2(k))$. Thus, by Fact 3.9 for every $k \geq 1$ one has a short exact sequence
\[ 0 \to H^k(C_2, \mathbb{I}_2(k)) \to H^k(U, \mathbb{I}_2(k)) \to H^{k-1}(C_2, \mathbb{I}_2(k-1)) \to 0, \]
and $H^k(C_2, \mathbb{I}_2(k)) = 0$ by (2.6). Hence, $(U, \theta|_U)$ is cyclotomically 2-oriented by Proposition 3.1. If $V \simeq \mathbb{Z}_2^n$ with $n > 1$, then $H^k(U, \mathbb{I}_2(k)) = 0$ by induction on $n$ and the previously mentioned argument. Finally, Corollary 3.3 yields the claim in case $V$ not finitely generated.

\section{Cyclotomically oriented pro-$p$ groups}

For a cyclotomically oriented pro-$2$ group $(G, \theta)$ satisfying $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ one has the following.

**Fact 7.1.** Let $(G, \theta)$ be a pro-2 group with a cyclotomic orientation satisfying $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Then $\chi \cup \chi = 0$ for all $\chi \in H^1(G, \mathbb{F}_2)$, i.e., the first Bockstein morphism $\beta^1: H^1(G, \mathbb{F}_2) \to H^2(G, \mathbb{F}_2)$ vanishes.

**Proof.** Since $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$, the action of $G$ on $\mathbb{F}_2(1)$ is trivial. The epimorphism of $\mathbb{Z}_2[G]$-modules $\mathbb{Z}_2(1)/4 \to \mathbb{F}_2$ induces a long exact sequence
\begin{equation}
\cdots \longrightarrow H^1(G, \mathbb{Z}_2(1)/4) \xrightarrow{\pi_{2,1}^1} H^k(G, \mathbb{F}_2) \xrightarrow{\beta^1} H^2(G, \mathbb{F}_2) \xrightarrow{2} H^2(G, \mathbb{Z}_2(1)/4) \longrightarrow \cdots
\end{equation}
where the connecting homomorphism is the first Bockstein morphism. Since $\theta$ is cyclotomic, the map $\pi_{2,1}^1$ is surjective, and thus $\beta^1$ is the 0-map. \hfill \Box

**Remark 7.2.** As before for a finitely generated pro-$p$ group $G$ let $d(G)$ denote its minimum number of generators. If $p$ is odd and $G$ is a finitely generated Bloch-Kato pro-$p$ group, the cohomology ring $(H^\bullet(G, \mathbb{F}_p), \cup)$ is a quotient of the exterior $\mathbb{F}_p$-algebra $\Lambda_\bullet = \Lambda_\bullet(H^\bullet(G, \mathbb{F}_p))$. In particular, $cd_p(G) \leq d(G)$. Moreover, $\Lambda_{d(G)}$ is the unique minimal ideal of $\Lambda_\bullet$. Hence equality of $cd_p(G)$ and $d(G)$ is equivalent to $H^\bullet(G, \mathbb{F}_p)$ being isomorphic to $\Lambda_\bullet$. It is well known that this implies that $G$ is uniformly powerful (cf. [33 Thm. 5.1.6]), and that there exists a $p$-orientation $\theta: G \to \mathbb{Z}_p^\times$ such that $G$ is $\theta$-abelian (cf. [25 Thm. 4.6]).
Let $p = 2$, and let $(G, \theta)$ be a cyclotomically oriented Bloch-Kato pro-$2$ group satisfying $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Then Proposition 7.1 implies that the cohomology ring $(\mathbb{H}^*(G, \mathbb{F}_2), \cup)$ is a quotient of the exterior $\mathbb{F}_2$-algebra $\Lambda_\bullet = \Lambda_\bullet(\mathbb{H}^1(G, \mathbb{F}_2))$, and hence $\text{cd}_2(G) \leq d(G)$. If $\text{cd}_2(G) = d(G)$, the previously mentioned argument, Proposition 7.1 and [42] imply that $G$ is uniformly powerful. Finally, [25, Thm. 4.11] yields that $G$ is $\theta'$-abelian for some orientation $\theta' : G \to \mathbb{Z}^\times_2$. Thus, if $d(G) \geq 2$, one has $\theta = \theta'$ by Corollary 5.4(c).

From the above remark and J-P. Serre’s theorem (cf. [30]) one concludes the following fact.

**Fact 7.3.** Let $(G, \theta)$ be a finitely generated cyclotomically oriented torsion free Bloch-Kato pro-$2$ group. Then $\text{cd}_2(G) < \infty$.

### 7.1. Tits’ alternative

From Remark 7.2 one concludes the following.

**Fact 7.4.** (a) Let $p$ be odd, and let $G$ be a Bloch-Kato pro-$p$ group satisfying $d(G) \leq 2$. Then $G$ is either isomorphic to a free pro-$p$ group, or $G$ is $\theta$-abelian for some orientation $\theta : G \to \mathbb{Z}^\times_p$.

(b) Let $p = 2$, and let $(G, \theta)$ be a cyclotomically oriented Bloch-Kato pro-$2$ group satisfying $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ and $d(G) \leq 2$. Then $G$ is either isomorphic to a free pro-$2$ group, or $G$ is $\theta$-abelian.

In [25, Thm. 4.6] it was shown, that for $p$ odd any Bloch-Kato pro-$p$ group satisfies a strong form of Tits’ alternative (cf. [35]), i.e., either $G$ contains a closed non-abelian free pro-$p$ subgroup, or there exists a $p$-orientation $\theta : G \to \mathbb{Z}^\times_p$ such that $G$ is $\theta$-abelian. Using the results from the previous subsection and [25, Thm. 4.11], one obtains the following version of Tits’ alternative if $p$ is equal to 2.

**Proposition 7.5.** Let $(G, \theta)$ be a cyclotomically oriented virtual pro-$2$ group which is also Bloch-Kato, such that $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Then either $G$ contains a closed non-abelian free pro-$2$ subgroup; or $G$ is $\theta$-abelian.

**Proof.** As $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$, Proposition 6.1(a) implies that $G$ is torsion free. From Proposition 7.1 one concludes that the first Bockstein morphism $\beta^1$ vanishes. Thus, the hypothesis of [25, Thm. 4.11] are satisfied (cf. Remark 7.2), and this yields the claim. \qed

**Remark 7.6.** Note that Proposition 7.5 without the hypothesis $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ does not remain true (cf. Remark 5.8).

### 7.2. The $\theta$-center

One has the following characterization of the $\theta$-center for a cyclotomically oriented Bloch-Kato pro-$p$ group $(G, \theta)$.

**Theorem 7.7.** Let $(G, \theta)$ be a cyclotomically oriented torsion free Bloch-Kato pro-$p$ group. If $p = 2$ assume further that $\text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Then $\mathbb{Z}_\theta(G)$ is the unique maximal closed abelian normal subgroup of $G$ contained in $\ker(\theta)$.
Proof. Let \( A \subseteq \ker(\theta) \) be a closed abelian normal subgroup of \( G \), let \( z \in A \), \( z \neq 1 \), and let \( x \in G \) be an arbitrary element. Put \( C = \text{cl}(\langle x, z \rangle) \subseteq G \). Then either \( C \cong \mathbb{Z}_p \) or \( C \) is a 2-generated pro-\( p \) group. Thus, by Fact 7.4, one has to distinguish three cases:

(i) \( d(C) = 1 \);
(ii) \( d(C) = 2 \) and \( C \) is isomorphic to a free pro-\( p \) group; or
(iii) \( d(C) = 2 \) and \( C \) is \( \theta' \)-abelian for some \( p \)-orientation \( \theta' : C \to \mathbb{Z}_p^\times \).

In case (i), \( x \) and \( z \) commute. If \( C \) is generated by \( z \), then \( C \subseteq \ker(\theta) \) and \( \theta(x) = 1 \). If \( C \) is generated by \( x \), then \( z = x^\lambda \) for some \( \lambda \in \mathbb{Z}_p \), and \( 1 = \theta(z) = \theta(x)^\lambda \). Hence \( \theta(x) = 1 \), as \( \text{im}(\theta) \) is torsion free. In both cases

\[
xyz^{-1} = z = z^{\theta(x)}.
\]

Case (ii) cannot hold: by hypothesis, \( A \cap C \neq \{1\} \), but free pro-\( p \) groups of rank 2 do not contain non-trivial closed abelian normal subgroups.

Suppose that case (iii) holds. Then \( \theta' = \theta|_C \) by Corollary 5.4(c), and \( z \in \ker(\theta|_C) = \mathbb{Z}_{\theta|_C}(C) \). Therefore,

\[
xyz^{-1} = z^{\theta|_C(x)} = z^{\theta(x)}.
\]

Hence we have shown that for all \( z \in A \) and all \( x \in G \) one has that

\[
xyz^{-1} = z^{\theta(x)}.
\]

This yields the claim. \( \square \)

The above result can be seen as the group theoretic generalization of [12, Corollary 3.3] and [13, Thm. 4.6]. Note that in the case \( p = 2 \) the additional hypothesis in Theorem 7.7 is necessary (cf. Remark 5.8). Indeed, if \( G \) is the Klein bottle pro-2 group then \( \langle x^2 \rangle \) is another maximal closed abelian normal subgroup of \( G \) contained in \( \ker(\partial_G) \).

Remark 7.8. Let \( \mathbb{K} \) be a field containing a primitive \( p^{th} \)-root of unity. Theorem 7.7 together with [12, Thm. 3.1] and [13, Thm. 4.6], implies that the \( \theta_{\mathbb{K},p} \)-center of the maximal pro-\( p \) Galois group \( G_{\mathbb{K}}(p) \) is the inertia group of the maximal \( p \)-henselian valuation admitted by \( \mathbb{K} \).

7.3. Isolated subgroups

Let \( G \) be a pro-\( p \) group, and let \( S \subseteq G \) be a closed subgroup of \( G \). Then \( S \) is called isolated, if for all \( g \in G \) for which there exists \( k \geq 1 \) such that \( g^p^k \in S \) follows that \( g \in S \). Hence a closed normal subgroup \( N \) of \( G \) is isolated if, and only if, \( G/N \) is torsion free.

Proposition 7.9. Let \( (G, \theta) \) be an oriented Bloch-Kato pro-\( p \) group. In the case \( p = 2 \) assume further that \( \text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2 \) and that \( \theta \) is 1-cyclotomic. Then \( \mathbb{Z}_\theta(G) \) is an isolated subgroup of \( G \).

Proof. Suppose there exists \( x \in G \setminus \mathbb{Z}_\theta(G) \) and \( k \geq 1 \) such that \( x^{p^k} \in \mathbb{Z}_\theta(G) \). By changing the element \( x \) if necessary, we may assume that \( k = 1 \), i.e., \( x^p \in \mathbb{Z}_\theta(G) \). As \( G \) is torsion free (cf. Corollary 6.3), one has that \( x^p \neq 1 \).

For an arbitrary \( g \in G \), the subgroup \( C(g) = \text{cl}(\langle g, x \rangle) \subseteq G \) is not free, as \( gx^pg^{-1} = x^{p\theta(g)} \). Thus, from Fact 7.4 one concludes that \( C(g) \) is
\( \theta_{\left( C(g) \right)} \)-abelian. Moreover, as \( \text{im}(\theta) \) is torsion-free, \( \theta(x^p) = \theta(x)p = 1 \) implies that

\[
x \in \ker(\theta|_{\left( C(g) \right)}) = Z_{\theta|_{\left( C(g) \right)}}(C(g)).
\]

Thus, \( x \in \bigcap_{g \in G} Z_{\theta|_{\left( C(g) \right)}}(C(g)) \subseteq Z_{\theta}(G). \)

Proposition \ref{thm:2} generalises to profinite groups as follows.

**Corollary 7.10.** Let \((G, \theta)\) be a torsion free \( p \)-oriented Bloch-Kato profinite group. For \( p = 2 \) assume also that \( \text{im}(\theta) \subseteq 1+4\mathbb{Z}_2 \) and that \( \theta \) is 1-cyclotomic. Then \( Z_{\theta}(G) \) is an isolated subgroup of \( G \).

**Proof.** Let \( x \in Z_{\theta}(G), y \in G \) and \( n \in \mathbb{N} \) such that \( x = y^n \). Then \( Y = \text{cl}(\langle y \rangle) \) is pro-cyclic and virtually pro-\( p \). Thus, as \( G \) is torsion free by hypothesis, \( Y \) is a cyclic pro-\( p \) group, and \( n \) is a \( p \)-power. Let \( P \in \text{Syl}_p(G) \) be a \( p \)-Sylow subgroup of \( G \) containing \( Y \). Then \((P, \theta|_P)\) satisfies the hypothesis of Proposition \ref{thm:2} which yields the claim. \( \square \)

**7.4. Split extensions**

**Proposition 7.11.** Let \((G, \theta)\) be a \( p \)-oriented Bloch-Kato pro-\( p \) group of finite cohomological dimension satisfying \( \text{im}(\theta) \subseteq 1 + p\mathbb{Z}_p \) (resp. \( \text{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2 \) if \( p = 2 \)), and let \( Z \) be a closed normal subgroup of \( G \) isomorphic to \( \mathbb{Z}_p \) such that \( G/Z \) is torsion free. Then \( Z \not\subseteq G^p[G, G] \).

**Proof.** Let \( d = \text{cd}_p(G) \). As \( \text{cd}(Z) = 1 \), and as \( H^1(Z, \mathbb{F}_p) \simeq \mathbb{F}_p \), one has \( \text{vcd}_p(G/Z) = d - 1 \) (cf. [33]). Thus, as \( G/Z \) is torsion free, J-P. Serre’s theorem (cf. [33]) implies that \( \text{cd}_p(G/Z) = d - 1 \).

Suppose that \( Z \not\subseteq G^p[G, G] \). Then \( \inf_{G, Z}^1: H^1(G/Z, \mathbb{F}_p) \rightarrow H^1(G, \mathbb{F}_p) \) is an isomorphism. For \( \chi \in H^1(G, \mathbb{F}_p) \), set \( \bar{\chi} \in H^1(G/Z, \mathbb{F}_p) \) such that \( \chi = \inf_{G, Z}^1(\bar{\chi}) \). Then, by [23] Prop. 1.5.3 one has

\[
\chi_1 \cup \ldots \cup \chi_k = \inf_{G, Z}^1(\bar{\chi}_1) \cup \ldots \cup \inf_{G, Z}^1(\bar{\chi}_k) = \inf_{G, Z}^k(\bar{\chi}_1 \cup \ldots \cup \bar{\chi}_k)
\]

for any \( \chi_1, \ldots, \chi_k \in H^1(G, \mathbb{F}_p) \), i.e.,

\[
\inf_{G, Z}^k: H^k(G/Z, \mathbb{F}_p) \longrightarrow H^k(G, \mathbb{F}_p)
\]

is surjective for all \( k \geq 0 \). Let

\[
(E_{r}^{st}, d_r) \Rightarrow H^{s+t}(G, \mathbb{F}_p), \quad E_2^{st} = H^s(G/Z, H^t(Z, \mathbb{F}_p))
\]

denote the Hochschild-Serre spectral sequence associated to the extension of pro-\( p \) groups \( Z \rightarrow G \rightarrow G/Z \) with coefficients in the discrete \( G \)-module \( \mathbb{F}_p \).

We claim that \( E_\infty^{st} \) is concentrated on the bottom row, i.e., \( E_\infty^{st} = 0 \) for all \( t \geq 1 \). Since \( \text{cd}_p(Z) = 1 \) and \( \text{cd}_p(G/Z) = d - 1 \), one has \( E_2^{st} = 0 \) for \( t \geq 2 \) or \( s \geq d \). Hence, \( d_r^{st} \) is the 0-map for every \( s, t \geq 0 \) and \( r \geq 3 \), i.e., \( E_\infty^{st} \simeq E_3^{st} \).

The total complex \( \text{tot}_\bullet(E_{\infty}^{\bullet \bullet}) \) of the graded \( \mathbb{F}_p \)-bialgebra \( E_{\infty}^{\bullet \bullet} \) coincides with \( H^\bullet(G, \mathbb{F}_p) \), which is quadratic by hypothesis. Thus \( E_{\infty}^{\bullet \bullet} \) is generated by

\[
\text{tot}_1(E_{\infty}^{\bullet \bullet}) = E_{1,0}^{\infty} = E_{2,0}^{1,0}.
\]

Hence, \( E_3^{st} = 0 \) for \( t \geq 1 \).
On the other hand, \( H^1(Z, \mathbb{F}_p) \) is a trivial \( G/Z \)-module isomorphic to \( \mathbb{F}_p \), and thus, as \( \text{cd}_p(G/Z) = d - 1 \), one has
\[
E_2^{d-1,1} = H^{d-1}(G/Z, H^1(Z, \mathbb{F}_p)) \neq 0.
\]
Moreover, \( d_2^{d-1,1} \) is the 0-map, thus \( E_3^{d-1,1} = \ker(d_2^{d-1,1}) = E_\infty^{d-1,1} \neq 0 \), a contradiction, and this yields the claim. \( \square \)

Proposition 7.11 has the following consequence.

**Proposition 7.12.** Let \((G, \theta)\) be a \( p \)-oriented Bloch-Kato pro-

Hence there exists a maximal closed subgroup \( C_1 \) of \( G \) such that
\[
C_1 Z = G \quad \text{and} \quad Z_1 = C_1 \cap Z = Z^p.
\]
Moreover, \( Z_1 \) is a closed normal subgroup in \( C_1 \) such that \( C_1/Z_1 \) is torsion free and \( Z_1 \cong Z_p \). From Proposition 7.11 again, one concludes that \( Z_1 \not\subseteq \Phi(C_1) \).

Thus repeating this process one finds open subgroup \( C_k \) of \( G \) of index \( p \) such that
\[
C_k Z = G \quad \text{and} \quad Z_k = C_k \cap Z = Z^{p^k}.
\]
Hence \( C = \bigcap_{k \geq 1} C_k \) is a \( Z \)-complement in \( G \).

If \( \hat{G} \) is a \( p \)-oriented virtual pro-

G is a maximal subgroup \( C_1 \) and hence all closed subgroups \( C_k \) can be chosen to be \( \hat{\Sigma} \)-invariant (cf. Proposition 4.5). Hence \( C = \bigcap_{k \in \mathbb{N}} C_k \) carries canonically a left \( \hat{\Sigma} \)-action, and thus defines a \( Z \)-complement \( H = C \rtimes \hat{\Sigma} \) in \( G \). \( \square \)

The proof of Theorem 1.2 can be deduced from Proposition 7.12 as follows.

**Proof of Theorem 1.2.** Assume first that \( G \) is either pro-

To prove statement (i) (and (ii)), we proceed by induction on \( d = \text{cd}_p(G) = \text{cd}(G) \). For \( d = 1 \), \( G \) is free (resp. virtually free) (cf. [23, Prop. 3.5.17]), and thus \( Z_\theta(G) = \{1\} \). So assume that \( d \geq 1 \), and that the claim holds for \( d - 1 \). Note that \( Z_\theta(G) \) is a finitely generated abelian pro-

If \( d_\circ = 0 \), there is nothing to prove. If \( d_\circ \geq 1 \), \( Z_\theta(G) \) contains an isolated closed subgroup \( Z \) satisfying \( d(Z) = 1 \). By definition, \( Z \) is normal in \( G \). Hence Proposition 7.12 implies that there exists a subgroup \( C \subseteq G \) satisfying
\( C \cap Z = \{1\} \) and \( C Z = G \). As \( C \simeq G/Z \), the main result of [43] implies that \( \text{cd}(C) = \text{vcd}(C) = d - 1 \). Since \( \mathbb{Z}_\theta|_C(C) Z = \mathbb{Z}_\theta(G) \), the claim then follows by induction.

To prove statement (iii), let \( G^\circ = \ker(\hat{\theta}: G \to \mathbb{F}_p^\times) \) and \( \bar{G}^\circ = \ker(\hat{\theta}: \bar{G} \to \mathbb{F}_p^\times) \), and put \( \bar{O} = O^p(\bar{G}^\circ) \) and

\[
\bar{O} = \{ g \in G^\circ \mid g\mathbb{Z}_\theta(G) \in \bar{O}^p(\bar{G}) \}.
\]

(7.6)

Then, by construction, \( \text{im}(\hat{\theta}|_\bar{O}) \) is a pro-\( p \) group and hence trivial. In particular, the left \( \mathbb{F}_p[\bar{O}] \)-module \( \mathbb{F}_p(1) \) is the trivial module. Thus, as \( \bar{O} \) is \( p \)-perfect, one concludes that

\[
H^1(\bar{O}, \mathbb{F}_p(1)) = 0.
\]

(7.7)

By hypothesis, \( (\bar{G}, \bar{\theta}) \) is Bloch-Kato, and therefore \( (\bar{O}, 1) \) is Bloch-Kato. Hence \( (7.7) \) yields that

\[
H^k(\bar{O}, \mathbb{F}_p(j)) = H^k(\bar{O}, \mathbb{F}_p(0)) = 0
\]

for all positive integers \( k, j \). Note that \( \mathbb{Z}_p(1) \) is the trivial \( \mathbb{Z}_p[\bar{O}] \)-module isomorphic to \( \mathbb{Z}_p \) as abelian pro-\( p \) group. The cyclotomicity of \( (\bar{O}, 1) \) implies that \( H^2(\bar{O}, \mathbb{Z}_p(1)) \) is \( p \)-torsion free, and from the exact sequence

\[
0 \to H^2(\bar{O}, \mathbb{Z}_p(1)) \xrightarrow{\text{p}} H^2(\bar{O}, \mathbb{Z}_p(1)) \to H^2(\bar{O}, \mathbb{F}_p(1)) \to 0
\]

(7.9)

one concludes that

\[
H^2(\bar{O}, \mathbb{Z}_p(1)) = 0.
\]

(7.10)

By hypothesis, \( \text{cd}_p(\mathbb{Z}_\theta(G)) \leq \text{cd}_p(G) < \infty \), and thus \( \mathbb{Z}_\theta(G) \simeq \mathbb{Z}_p(1)^r \) is a trivial left \( \mathbb{Z}_p[\bar{O}] \)-module and a finitely generated free (abelian pro-\( p \) group). Hence

\[
H^2(\bar{O}, \mathbb{Z}_\theta(G)) = 0,
\]

(7.11)

which implies that

\[
\{1\} \to \mathbb{Z}_\theta(G) \to \bar{O} \to \{1\}
\]

(7.12)

is a split short exact sequence of profinite groups. From this fact one concludes that

\[
\bar{O} = \mathbb{Z}_\theta(G) \cdot O^p(G^\circ) \quad \text{and} \quad \mathbb{Z}_\theta(G) \cap O^p(G^\circ) = \{1\}.
\]

(7.13)

Let \( \tilde{G} = G/O^p(G^\circ) \). Then for all abelian pro-\( p \) groups \( M \) with a continuous left \( \mathbb{Z}_p[\tilde{G}] \)-action inflation induces an isomorphism in cohomology

\[
\text{inf}_{\tilde{M}}^\tilde{G}(-): H^k_{\text{cts}}(\tilde{G}, M) \to H^k_{\text{cts}}(G, M)
\]

(7.14)

(cf. Proposition 4.6). Moreover, as \( \theta|_\bar{O} = 1 \) is the constant 1 function, \( \theta \) induces a \( p \)-orientation \( \hat{\theta}: \tilde{G} \to \mathbb{Z}_p^\times \) on \( \tilde{G} \). In particular, from \( (7.14) \) one concludes that \( \text{cd}_p(\tilde{G}) < \infty \), and that \( (\tilde{G}, \hat{\theta}) \) is cyclotomic and Bloch-Kato. Thus, by part (i), the exact sequence of virtual pro-\( p \) groups

\[
\{1\} \to \mathbb{Z}_\theta(G)O^p(G^\circ) \to \hat{\tilde{G}} \to \hat{\tilde{G}}/\bar{O} \to \{1\}
\]

(7.15)
splits. Let $\tilde{H} \subset \tilde{G}$ be a complement for $Z_{\theta}(G)O^p(G^\circ)/O^p(G^\circ)$ in $\tilde{G}$, and let
\[ H = \{ g \in G^\circ \mid gO^p(G^\circ) \in \tilde{H} \}. \tag{7.16} \]
Then, by construction, $H \cap Z_{\theta}(G)O^p(G^\circ) \subseteq O^p(G^\circ)$. Thus $HO^p(G^\circ)$ is a complement of $Z_{\theta}(G)$ in $G$. □

Finally, we ask whether the converse of Theorem 3.13 holds true.

**Question 7.13.** Let $(G, \theta)$ be a cyclotomically $p$-oriented Bloch-Kato pro-$p$ group, and suppose that
\[ H^\bullet(G, \mathbb{F}_p) \simeq H^\bullet(C, \mathbb{F}_p) \otimes \Lambda^\bullet(V), \]
for some subgroup $C \subseteq G$ and some nontrivial subspace $V \subseteq H^1(G, \mathbb{F}_p)$. Does there exist an isolated closed subgroup $Z \subseteq Z_{\theta}(G)$ such that $G = CZ$ and $Z/Z^p \simeq V^* = \text{Hom}(V, \mathbb{F}_p)$?

### 7.5. The elementary type conjecture

In order to formulate a conjecture concerning the maximal pro-$p$ Galois groups of fields, I. Efrat introduced in [9] the class $\mathcal{C}_{FG}$ of $p$-oriented pro-$p$ pairs of elementary type.

This class consists of all finitely generated $p$-oriented pro-$p$ groups which can be constructed from $\mathbb{Z}_p$ and Demuškin groups using coproducts and fibre products (cf. [9, §3]).

Efrat’s *elementary type conjecture* asks whether every pair $(G_K(p), \theta_K, p)$ for which $K$ contains a primitive $p^{th}$-root of unity and $G_K(p)$ is finitely generated, belongs to $\mathcal{C}_{FG}$ (see [7], and also [15] for the case $p = 2$). This conjecture originates from the theory of quadratic forms (cf. [20], [10, p. 268]).

One may extend slightly Efrat’s class by defining the class $E_{CO}$ of cyclotomically $p$-oriented Bloch-Kato pro-$p$ groups of elementary type to be the smallest class of cyclotomically $p$-oriented pro-$p$ groups containing

(a) $(F, \theta)$, with $F$ a finitely generated free pro-$p$ group and $\theta : F \to \mathbb{Z}_p^\times$ any $p$-orientation;
(b) $(G, \delta_G)$, with $G$ a Demuškin pro-$p$ group;
(c) $(\mathbb{Z}/2\mathbb{Z}, \theta)$, with $\text{im}(\theta) = \{ \pm 1 \}$ in case that $p = 2$;
and which is closed under coproducts and under fibre products with respect to finitely generated split $\theta$-abelian pro-$p$ groups, i.e., if $(G_1, \theta_1)$ and $(G_2, \theta_2)$ are contained in $E_{CO}$, then
\[ (G, \theta) = (G_1, \theta_1) \amalg (G_2, \theta_2) \in E_{CO}; \]
and
\[ (G, \theta) = \mathbb{Z}_p \ltimes_{\theta_1} (G_1, \theta_1) \in E_{CO}. \]

**Question 1.5** asks whether every finitely generated cyclotomically $p$-oriented Bloch-Kato pro-$p$ group belongs to the class $E_{CO}$. By Theorem 1.1, **Question 1.5** is stronger than Efrat’s elementary type conjecture. Nevertheless, it is stated in purely group theoretic terms.

**Remark 7.14.** Recently, **Question 1.5** has received a positive solution in the class of *trivially $p$-oriented right-angled Artin pro-$p$ groups*: I. Snopce and P.A. Zalesskiü proved that the only indecomposable right-angled Artin pro-$p$ group which is Bloch-Kato and cyclotomically $p$-oriented is $(\mathbb{Z}_p, 1)$ (cf. [32]).
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