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LINEAR RATIONAL INSURANCE MODEL
&
ECONOMIC SCENARIO GENERATOR

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Contents

Abstract

Linear Rational Insurance Model

The aim of the first work is to provide a closed pricing formula for insurance contracts in a linear rational framework, which consists in assuming the existence of a linear drift diffusion process and a state price density which is a linear function of it.

The main advantage of this process is that we can compute the conditional expectation of polynomials function of this diffusion: in fact a polynomial can be see as a linear combination of an enlarged set of variables with a linear drift, this can be proofed by means of the Itô's formula.
This result is very important because it allows us, under the hypothesis that the diffusion part is a martingale, to use all the results we have about linear drift diffusions for this new set of variables.

As a consequence, we are able to give the price of three important life insurance contracts: the survival and death benefit and the guaranteed annuity option (also called GAO).
It is about the GAO that we can see the advantage of the framework we are using: actually the payoff of the GAO is not an affine or a polynomial function, so the only way to treat it is by performing a change of measure or a Monte Carlo simulation. We show that, under the assumption that the state space is compact, we are able to approximate the GAO payoff by a polynomial, which will allow us to find a closed formula for the price of this contract.

The end of this work is dedicated to some numerical experiments which have the aim to point out the importance of the choice of the degree of the approximated polynomials in order to have reliable results. We show that a ten degree polynomial is able to estimate with a small error the Monte Carlo price of the GAO.

This work extend the existing literature concerning polynomial models and their application in life insurance, proposing a pricing method also for liabilities which are not necessarily building blocks, but more complicated functions, like the guaranteed annuity option.

Economic Scenario Generator

The aim of this second work is to build an economic scenario generator with the intention of improving the portfolio allocation of Bpifrance.
In order to do that, we have to pass through a different number of steps.

The first thing is to study, by a principal component analysis, the present portfolio of Bpifrance, in order to find the variables which explains the most of its variability.

A second step consists in selecting from the market the financial instruments that allows us to replicate the components we retained from the step before. This part is then completed by both an univariate and multivariate analysis of these assets, finding in this way the stylized facts that we need to take into account when choosing a model for the diffusion of the price of these financial factors.

The third step, and last concerning our work, is to estimate the parameters of the models we retained and see if they are able to fit the empirical data and, as a consequence, if they could be used as a part of our future economic scenario generator.
In order to achieve this point, we focus only on the diffusion of the equity indices, proposing also a model who takes into account the dependency on the inflation. We will see that on the basis of our data there is no evidence to link the return on equity indices on the realisations of this macroeconomic factor.

Part I

Linear Rational Insurance Model

Introduction

Actuarial valuations involve the consideration of two type of risks: the financial risk (concerning the interest rates) and the mortality risk. Mortality is generally considered to be less difficult to be modelled than interest rates. The traditional approach of actuaries consisted in modelling mortality in a deterministic way either assuming a suitable analytical model, either adopting adjusted or projected mortality tables. Nowadays, it is widely admitted that mortality intensities behave in a stochastic way and that they have important similarities with interest rates (see [6]). In the literature, one of the most dominant framework, when it comes to model interest rates and mortality force, is the affine one, as highlighted by [19]. In such a framework one can ensure the non negativity of the quantity of interest and it admits analytical solutions for the price of financial and life insurance products traded in the market, as proposed in [6] and [15], where they exploit these advantages.
Another class of models, that can be used and allows to have closed analytical formula, is the one where the risk factor is driven by a Lévy process, as suggested in [2], where they provide a framework for the modelling of the mortality risk and the market consistent valuation of variable annuity contracts.
If it is true that these models (affine and Lévy) has good features (analytical tractability and market consistent valuation), it is also true that the price to pay is high in term of complexity. In fact, in order to obtain a result, one has to perform either a change of measure (which in some cases gives a new dynamics that cannot be easily computed and treated) or a Monte Carlo simulation (which can be very slow and the results depend on the number of simulations) or both.
The goal of this work is to analyse the problem of pricing life insurance liabilities using a linear rational framework: which consists on assuming the existence of a state price density ζ and a factor process Z with linear drift, as suggested by [19]. The main advantage of this framework is that, under some suitable assumptions, it gives us explicit and tractable formula for conditional expectation of polynomial functions of the state variable, see [18].
We assume that the factor process Z takes values in a compact state space E (as proposed in [5]). This will guarantee on the one hand the computability of the aforementioned conditional expectation, and on the other hand the possibility of using polynomial approximation in order to price life insurance contracts with more complex payoff functions, such as the guaranteed annuity option for instance.
This is the main feature of our setting: there is no need to perform a Monte Carlo simulation or a change of measure if we can provide a good approximation of the conditional expectation we want to compute. This will be showed under two particular model specifications, as proposed also in [1].
This work will be developed in the following way: in the first chapter we will introduce the main concepts about state price density, linear rational and polynomial diffusion models; in the second and last chapter we will present our main work. This chapter can be divided into four main sections: the first two sections concern respectively the setting of our financial market and mortality model; the third section is dedicated to the valuation of the life insurance contracts we decided to price in this model; the last section is about a numerical application of a particular linear model, the Linear Hypercube, in order to show the advantages and drawbacks of this framework.

Chapter 1

Essential notions

Let us start with some fundamental tools which are essential for the understanding of the main part of this work.
In Section 1.1 we will introduce some notions about the state price density, Sections 1.2 and 1.3 state the results we are going to need about the linear rational and polynomial diffusion models.

1.1 State Price Density

We consider a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$, where T is a fixed time horizon. This filtered probability space is assumed to satisfy the usual conditions: it is complete ($\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) and right-continuous ($\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$ for all times t).

Assumption 1.1. *We can define the state price density as the strictly positive process ζ such that the t-price of a contingent claim with value Y in T is*

$$\Pi(t,T,Y) = \mathbb{E}\left[\frac{\zeta_T}{\zeta_t}Y\middle|\mathcal{F}_t\right],\ t \leq T.$$

In the case $Y = 1$, we obtain the price of a zero coupon bond

$$P(t,T) = \mathbb{E}\left[\frac{\zeta_T}{\zeta_t}\middle|\mathcal{F}_t\right],\ t \leq T.$$

Remark 1.2. *Following* [38]*, the state price density, or also called the pricing kernel, can be defined using a reference probability $\mathbb{Q}$ (a risk neutral probability equivalent to $\mathbb{P}$) and the inverse of the money market account $B_t = exp(-\int_0^t r_s ds)$ (which represents the value at time 0 from investing one unit in risk free deposits until t) as follows*

$$\zeta_t = exp\Big(-\int_0^t r_s ds\Big)\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = exp\Big(-\int_0^t r_s ds\Big)V_t, \tag{1.1}$$

where V denotes the density process and r is an adapted process representing the short rate, such that $\int_0^t |r_s|ds < \infty$ for all $t \geq 0$.

Remark 1.3. *It is important also to note that assuming the existence of a state price density under which the prices are martingales is a sufficient condition to exclude arbitrage in our market, which is the possibility of constructing a portfolio with negative price at time $t = 0$ and a positive payoff at maturity in at least one state of the world (see* [29]*).*

In the following we will work in a linear rational framework. In this way we are able to exploit the analytical tractability of linear drift processes in the context of mortality risk modelling. The main advantage of this model, as we will see in Section 2.3.4, is that we will be able to give, under some suitable assumptions, approximated closed formula for the price of more complex life insurance products (as the guaranteed annuity option), without performing neither a change of measure nor a Monte Carlo simulation (see [15] and [2]).

1.2 Linear Rational Term Structure Models

A linear rational term structure model consists of two components:

i) a multivariate factor process Z with a linear drift and state space $E \subset \mathbb{R}^d$. We can take the following dynamics (see [19])

$$dZ_t = \kappa(\theta - Z_t)dt + dM_t,$$

where M is a m-dimensional martingale, $\kappa \in \mathbb{R}^{m\times m}$ and $\theta \in \mathbb{R}^m$.

ii) a state price density ζ_t defined as a linear function of Z_t

$$\zeta_t = \alpha(t) + \omega(t)Z_t,$$

for some deterministic functions $\alpha(t)$ and $\omega(t)$ such that $\alpha(t) + \omega(t)z > 0$ for all $z \in E$.

Remark 1.4. *The linear drift of the process Z implies that its conditional expectations are of the linear form (see* [19]*), that is*

$$\mathbb{E}[Z_T|\mathcal{F}_t] = \theta + exp\big(-\kappa(T-t)\big)(Z_t - \theta),\ t \le T.$$

Lemma 1.5. *The zero coupon bond prices and the short rate are linear rational functions of Z_t, that is*

$$P(t,T) = F(T-t, Z_t) = \frac{\alpha(T) + \theta\omega(T) + \omega(T)exp\big(-k(T-t)\big)(Z_t-\theta)}{\alpha(t)+\omega(t)Z_t},$$

$$r_t = -\partial_T \log P(t,T)|_{T=t} = \frac{k\omega(t)(Z_t-\theta) - \alpha'(t) - \omega'(t)Z_t}{\alpha(t)+\omega(t)Z_t}.$$

Proof. Using Assumption 1.1, we can compute the price of a zero coupon bond as follows

$$\begin{aligned} P(t,T) = \mathbb{E}\left[\frac{\zeta_T}{\zeta_t}\middle|\mathcal{F}_t\right] &= \frac{\mathbb{E}[\alpha(T)+\omega(T)Z_T|\mathcal{F}_t]}{\alpha(t)+\omega(t)Z_t} \\ &= \frac{\alpha(T)+\omega(T)\mathbb{E}[Z_T|\mathcal{F}_t]}{\alpha(t)+\omega(t)Z_t} \\ &= \frac{\alpha(T)+\theta\omega(T)+\omega(T)exp\big(-k(T-T)\big)(Z_t-\theta)}{\alpha(t)+\omega(t)Z_t}. \end{aligned}$$

The last equality follows from Remark 1.4.
To compute r_t, we can proceed by computing

$$\begin{aligned} -\log P(t,T) &= -\log\left(\frac{\alpha(T)+\theta\omega(T)+\omega(T)exp\big(-k(T-T)\big)(Z_t-\theta)}{\alpha(t)+\omega(t)Z_t}\right), \\ -\partial_T \log P(t,T) &= -\frac{\alpha'(T)+\theta\omega'(T)+\omega'(T)exp\big(-\kappa(T-t)\big)(Z_t-\theta) - \kappa\omega(T)exp\big(-\kappa(T-t)\big)(Z_t-\theta))}{(\alpha(t)+\omega(t)Z_t)P(t,T)}, \\ -\partial_T \log P(t,T)|_{T=t} &= -\frac{\alpha'(t)+\theta\omega'(t)+\omega'(t)(Z_t-\theta)-\kappa\omega'(t)(Z_t-\theta)}{\alpha(t)+\omega(t)Z_t} \\ &= \frac{\kappa\omega(t)(Z_t-\theta)-\alpha'(t)-\omega'(t)Z_t}{\alpha(t)+\omega(t)Z_t}. \end{aligned}$$

□

1.3 Polynomial Diffusion Models

In this section we will give an overview of the most important results about polynomial models which will be used in our work. Let $\mathbb{S}^d$ be the space of real symmetric matrices of order d and let $\mathbb{S}^d_+$ be the convex cone of positive semidefinite symmetric matrices. For any $n \in \mathbb{N}$ we denote by $Pol_n(E)$ the space

of the polynomial of degree less or equal than n on E, and by N_n the dimension of $Pol_n(E)$.
We define an E-valued process Z with the following dynamics

$$dZ_t = b(Z_t)dt + \sigma(Z_t)dW_t \tag{1.2}$$
$$Z_0 = z_0 \in E, \tag{1.3}$$

where W is a d-dimensional $\mathbb{F}$-adapted standard Brownian motion, and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a continuous function such that $a = \sigma\sigma'$, and a and b are two fixed maps such that

$$a : \mathbb{R}^d \to \mathbb{S}^d \quad a_{ij}|_E \in Pol_2(E) \tag{1.4}$$
$$b : \mathbb{R}^d \to \mathbb{R}^d \quad b_i|_E \in Pol_1(E), \tag{1.5}$$

for all i and j.
Let us consider the partial differential operator $\mathcal{G} : \mathcal{C}^2(\mathbb{R}^d) \to \mathbb{R}$ associated to Z defined by

$$\mathcal{G}f(z) = \frac{1}{2}tr\big(a(z)\nabla^2 f(z)\big) + b(z)'\nabla f(z),\ z \in \mathbb{R}^d.$$

The process Z is a polynomial diffusion on E if $\mathcal{G}Pol_n(E) \subseteq Pol_n(E)$ for every $n \in \mathbb{N}$.
In particular, Z is a polynomial diffusion if and only if conditions (1.4) and (1.5) hold (see Lemma 2.2 of [18]). In this case there exists a unique matrix representation $G_n \in \mathbb{R}^{N_n\times N_n}$ restricted to $Pol_n(E)$. In other words, for each $p \in Pol_n(E)$ with coordinate representation

$$p(z) = H_n(z)\overline{p},\ z \in \mathbb{R}^d,$$

where $H_n(z)$ is a fixed basis vector of $Pol_n(E)$ and $\overline{p} \in \mathbb{R}^{N_n}$, one gets

$$\mathcal{G}p(z) = H_n(z)'G_n\overline{p},\ z \in \mathbb{R}^d.$$

The following theorems state some fundamental results about the computation of the conditional expectation of $p(Z_T)$.
These results are a special cases of the results of [18] taking a deterministic initial value Z_0.

Theorem 1.6. *For any $p \in Pol_n(E)$ with coordinate representation $\overline{p} \in \mathbb{R}^{N_n}$, we have*

$$\mathbb{E}[p(Z_T)|\mathcal{F}_t] = H_n(Z_t)'e^{(T-t)G_n}\overline{p},\ t \leq T.$$

Proof. See section B of [18]. □

The next theorem provides conditions under which the process Z admits finite exponential moments.

Theorem 1.7. *If $||a(z)|| \leq C(1 + ||z||)$ for all $z \in E$ and for some constant C, then*

$$\textit{for any } t \geq 0, \textit{ there exists } \epsilon > 0 \textit{ such that } \mathbb{E}[e^{\epsilon||Z_t||}] < \infty.$$

Proof. See section C of [18]. □

The following two theorems give some sufficient conditions on the state space E, under which (1.2) – (1.3) admits a unique weak solution.

Theorem 1.8. *Let Z be an E-valued solution to* (1.2) – (1.3)*. If E is compact, then Z is weakly unique, i.e. any other E-valued solution to* (1.2) – (1.3)*, with initial value Z_0, has the same law as Z.*

Proof. See Theorem 5.1 of [18]. □

Remark 1.9. *For $d = 1$ we have also strong uniqueness (see* [5]*).*

Theorem 1.10. *Let $\mathcal{P}$ be a family of polynomials on $\mathbb{R}^d$. If the boundary of the state space E is defined by these polynomials, i.e. $E = \{z \in \mathbb{R}^d : p(z) \leq 0\ \forall p \in \mathcal{P}\}$, then the following conditions on a and b guarantee the existence of an E-valued solution to* (1.2) – (1.3)*:*

i) $a \in \mathbb{S}^d_+$,

ii) $a\nabla p = 0$ *on* $\{p = 0\}$ *for each* $p \in \mathcal{P}$,

iii) $\mathcal{G}p > 0$ *on* $E \cap \{p = 0\}$ *for each* $p \in \mathcal{P}$.

Proof. See section E of [18]. □

Remark 1.11. *It is important to note that every E-solution to* $(1.2) - (1.3)$ *is a strong Markov process thanks to the compactness of the state space E and Theorem* 1.8 *(see Remark* 2.6 *of* [5]*).*

Chapter 2

Linear Rational Insurance Model

2.1 Financial Market

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, with $\mathbb{P}$ the historical measure and $\mathcal{F}_t$ the economic background information available at time t. We will denote by $\mathbb{E}[\cdot]$ the expectation under the measure $\mathbb{P}$.

We consider a framework where the prices are determined by the state price density (Assumption 1.1), which has been inspired by [19]. We suppose also a linear rational term structure, which, recalling from Section 1.2, consists of two components: a multivariate process Z with linear drift, a state space $E \subset \mathbb{R}^{m+d}$ and a state price density ζ_t given as a linear function of Z_t.
In particular

$$\begin{aligned} dZ_t &= AZ_t dt + dM_t^Z \\ \zeta_t &= \alpha(t) + \omega(t) Z_t, \end{aligned}$$

for some

i) $A \in \mathbb{R}^{(m+d)\times(m+d)}$ such that $A = \begin{bmatrix} c & \gamma \\ b & \beta \end{bmatrix}$, with $c \in \mathbb{R}^{d\times d}$, $b \in \mathbb{R}^{m\times d}$, $\gamma \in \mathbb{R}^{d\times m}$, $\beta \in \mathbb{R}^{m\times m}$,

ii) deterministic functions $\alpha(t)$ and $\omega(t)$ such that $\zeta_t > 0$ for every $z \in E$,

iii) $(m+d)$-dimensional martingale M^Z.

Remark 2.1. *The $\mathcal{F}_t$-conditional expectation of Z_T is linear in Z_t,*

$$\mathbb{E}[Z_T|\mathcal{F}_t] = exp\big(A(T-t)\big)Z_t, \quad t \leq T. \tag{2.1}$$

This remark is a generalisation of Remark 1.4.

Lemma 2.2. *The price of a zero coupon bond is given by the following closed formula*

$$P(t,T) = F(T-t, Z_t) = \frac{\alpha(T) + \omega(T) exp\big(A(T-t)\big)Z_t}{\alpha(t) + \omega(t)Z_t}. \tag{2.2}$$

Proof. See proof of Lemma 1.5. □

2.2 Mortality Model

In this section we will present, following [1], the choice of the model we are going to use to describe the mortality intensity in our population. In the first part we will introduce the general setting that will be used in the remaining sections of this work. In the last part, we will discuss about different ways to generalise the framework we presented in order to take into account empirical evidences concerning the modelling of the mortality force.

We consider a survival process S of an insured person, which is a right-continuous $\mathbb{F}$-adapted and non-increasing strictly positive process, with $S_0 = 1$.
Let $U \sim \mathcal{U}([0,1])$ independent from $\mathcal{F}_T$. We can define the random death time of a policyholder in this way

$$\tau = inf\{t \geq 0 : S_t \leq U\}, \tag{2.3}$$

which is infinite if the set is empty.
Let $\mathbb{D}$ be the filtration generated by the indicator process D defined by $D_t = \mathbf{1}_{\{\tau \geq t\}}$.
The random time τ is a stopping time in the enlarged filtration $\mathbb{F} \vee \mathbb{D}$ and it is doubly stochastic (see [1]), in the sense that

$$\mathbb{P}[\tau > t|\mathcal{F}_T] = \mathbb{P}[S_t > U|\mathcal{F}_T] = S_t.$$

The filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$ contains all the economic and mortality information.

In our study we suppose that the survival process is modelled by $S_t = a'Y_t$ for some $a \in \mathbb{R}^d_+$ s.t. $a'\underline{1} = 1$ and the process Y is defined below.

Assumption 2.3. *The process Z will be decomposed into two separate processes (Y, X) with values in $\mathbb{R}^d_+ \times \mathbb{R}^m$ with linear drift of the form*

$$dY_t = (cY_t + \gamma X_t)dt + dM^Y_t \tag{2.4}$$
$$dX_t = (bY_t + \beta X_t)dt + dM^X_t, \tag{2.5}$$

for some $c \in \mathbb{R}^{d\times d}$, $b \in \mathbb{R}^{m\times d}$, $\gamma \in \mathbb{R}^{d\times m}$, $\beta \in \mathbb{R}^{m\times m}$, m-dimensional $\mathcal{F}_t$-martingale M^X_t and d-dimensional $\mathcal{F}_t$-martingale M^Y_t.

The process S being positive and non-increasing, we necessarily have that its martingale component $M^S_t = a'M^Y_t$ is of finite variation and thus purely discontinuous, and that $-S_{t^-} < \Delta M^S_t \leq 0$ because $\Delta S_t = \Delta M^S_t$. This is the reason of the decomposition of Z into a component X and a component Y of finite variation (see [1]).

We will make use of the following results

Lemma 2.4. *Let Y be a non negative $\mathcal{F}_T$-measurable random variable. For any $t \leq t_M \leq T$,*

$$\mathbb{E}[\mathbf{1}_{\{\tau > t_M\}} Y|\mathcal{G}_t] = \mathbf{1}_{\{\tau > t\}} \frac{1}{S_t} \mathbb{E}[S_{t_M} Y|\mathcal{F}_t].$$

Proof. See Lemma A.1 of [1]. □

Lemma 2.5. *Let Z be a bounded $\mathbb{F}$-predictable process. For any $t \leq t_M < T$,*

$$\mathbb{E}[\mathbf{1}_{\{t<\tau\leq t_M\}} Z_\tau|\mathcal{G}_t] = \frac{\mathbf{1}_{\{\tau>t\}}}{S_t} \mathbb{E}\left[\int_t^{t_M} -Z_u dS_u \middle| \mathcal{F}_t\right].$$

Proof. See Lemma A.2 of [1]. □

Proposition 2.6. *The process S is non increasing if and only if the following conditions are satisfied*

i) $a' < M^Y >= 0$ *almost surely, for the continuous part of the martingale M^Y,*

ii) $a'(cY_t + \gamma X_t) \leq 0$ *almost surely for almost every t,*

iii) $a'\Delta M^Y_t \leq 0$ *almost surely.*

Proof. To prove this proposition we can compute the variation of S,

$$\begin{aligned}\Delta S_t &= a'\Delta Y_t \\ &= a'(cY_t + \gamma X_t)\Delta t + a'\Delta M_t^Y \\ &\leq 0\end{aligned}$$

The inequality follows from the conditions i, ii and iii of the proposition. This prove the sufficiency of this conditions.
For the necessary part: if S is decreasing for every t, this means that S is a finite variation process (condition i). In addition, since S is decreasing, its drift and its jumps have to be negative (conditions ii and iii). This concludes our proof. □

In addition, we will suppose the following covariances between our variables

$$d < Y^i, Y^j >_t = \big(\nu_{1,y}^{ij} Y_t + \nu_{2,y}^{ij}(Y_t)^2 + \delta_{1,y}^{ij} X_t + \delta_{2,y}^{ij}(X_t)^2 + \eta_y^{ij}\big)dt, \quad i, j = 1, \ldots, d \tag{2.6}$$

$$d < X^i, X^j >_t = \big(\nu_{1,x}^{ij} Y_t + \nu_{2,x}^{ij}(Y_t)^2 + \delta_{1,x}^{ij} X_t + \delta_{2,x}^{ij}(X_t)^2 + \eta_x^{ij}\big)dt, \quad i, j = 1, \ldots, m \tag{2.7}$$

$$d < Y^i, X^j >_t = \big(\nu_{1,xy}^{ij} Y_t + \nu_{2,xy}^{ij}(Y_t)^2 + \delta_{1,xy}^{ij} X_t + \delta_{2,xy}^{ij}(X_t)^2 + \eta_{xy}^{ij}\big)dt, \quad i = 1, \ldots d, \ j = 1, \ldots, m, \tag{2.8}$$

where $(Y_t)^2$ and $(X_t)^2$ denote the vectors Y_t and X_t where all their components are taken to the second power, ν^{ij} is a $d - dimensional\ vector$, δ^{ij} is a $m - dimensional\ vector$ and η^{ij} is a real number.

Remark 2.7. *The information about the covariances is contained in the following matrices,*

$$\begin{aligned} < Y >_t &\in \mathbb{R}^{d\times d}, \\ < X >_t &\in \mathbb{R}^{m\times m}, \\ < Y, X >_t &\in \mathbb{R}^{d\times m}. \end{aligned}$$

Lemma 2.8. *If S is absolutely continuous (i.e. $a'\Delta M_t^Y = 0$) then, the mortality intensity μ_t, which is derived from the relation $S_t = exp(-\int_0^t \mu_s ds)$, is linear rational in (Y_t, X_t) and takes this expression,*

$$\mu_t = -\frac{a'(cY_t + \gamma X_t)}{a'Y_t}. \tag{2.9}$$

Proof. Starting from $S_t = exp(-\int_0^t \mu_s ds)$ and taking the derivative with respect to the variable t, we obtain

$$S_t(-\mu_t)dt = dS_t,$$

which leads us to

$$\begin{aligned} S_t(-\mu_t) &= a'(cY_t + \gamma X_t) \\ \mu_t &= -\frac{a'(cY_t + \gamma X_t)}{a'Y_t}. \end{aligned}$$

□

2.2.1 Modelling of the Mortality Force

In this section we will show how we can choose an appropriate linear rational model in order to take into account different demographic features of the mortality intensity.

According to [35] and [44] we can observe different trends in mortality, one among them is the so called rectangularization phenomenon:
this aspect is associated with a reduction in the variability of the death time. As deaths start to concentrate in a more and more narrower age range, the slope of the survival function becomes steeper in that

range, and the curve starts to appear more rectangular. According to [21], this phenomenon, which can be observed in industrialised countries, suggests that human life expectancy is approaching its maximum potential value.

It is important to note also that the rectangularization may depend on the age range and the time frame of analysis. Actually, the variability of age at death may decline in some periods when the entire population is considered, but may remain stable if the analysis is limited to older ages ([37]). Moreover, there is no reason to expect that any process of compression of mortality should continue indefinitely.

Another important aspect to take into account is that the average length of life (the life expectancy at birth) is in general independent of its variability. Therefore limits to life expectancy and the rectangularization phenomenon should be considered as two different issues. By the way, in some particular cases, [21] arguments that these two concepts may be interrelated. He establishes an upper limit of the life expectancy and believes that the maximum human life span is fixed. This is a very important statement, because if it were true, then it would mean that the distribution of ages at death is bounded on the right with the consequence that if mortality declines and death is delayed, the distribution of the ages at death must be compressed and the survival function must become more rectangular. Despite this statement seems to be quite logical, [37], [42] and [45] showed that from an empirical point of view the rectangularization has not continued on recent decades for elderly population and the proportion of centenarians in a population has not been constant and unchanging over time. Even if those studies argue that rectangularization is not necessarily associated with the mortality decline, the connection between these two aspects remains indeterminate ([43]).

Another important trend that can be derived from [44] concerns the fact that the variability of age at death has decreased during the past 250 years. This reflects three important phases of mortality decline:

i) period of slow mortality reduction across the age range, associated with high but stable levels of variability in age of death.

ii) an era of rapid reduction in infant and child mortality, resulting in an enormous and rapid compression of mortality.

iii) a period of unprecedented reduction in late-adult mortality, followed by near-constant levels of variability in ages at death.

According to [44], the current pattern of mortality suggests that the level of variability observed today could be maintained for the foreseeable future.

This aspects give rise to a reduction of the risk of mortality and an augmentation of the risk of longevity (the risk that a person lives beyond its life expectancy), see [6].
So, when choosing a model, we need to take into account this new kind of risk.

The first thing to do, as suggested in [6], is to calculate some indicators that could help to understand the shape of the survival and death curves. We can compute:

i) the entropy of the survival function:

$$\mathcal{H} = -\frac{\int_0^\omega \log(S_t)/S_t dt}{\int_0^\omega S_t dt},$$

where ω is the maximal age of a policyholder. This indicator measures the rectangularization of the survival function implied by the mortality force.
If $\mathcal{H} \to 0$ it means that the shape of the survival function tends to be rectangular.

ii) the expectation of life for a policyholder of age x:

$$\mathbb{E}[\tau_x] = \int_0^{\omega - x} S_t dt.$$

This result can be compared with the benchmarks provided by UN and OECD.

It is known (see [6]) that processes with non time dependent coefficient can hardly capture these demographic features we stated. That is why it would be interesting to propose some special cases of our framework.
We present now two examples of how we can integrate these empirical evidences (the longevity risk and the rectangularization phenomenon) in our linear rational framework. This can be done either modifying the survival process S or the process (Y, X). These time dependent coefficients could represent for instance an available mortality table, a prudential demographic basis when carrying out risk analysis or a realistic assumption on μ (see [6]).

Examples

1) We can treat the case where the coefficients of the survival process are time dependent, i.e. $S_t = a'(t)Y_t$, such that $a'(t)\underline{1} = 1$.

Lemma 2.9. *In this case the mortality intensity takes the following expression*

$$\mu_t = -\frac{a'(t)Y_t + a(t)(cY_t + \gamma X_t)}{a'(t)Y_t}. \tag{2.10}$$

Proof. From the application of the Itô's lemma on the process $S = a'(t)Y$, we have that $dS_t = a'(t)Y_t dt + a'(t)dY_t$. Then, by using Lemma 2.8, we get

$$\begin{aligned} dS_t &= -\mu_t S_t dt \\ \mu_t &= -\frac{a'(t)Y_t + a'(t)(cY_t + \gamma X_t)}{a'(t)Y_t}. \end{aligned}$$

□

2) Another interesting case would be the one with time dependent coefficients in the dynamics of (Y, X), that is

$$dY_t = \big(c(t)Y_t + \gamma(t)X_t\big)dt + dM_t^Y \tag{2.11}$$
$$dX_t = \big(b(t)Y_t + \beta(t)X_t\big)dt + dM_t^X. \tag{2.12}$$

Remark 2.10. *In this case equation* (2.1) *becomes*

$$\mathbb{E}\left[\begin{bmatrix} Y_T \\ X_T \end{bmatrix} \middle| \mathcal{F}_t\right] = exp\left(\int_t^T A(u)du\right)\begin{bmatrix} Y_t \\ X_t \end{bmatrix}. \tag{2.13}$$

Lemma 2.11. *The new mortality force is defined as*

$$\mu_t = -\frac{a'\big(c(t)Y_t + \gamma(t)X_t\big)}{a'Y_t}. \tag{2.14}$$

Proof. See Lemma 2.8. □

In this work we will principally focus on the case of non time dependent coefficients, nevertheless we will extend our results to these two special cases whenever this would lead to a different or interesting result.

2.3 Valuation of insurance contracts

In this section we will evaluate the survival and death benefits of standard insurance contracts under the assumptions of model (2.4) – (2.5).
We are using an $\mathbb{F}$-adapted state price density to price claims which depend on the larger filtration $\mathbb{G}$. We know that τ is doubly stochastic, so every $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale. On the other hand, for the claims that are $\mathcal{G}_T$-measurable (and not $\mathcal{F}_T$-measurable), we need the following assumption

Assumption 2.12. *The arbitrage-free prices of $\mathcal{G}_T$-measurable claims are generated by the same state price density ζ of the claims that are $\mathcal{F}_T$-measurable, which implies that this state price density will satisfy the same stochastic differential equation, i.e. dynamics, in $\mathbb{G}$.*

Before presenting the main result, we will introduce some important notation.

Definition 2.13. *We will note:*

i) $H_n(y) = \Big[y_1\ y_2\ \dots\ (y_1)^n\ (y_2)^n\ \dots\Big]$ *is a polynomial basis of* $Pol_n(\mathbb{R}^d_+)$ *and it will be also denoted by* $\mathcal{Y}_n$.

ii) $H^*_n(y,x) = \Big[x_1\ x_2\ \dots\ x_1^2\ x_1 y\ x_2^2\ x_2 y\ \dots\ x_m\ \dots\Big]$ *is a vector which contains only the polynomials of order n which are function of x and xy. It will be also denoted by* $\mathcal{X}_n$.

iii) $H_n(y,x) = \Big(H_n(y), H^*_n(y,x)\Big)$ *is a polynomial basis of the whole space E.*

iv) $n \in \mathbb{N}^+$.

The following theorem state an important result, which will allow us to price life insurance contracts even when their payoffs are not a linear function of the factor process (Y,X), but a polynomial. The idea is to consider this polynomial as a linear function of an enlarged set of variables, which will enable us, under our assumptions, to use all the results we have concerning linear drift processes.

Theorem 2.14. *Let us consider the process (Y,X) as defined in* (2.4)$-$(2.5), *then the process* $H_n(Y,X) = (\mathcal{Y}_n, \mathcal{X}_n)$ *has linear drift in* $\mathcal{Y}_{t,n}$ *and* $\mathcal{X}_{t,n}$.

An analogous general result can be found in [18].
For brevity of notation we will not indicate explicitly the order of the polynomials (as we assumed that it is n) and the dependence on time in the gradient and the Hessian matrix.

Proof. To prove this theorem we will proceed by computing the dynamics of $\mathcal{Y}$ and of $\mathcal{X}$.
In order to compute the entire dynamics of the process, it will be sufficient to calculate the dynamics of their generic components.

i) we know that $\mathcal{Y} = H_n(Y) = \Big[Y_1\ Y_2\ \dots\ Y_1^n\ Y_2^n\ \dots\Big]$, we need to show that the drift of the generic component h is linear in $\mathcal{Y}_t$ and $\mathcal{X}_t$.
So we take $\mathcal{Y}^h = (Y^1)^{q_1}\dots(Y^d)^{q_d}$, such that $\sum_{i=1}^d q_i \le n$ and $q_i \ge 0$ $for\ any\ i = 1,\dots,d$.
Then, by application of Itô's lemma, we have

$$d\mathcal{Y}^h_t = \nabla'_{h,y} dY_t + \frac{1}{2} tr\big(\mathbf{H}_{h,y} d < Y >_t\big),$$

where $\nabla_{h,y}$ and $\mathbf{H}_{h,y}$ are respectively the gradient and the Hessian matrix of $\mathcal{Y}^h$ (taken with respect to the variable y), and $tr(\cdot)$ is the trace operator.
In particular they are equal to

$$\nabla_{h,y} = \left(q_i (Y^i_t)^{q_i-1} \prod_{j\neq i} (Y^j_t)^{q_j}\right)_{i=1,\dots,d}$$

$$\mathbf{H}_{h,y} = \left(q_i(q_i-1)(Y^i_t)^{q_i-2}\mathbf{1}_{\{i=j\}} \prod_{k\neq i}(Y^k_t)^{q_k} + (q_i q_j (Y^i_t)^{q_i-1}(Y^j_t)^{q_j-1}\mathbf{1}_{\{i\neq j\}} \prod_{k\neq i,j}(Y^k_t)^{q_k}\right)_{i,j=1,\dots d}.$$

Then, replacing the dynamics of Y, we have

$$\begin{aligned} d\mathcal{Y}^h_t &= \nabla'_{h,y}(cY_t + \gamma X_t)dt + \frac{1}{2} tr(\mathbf{H}_{h,y} d < Y >_t) + \nabla'_{h,y} dM^Y_t \\ &= (\nabla'_{h,y} cY_t + \nabla'_{h,y}\gamma X_t + \lambda^Y(Y_t))dt + \nabla'_{h,y} dM^Y_t, \end{aligned}$$

where

$$\lambda^Y(Y_t) = \frac{1}{2} tr\big(\mathbf{H}_{h,y} d < Y >_t\big).$$

By regrouping the dt-terms and observing that they are a linear transformation of $\mathcal{Y}_t$ and $\mathcal{X}_t$, one has

$$d\mathcal{Y}_t^h = (\tilde{c}_h \mathcal{Y}_t + \tilde{\gamma}_h \mathcal{X}_t)dt + \nabla'_{h,y} dM_t^Y,$$

where $\tilde{c}_h$ is a row vector of dimension $d^{\mathcal{Y}} = dim(\mathcal{Y})$ and $\tilde{\gamma}_h$ is a another row vector of dimension $d^{\mathcal{X}} = dim(\mathcal{X})$ such that they solve the following equation

$$\tilde{c}_h \mathcal{Y}_t + \tilde{\gamma}_h \mathcal{X}_t = \nabla'_{h,y} c Y_t + \nabla'_{h,y} \gamma X_t + \lambda^Y(Y_t).$$

Finally, combining the dynamics of all the components of $\mathcal{Y}$, we obtain

$$d\mathcal{Y}_t = (\tilde{c}\mathcal{Y}_t + \tilde{\gamma}\mathcal{X}_t)dt + \nabla'_y dM_t^Y, \tag{2.15}$$

where $\tilde{c} \in \mathbb{R}^{d^{\mathcal{Y}} \times d^{\mathcal{Y}}}$, $\tilde{\gamma} \in \mathbb{R}^{d^{\mathcal{Y}} \times d^{\mathcal{X}}}$ and ∇'_y is the matrix of dimension $d^{\mathcal{Y}} \times d$ which contains the transposed gradients of every component of $\mathcal{Y}$.

ii) We know that $\mathcal{X} = H_n^*(Y, X) = \begin{bmatrix} X_1 & \dots X_m & \dots X_1 Y_1 & \dots \end{bmatrix}$.
As before, we take the generic component h,
$\mathcal{X}^h = (Y^1)^{p_1} \dots (X^m)^{p_{d+m}}$, such that $\sum_{i=1}^{d+m} p_i \le n$, $p_i \ge 0$ for $i = 1, \dots, d$ and $p_i > 0$ for $i = d+1, \dots, m$.
By Itô's lemma, we have

$$d\mathcal{X}_t^h = \nabla'_{h,x} \begin{bmatrix} dY_t \\ dX_t \end{bmatrix} + \frac{1}{2} tr\big(\mathbf{H}_{h,x} d < Y, X >_t\big),$$

where $\nabla_{h,x}$ is the gradient of $\mathcal{X}_t^h$ and $\mathbf{H}_{h,x}$ is the Hessian matrix of $\mathcal{X}^h$ (both taken with respect to the variables x and y), which are defined as follows

$$\nabla_{h,x} = \left(p_i (Z_t^i)^{p_i - 1} \prod_{j \ne i} (Z_t^j)^{p_j} \right)_{i=1,\dots,d+m}$$

$$\mathbf{H}_{h,x} = \left(p_i(p_i - 1)(Z_t^i)^{p_i - 2} \mathbf{1}_{\{i=j\}} \prod_{k \ne i} (Z_t^k)^{p_k} + (p_i p_j (Z_t^i)^{p_i - 1} (Z_t^j)^{p_j - 1} \mathbf{1}_{\{i \ne j\}} \prod_{k \ne i,j} (Z_t^k)^{p_k} \right)_{i,j=1,\dots d+m},$$

where $Z_t^j = \begin{cases} Y_t^j \ if \ j \le d \\ X_t^{j-d} \ otherwise \end{cases}$.
Then, we get

$$\begin{aligned} d\mathcal{X}_t^h &= \nabla'_{h,x} \begin{bmatrix} cY_t + \gamma X_t \\ bY_t + \beta X_t \end{bmatrix} dt + \frac{1}{2} tr\big(\mathbf{H}_{h,x} d < Y, X >_t\big) + \nabla'_{h,x} \begin{bmatrix} dM_t^Y \\ dM_t^X \end{bmatrix} \\ &= \left(\nabla'_{h,x} \begin{bmatrix} cY_t + \gamma X_t \\ bY_t + \beta X_t \end{bmatrix} + \lambda^{XY}(Y_t, X_t) \right) dt + \nabla'_{h,x} \begin{bmatrix} dM_t^Y \\ dM_t^X \end{bmatrix}, \end{aligned}$$

where

$$\lambda^{XY}(Y_t, X_t) = \frac{1}{2} tr\big(\mathbf{H}_{h,x} d < Y, X >_t\big).$$

Now, we can observe that the dt-terms are a linear transformation of $\mathcal{Y}_t$ and $\mathcal{X}_t$ and obtain

$$d\mathcal{X}_t^h = (\tilde{b}_h \mathcal{Y}_t + \tilde{\beta}_h \mathcal{X}_t)dt + \nabla'_{h,x} \begin{bmatrix} dM_t^Y \\ dM_t^X \end{bmatrix},$$

where $\tilde{b}_h$ is a row vector of dimension $d^{\mathcal{Y}}$ and $\tilde{\beta}_h$ is another row vector of size $d^{\mathcal{X}}$, which solve the following equation

$$\tilde{b}_h \mathcal{Y}_t + \tilde{\beta}_h \mathcal{X}_t = \nabla'_{h,x} \begin{bmatrix} cY_t + \gamma X_t \\ bY_t + \beta X_t \end{bmatrix} + \frac{1}{2} tr\big(\mathbf{H}_{h,x} d < Y, X >_t \big).$$

Finally, the dynamics of $\mathcal{X}$ is

$$d\mathcal{X}_t = (\tilde{b}\mathcal{Y}_t + \tilde{\beta}\mathcal{X}_t)dt + \nabla'_x \begin{bmatrix} dM_t^Y \\ dM_t^X \end{bmatrix}, \tag{2.16}$$

with $\tilde{b} \in \mathbb{R}^{d^{\mathcal{X}} \times d^{\mathcal{Y}}}$, $\tilde{\beta} \in \mathbb{R}^{d^{\mathcal{X}} \times d^{\mathcal{X}}}$ and ∇'_x is the matrix of dimension $d^{\mathcal{X}} \times (d+m)$ which contains the transposed gradients of all components of $\mathcal{X}^h$.

□

The dynamics of the process $(\mathcal{Y}, \mathcal{X})$ is

$$\begin{aligned} d\mathcal{Y}_t &= (\tilde{c}\mathcal{Y}_t + \tilde{\gamma}\mathcal{X}_t)dt + \nabla'_y dM_t^Y \\ d\mathcal{X}_t &= (\tilde{b}\mathcal{Y}_t + \tilde{\beta}\mathcal{X}_t)dt + \nabla'_x \begin{bmatrix} dM_t^Y \\ dM_t^X \end{bmatrix}. \end{aligned}$$

Lemma 2.15. *Let $(\mathcal{Y}, \mathcal{X})$ be a process defined by $(2.15) - (2.16)$, then if $\int_0^t \nabla'_y dM_s^Y$ and $\int_0^t \nabla'_x \begin{bmatrix} dM_s^Y \\ dM_s^X \end{bmatrix}$ are martingale for $t \leq T$, one has*

$$\mathbb{E}\left[\begin{bmatrix} \mathcal{Y}_T \\ \mathcal{X}_T \end{bmatrix} \middle| \mathcal{F}_t \right] = exp\big(\tilde{A}(T-t)\big) \begin{bmatrix} \mathcal{Y}_t \\ \mathcal{X}_t \end{bmatrix}, \ t \leq T. \tag{2.17}$$

where $\tilde{A} = \begin{bmatrix} \tilde{c} & \tilde{\gamma} \\ \tilde{b} & \tilde{\beta} \end{bmatrix}$

Proof. The proof follows by Remark 2.1 and by linearity of the drift of our new process $(\mathcal{Y}, \mathcal{X})$. □

Conditions such that the martingale condition is verified will be specified later in some particular cases.

Remark 2.16. *It is important to notice that in the case where the process (Y, X) has time dependent coefficients, assumed in $(2.11) - (2.12)$, the result of Theorem 2.14 is still valid, in particular*

$$\begin{aligned} d\mathcal{Y}_t &= \big(\tilde{c}(t)\mathcal{Y}_t + \tilde{\gamma}(t)\mathcal{X}_t\big)dt + \nabla'_y dM_t^Y \\ d\mathcal{X}_t &= \big(\tilde{b}(t)\mathcal{Y}_t + \tilde{\beta}(t)\mathcal{X}_t\big)dt + \nabla'_x \begin{bmatrix} dM_t^Y \\ dM_t^X \end{bmatrix}, \end{aligned}$$

and, provided that the condition stated in Lemma 2.15 is satisfied,

$$\mathbb{E}\left[\begin{bmatrix} \mathcal{Y}_T \\ \mathcal{X}_T \end{bmatrix} \middle| \mathcal{F}_t \right] = exp\bigg(\int_0^t \tilde{A}(u)du \bigg) \begin{bmatrix} \mathcal{Y}_t \\ \mathcal{X}_t \end{bmatrix}, \ t \leq T.$$

From now on we will focus on the pricing of some life insurance liabilities, which can be considered as building blocks for other products, using the model we specified.

Assumption 2.17. *For the remaining part of Section 2.3 we will assume that the martingale condition stated in Lemma 2.15 is always satisfied in order to use its result and do our computations. This condition can be easily verified in some model specification, as we will see in the last section of this work.*

2.3.1 Survival benefit

A survival benefit is the benefit C_T, $\mathbb{F}$-measurable, given to the policyholder at the end of his policy tenure (the time T).
The value at time $t \leq T$ of this insurance contract is given by

$$\begin{aligned} SB_t(C_T, T) &= \mathbb{E}\left[\frac{\zeta_T}{\zeta_t}\mathbf{1}_{\{\tau>T\}}C_T\middle|\mathcal{G}_t\right] = \frac{\mathbf{1}_{\{\tau>t\}}}{\zeta_t S_t}\mathbb{E}\big[\zeta_T\mathbf{1}_{\{\tau>T\}}C_T\big|\mathcal{F}_t\big] \\ &= \frac{\mathbf{1}_{\{\tau>t\}}}{\zeta_t S_t}\mathbb{E}\big[\zeta_T\mathbb{E}[\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_T]C_T\big|\mathcal{F}_t\big] = \frac{\mathbf{1}_{\{\tau>t\}}}{\zeta_t S_t}\mathbb{E}\big[\zeta_T S_T C_T\big|\mathcal{F}_t\big], \end{aligned} \tag{2.18}$$

where τ is the random death time (see formula (2.3)). The second equality follows from Lemma 2.4.
In the following sections we will treat two different cases: the one where the benefit is constant (insurances with amount fixed at the start of the contract, see [5]), and the one where the benefit is a polynomial function of order k of Y.
The interest in considering constant benefits, besides its simplicity from a mathematical point of view, lies in the fact that this kind of contracts are actually traded on the market and they are preferred to the ones with time dependent or random benefits by the average policyholder. An example is a term life insurance which pays out a fixed amount of money to the beneficiary in case of death.

A) $C_T = C \in \mathbb{R}^+$
Using the representation of ζ_T and S_T, the conditional expectation in (2.18) becomes

$$\begin{aligned} \mathbb{E}\big[\zeta_T S_T\big|\mathcal{F}_t\big] &= \mathbb{E}\left[\left(\alpha(T) + \omega(T)\begin{bmatrix} Y_T \\ X_T \end{bmatrix}\right)a'Y_T\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\big[\alpha(T)a'Y_T + \omega_1(T)Y_T a'Y_T + \omega_2(T)X_T a'Y_T\big|\mathcal{F}_t\big], \end{aligned}$$

where $\omega_1(\cdot)$ is the vector consisting of the first d components of $\omega(\cdot)$,and $\omega_2(\cdot)$ is the one containing the last m components of $\omega(\cdot)$.
Then, applying Theorem 2.14 and by linearity, we have

$$\mathbb{E}\big[\zeta_T S_T\big|\mathcal{F}_t\big] = a_s'(T)exp\big(\tilde{A}^2(T-t)\big)\begin{bmatrix} \mathcal{Y}_{t,2} \\ \mathcal{X}_{t,2} \end{bmatrix}, \tag{2.19}$$

where $a_s(\cdot)$ is such that

$$a_s'(T)H_2(y,x) = \omega_1(T)ya'y + \alpha(T)a'y + \omega_2(T)xa'y$$

where $H_2(y,x)$ is a polynomial basis of the state space $Pol_2(\mathbb{R}^d_+ \times \mathbb{R}^m_+)$ (see Definition 2.13).

B) $C_T = a_c'H_k(Y_T)$
We can extend (2.19) to the case $C_T = a_c'H_k(Y_T)$.
The computations are the same, only the dimension of the problem increases, going from 2 to $2+k$, as we will treat polynomials of degree $k+2$ instead of polynomial of degree 2.

Proposition 2.18. *The value of the survival benefit in the hypothesis $C_T = a_c'H_k(Y_T)$ is*

$$SB_t(C_T, T) = \frac{\mathbf{1}_{\{\tau>t\}}}{S_t\zeta_t}a_s'(T)exp\big(\tilde{A}^{k+2}(T-t)\big)\begin{bmatrix} \mathcal{Y}_{t,k+2} \\ \mathcal{X}_{t,k+2} \end{bmatrix}. \tag{2.20}$$

where $a_s(\cdot)$ is such that

$$a_s'(T)H_{k+2}(y,x) = \alpha(T)a_c'H_k(y)a'y + \omega_1(T)ya_c'H_k(y)a'y + \omega_2(T)xa_c'H_k(y)a'y.$$

Proof. Starting from the conditional expectation in (2.18) and using the expression of C_T and ζ_T, we get

$$\begin{aligned} \mathbb{E}\big[\zeta_T C_T S_T\big|\mathcal{F}_t\big] &= \mathbb{E}\big[\big(\alpha(T) + \omega_1(T)Y_T + \omega_2(T)X_T\big)a_c'H_k(Y_T)a'Y_T\big|\mathcal{F}_t\big] \\ &= \mathbb{E}\big[\alpha(T)a_c'H_k(Y_T)a'Y_T + \omega_1(T)Y_T a_c'H_k(Y_T)a'Y_T + \omega_2(T)X_T a_c'H_k(Y_T)a'Y_T\big|\mathcal{F}_t\big] \\ &= a_s'(T)exp\big(\tilde{A}^{k+2}(T-t)\big)\begin{bmatrix} \mathcal{Y}_{t,k+2} \\ \mathcal{X}_{t,k+2} \end{bmatrix}. \end{aligned}$$

□

Remark 2.19. *In the case where the survival process S or the process (Y, X) have time dependent coefficients, the prices found in* (2.19) *and* (2.20) *are not valid anymore. Anyway, we are always able to compute the price of the survival benefit under the hypothesis that the benefit C is a polynomial of order k of the process Y, i.e. $C_T = a_c' H_k(Y_T)$.*

1) $S_t = a(t)Y_t$. In this case the expectation in (2.18) *changes into*

$$\begin{aligned}\mathbb{E}\big[\zeta_T S_T a_c' H_k(Y_T)\big|\mathcal{F}_t\big] &= \mathbb{E}\left[\left(\alpha(T) + \omega(T)\begin{bmatrix} Y_T \\ X_T \end{bmatrix}\right) a'(T) Y_T a_c' H_k(Y_T)\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\big[a_s'(T) H_{k+2}(Y_T, X_T)\Big|\mathcal{F}_t\big] \\ &= a_s'(T) exp\big(\tilde{A}^{k+2}(T-t)\big)\begin{bmatrix} \mathcal{Y}_{t,k+2} \\ \mathcal{X}_{t,k+2}, \end{bmatrix},\end{aligned}$$

where $a_s(\cdot)$ satisfies the following equation

$$a_s'(T) H_{k+2}(y, x) = \big(\alpha(T) + \omega_1(T)y + \omega_2(T)x\big)\big(a'(T) y a_c' H_k(y)\big)$$

2) (Y, X) has time dependent coefficients. Under this assumption, the expected value in (2.18) *becomes*

$$\begin{aligned}\mathbb{E}\big[\zeta_T S_T a_c' H_k(Y_T)\big|\mathcal{F}_t\big] &= \mathbb{E}\left[\left(\alpha(T) + \omega(T)\begin{bmatrix} Y_T \\ X_T \end{bmatrix}\right) a' Y_T a_c' H_k(Y_T)\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\big[a_s'(T) H_{k+2}(Y_T, X_T)\big|\mathcal{F}_t\big] \\ &= a_s'(T) exp\bigg(\int_t^T \tilde{A}^{k+2}(u)du\bigg) du \begin{bmatrix} \mathcal{Y}_{t,k+2} \\ \mathcal{X}_{t,k+2} \end{bmatrix},\end{aligned}$$

where $a_s(\cdot)$ satisfies

$$a_s'(T) H_{k+2}(y, x) = \big(\alpha(T) + \omega_1(T)y + \omega_2(T)x\big)\big(a' y a_c' H_k(y)\big).$$

2.3.2 Death benefit

A death benefit is a payout C_τ to the beneficiary of a life insurance policy, annuity, or pension when the insured or annuitant dies (before the time T).

Assumption 2.20. *The process S is assumed to be almost surely continuous.*

This means we exclude the possibility that in a given population the mortality could change due to an instantaneous shock.

The value at time $t \le T$ of the death benefit is given by

$$\begin{aligned}DB_t(C, T) = \mathbb{E}\left[\frac{\zeta_\tau}{\zeta_t}\mathbf{1}_{\{t<\tau\le T\}} C_\tau\middle|\mathcal{G}_t\right] &= \frac{\mathbf{1}_{\{\tau>t\}}}{\zeta_t S_t}\mathbb{E}\left[-\int_t^T \zeta_u C_u dS_u\middle|\mathcal{F}_t\right] \\ &= \frac{\mathbf{1}_{\{\tau>t\}}}{\zeta_t S_t}\mathbb{E}\left[\int_t^T \zeta_u C_u[-a'(cY_u + \gamma X_u)]du\middle|\mathcal{F}_t\right] \\ &= \frac{\mathbf{1}_{\{\tau>t\}}}{\zeta_t S_t}\int_t^T \mathbb{E}\big[\zeta_u C_u[-a'(cY_u + \gamma X_u)]\big|\mathcal{F}_t\big]du. \end{aligned} \tag{2.21}$$

The second equality follows from Lemma 2.5 and the last inequality follows from the application of Fubini-Tonelli Theorem, which is possible thanks to the positivity of the integrand.
As before we will treat the case where C_τ is a constant and the case where C_τ is a polynomial function of Y.

A) $C_\tau = C \in \mathbb{R}^+$

Following the same procedure as before, the integrand in (2.21) can be rewritten as

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\alpha(u)+\omega(u)\begin{bmatrix}Y_u\\X_u\end{bmatrix}\Big)\big[-a'(cY_u+\gamma X_u)\big]\Big|\mathcal{F}_t\Big]\\
&=\mathbb{E}\big[-\alpha(u)a'(cY_u+\gamma X_u)-\omega_1(u)Y_u a'(cY_u+\gamma X_u)-\omega_2(u)X_u a'(CY_u+\gamma X_u)\big|\mathcal{F}_t\big]\\
&=a_d'(u)exp\big(\tilde{A}^2(u-t)\big)\begin{bmatrix}\mathcal{Y}_{t,2}\\\mathcal{X}_{t,2}\end{bmatrix},
\end{aligned} \tag{2.22}$$

where $a_d(\cdot)$is such that

$$a_d'(u)H_2(y,x)=-\omega_1(u)ya'cy-\alpha(u)a'cy-\omega_1(u)ya'cx-\omega_2(u)xa'(cy+\gamma x)-\alpha(u)a'\gamma x.$$

B) $C_\tau = a_c'H_k(Y_\tau)$

The same procedure can be used to compute the death benefit under the hypothesis $C_\tau = a_c'H_k(Y_\tau)$.

Proposition 2.21. *The price of the death benefit when $C_\tau = a_c'H_k(Y_\tau)$ is*

$$DB_t(C_\tau,T)=\frac{\mathbf{1}_{\{\tau>t\}}}{S_t\zeta_t}\int_t^T\Big(a_d'(u)exp\big(\tilde{A}^{k+2}(u-t)\big)\begin{bmatrix}\mathcal{Y}_{t,k+2}\\\mathcal{X}_{t,k+2}\end{bmatrix}\Big)du, \tag{2.23}$$

where $a_d'(\cdot)$ satisfies the following equation

$$a_d'(u)H_{k+2}(y,x)=-\big(\alpha(u)+\omega_1(u)y+\omega_2(u)x\big)a_c'H_k(y)a'(cy+\gamma x).$$

Proof. From (2.21), focusing on the conditional expectation and using the hypothesis that C_τ is a polynomial, we have

$$\begin{aligned}
\mathbb{E}\big[\zeta_u C_u[-a'(cY_u+\gamma X_u)]\big|\mathcal{F}_t\big]&=\mathbb{E}\big[-\zeta_u a_c'H_k(Y_u)a'(cY_u+\gamma X_u)\big|\mathcal{F}_t\big]\\
&=\mathbb{E}\big[-(\alpha(u)+\omega_1(u)Y_u+\omega_2(u)X_u)a_c'H_k(Y_u)a'(cY_u+\gamma X_u)\big|\mathcal{F}_t\big]\\
&=a_d'(u)exp\big(\tilde{A}^2(u-t)\big)\begin{bmatrix}\mathcal{Y}_{t,k+2}\\\mathcal{X}_{t,k+2}\end{bmatrix},
\end{aligned}$$

□

Remark 2.22. *As we did in the last section, we can study the price of the death benefit under the time dependent coefficients framework, when the benefit C_τ is a polynomial function of order k of Y, that is $C_\tau = a_c'H_k(Y_\tau)$*

1) $S_t = a(t)Y_t$. The integrand of (2.21) becomes

$$\begin{aligned}
\mathbb{E}\big[\zeta_u a_c'H_k(Y_u)[-a'(u)(cY_u+\gamma X_u)]\big|\mathcal{F}_t\big]&=\mathbb{E}\Big[-\Big(\alpha(u)+\omega(u)\begin{bmatrix}Y_u\\X_u\end{bmatrix}\Big)a_c'H_k(Y_u)a'(u)(cY_u+\gamma X_u)\Big]\\
&=\mathbb{E}\big[a_d'(u)H_{k+2}(Y_u,X_u)\big|\mathcal{F}_t\big]\\
&=a_d'(u)exp\big(\tilde{A}^{k+2}(u-t)\big)\begin{bmatrix}\mathcal{Y}_{u,k+2}\\\mathcal{X}_{u,k+2,}\end{bmatrix},
\end{aligned}$$

where $a_d(\cdot)$ satisfies

$$a_d'(u)H_{k+2}(y,x)=-\big(\alpha(u)+\omega_1(u)y+\omega_2(u)x\big)a_c'H_k(y)a'(u)(cy+\gamma x).$$

2) (Y,X) has time dependent coefficients. In this case, the expectation in (2.21) changes into

$$\begin{aligned}
\mathbb{E}\big[\zeta_u a_c'H_k(Y_u)[-a'(cY_u+\gamma X_u)]\big|\mathcal{F}_t\big]&=\mathbb{E}\Big[-\Big(\alpha(u)+\omega(u)\begin{bmatrix}Y_u\\X_u\end{bmatrix}\Big)a_c'H_k(Y_u)a'(cY_u+\gamma X_u)\Big]\\
&=\mathbb{E}\big[a_d'(u)H_{k+2}(Y_u,X_u)\big|\mathcal{F}_t\big]\\
&=a_d'(u)exp\Big(\int_t^u\tilde{A}^{k+2}(s)ds\Big)\begin{bmatrix}\mathcal{Y}_{u,k+2}\\\mathcal{X}_{u,k+2}\end{bmatrix},
\end{aligned}$$

where $a_d(\cdot)$ is solution of

$$a_d'(u)H_{k+2}(y,x)=-\big(\alpha(u)+\omega_1(u)y+\omega_2(u)x\big)a_c'H_k(y)a'(cy+\gamma x).$$

2.3.3 Whole Life Annuity

Another life insurance contract which can be priced under our framework is the Whole Life Annuity. This contract allows the insured to get a periodic payoff until his death. This kind of insurance products are usually purchased by investors who want to secure an income stream during retirement.
If we suppose ω to be the maximal age of the policyholder and x his age at time t, then the whole life annuity can be seen as a sum of $\omega - x$ survival benefits which are dependent on C, which is the periodic payoff.

Proposition 2.23. *The price at time $t \leq T$ of a whole life annuity, which gives the right to the insured to get a sum C_j until his death, is*

$$a_x(t, C) = \sum_{j=0}^{\omega-x} SB_t(C_{t+j}, t+j). \tag{2.24}$$

Then, under the hypothesis $C_j = a_c' H_k(Y_j)$, (2.24) can be rewritten as

$$a_x(t, C) = \sum_{j=0}^{\omega-x} \frac{\mathbf{1}_{\{\tau>t\}}}{S_t \zeta_t} a_s'(j) exp\big(\tilde{A}^{k+2}(j-t)\big) \begin{bmatrix} \mathcal{Y}_{t,k+2} \\ \mathcal{X}_{t,k+2} \end{bmatrix}. \tag{2.25}$$

Proof. This is obtained by applying the result in Proposition 2.18 (which gives us the t-value of survival benefit when C is a polynomial function of Y) to (2.24). □

Remark 2.24. *We will denote by $a_x(t)$ the t-price of a whole life annuity which pays a unit of money until the death time of the insured (see Section* 3.1 *of* [15]*).*

2.3.4 Guaranteed annuity option

The guaranteed annuity option (GAO) is the contract who gives to the insured the right to choose, at time T, between an annual payment g until his death time, where g is the fixed rate called the guaranteed annuity rate, or a cash payment equal to the capital 1.
The value at time T of this contract is

$$V(T) = \max\big(g a_x(T), 1\big) = 1 + g \ \max\bigg(a_x(T) - \frac{1}{g}, 0\bigg). \tag{2.26}$$

Focusing on the second term, which is called the GAO option price entered by an x-year policyholder at time t, the value at time $t \leq T$ will be

$$GAO_x(t, T) = \mathbb{E}\bigg[\frac{\zeta_T}{\zeta_t} g \ \max\bigg(a_x(T) - \frac{1}{g}, 0\bigg)\bigg|\mathcal{F}_t\bigg]. \tag{2.27}$$

This payoff is very similar to the one of a basket option.
A way to treat this value is by approximation. The idea is to approximate the payoff function by a polynomial and then apply Theorem 2.14 to compute the expectation and obtain a closed formula for the price of this payoff.
This is a very important feature of our model: actually the complexity does not depend on the number of the terms in (2.25), because we will treat a polynomial, for which the computations are easier than affine models or models driven by a Lévy process. In fact, in these models, one could evaluate the price of a GAO only by performing a suitable change of measure and/or a Monte Carlo simulation (see [15] and [2]): if it is true that this kind of method is very flexible and easy to use, it is also true that it could be very slow and the solutions are not exact, but depend on the number of simulations. In our framework, we can propose an approximated closed formula for the price of this contingent claim, without necessarily performing neither a change of measure nor a Monte Carlo simulation.

Polynomial approximation

In this section we present the method we will apply to approximate the payoff function

$$f(X_T, Y_T) = \left(a_x(T) - \frac{1}{g}\right)^+ \tag{2.28}$$

by a polynomial.
To do that we will focus on the method proposed in [5] and used also in [19].
Let us consider a $\mathcal{F}_t$-conditional expectation of the form

$$\xi(t,T) = \mathbb{E}\big[f(Y_T, X_T)\big|\mathcal{F}_t\big],$$

for some continuous function $f(y,x)$ on the state space E.
We will make use of the following assumption on the state space E

Assumption 2.25. *The state space $E \subset \mathbb{R}^d$ is assumed to be compact.*

This assumption provides that we can always find a polynomial approximation $\{f_m\}_{m\in\mathbb{N}}$ of f on E of order $n \in \mathbb{N}$, i.e.

$$\lim_{m\to\infty} ||f_m - f||_\infty = 0,$$

where

$$||f||_\infty = \sup_{x\in E} |f(Y_T, X_T)|,$$

for any $f \in \mathcal{C}(E)$ (see [5]).

Lemma 2.26. *Let $(f_m)_{m\in\mathbb{N}}$ be a uniform polynomial approximation of the continuous function f on E. Then, we consider for every $m \in \mathbb{N}$*

$$GAO_x(t,T)_m = g\mathbb{E}\left[\frac{\zeta_T}{\zeta_t} f_m(Y_T, X_T)\middle|\mathcal{F}_t\right].$$

$\{GAO_x(t,T)_m\}_{m\in\mathbb{N}}$ provides both a pathwise and $\mathcal{L}^p$ approximation of $GAO_x(t,T)$ in (2.27) for any $p \geq 1$ uniformly in $t \in [0,T]$, i.e.

$$\sup_{t\in[0,T]} |GAO_x(t,T)_m - GAO_x(t,T)| \xrightarrow{m\to\infty} 0 \;\; a.s.$$
$$\sup_{t\in[0,T]} ||GAO_x(t,T)_m - GAO_x(t,T)||_{\mathcal{L}^p} \xrightarrow{m\to\infty} 0 \; for \; every \; p \geq 1.$$

Proof. See Lemma 4.2 of [5]. □

We are now able to compute the price of our approximated payoff

$$\begin{aligned} GAO_x(t,T)_m &= g\mathbb{E}\left[\frac{\zeta_T}{\zeta_t} f_m(Y_T, X_T)\middle|\mathbb{F}_t\right] = \frac{g}{\zeta_t}\mathbb{E}\big[\zeta_T f_m(Y_T, X_T)\big|\mathcal{F}_t\big] \\ &= \frac{g}{\zeta_t}\mathbb{E}\left[\left(\alpha(T) + \omega(T)\begin{bmatrix} Y_T \\ X_T \end{bmatrix}\right) f_m(Y_T, X_T)\middle|\mathcal{F}_t\right] \\ &= \frac{g}{\zeta_t}\mathbb{E}\big[a_g'(T) H_{n+1}(Y_T, X_T)\big|\mathcal{F}_t\big] \\ &= \frac{g}{\zeta_t} a_g'(T) exp\big(\tilde{A}^{n+1}(T-t)\big)\begin{bmatrix} \mathcal{Y}_{t,n+1} \\ \mathcal{X}_{t,n+1} \end{bmatrix}. \end{aligned} \tag{2.29}$$

The last equality is obtained by applying Lemma 2.15.
The deterministic function $a_g(\cdot)$ satisfies the following equation

$$a_g'(T) H_{n+1}(y,x) = \big(\alpha(T) + \omega_1(T)y + \omega_2(T)x\big) f_m(y,x).$$

Remark 2.27. *Like the survival and the death benefit, we can compute the price of the GAO also when the process* (Y, X) *has time dependent coefficients. We can notice also that the case* $S_t = a(t)Y_t$ *does not change the value of* (2.29)*, as the approximated payoff does not depend on the survival process* S*.*
Then, the price of the guaranteed annuity option turns into

$$\begin{aligned}
GAO_x(t,T)_m &= g\mathbb{E}\left[\frac{\zeta_T}{\zeta_t} f_m(Y_T, X_T)\middle|\mathbb{F}_t\right] = \frac{g}{\zeta_t}\mathbb{E}\big[\zeta_T f_m(Y_T, X_T)\big|\mathcal{F}_t\big] \\
&= \frac{g}{\zeta_t}\mathbb{E}\left[\left(\alpha(T) + \omega(T)\begin{bmatrix} Y_T \\ X_T \end{bmatrix}\right) f_m(Y_T, X_T)\middle|\mathcal{F}_t\right] \\
&= \frac{g}{\zeta_t}\mathbb{E}\big[a_g'(T) H_{n+1}(Y_T, X_T)\big|\mathcal{F}_t\big] \\
&= \frac{g}{\zeta_t} a_g'(T) exp\left(\int_t^T \tilde{A}^{n+1}(u)du\right)\begin{bmatrix} \mathcal{Y}_{t,n+1} \\ \mathcal{X}_{t,n+1} \end{bmatrix},
\end{aligned}$$

where $a_g(\cdot)$ *solve the following equation*

$$a_g'(T)H_{n+1}(y,x) = \big(\alpha(T) + \omega_1(T)y + \omega_2(T)x\big) f_m(y,x).$$

2.4 Application

We will focus now on some applications of the linear models in order to price the life insurance contracts we presented in the previous sections. We will treat the Linear Hypercube model (LHC) and in particular two special cases: the one-factor LHC and the two-factors cascade LHC (studied in [1]). This class of models is appropriate for the modelisation of the mortality, as they allow the mortality intensity to be a non negative process.

2.4.1 The Linear Hypercube Model

The Linear Hypercube Model is a model with $d = 1$ (so that $S_t = Y_t$), where the survival process is absolutely continuous and the factor process X is diffusive and takes values in a hypercube whose edges' length is given by Y (see [1]),

$$E = \{(y,x) \in \mathbb{R}^{1+m} : \ y \in (0,1] \ and \ x \in [0,y]^m\}. \tag{2.30}$$

The dynamics of (Y, X) is

$$dY_t = -\gamma' X_t dt \tag{2.31}$$
$$dX_t = (bY_t + \beta X_t)dt + \Sigma(Y_t, X_t)dW_t, \tag{2.32}$$

for some $\gamma \in \mathbb{R}^m_+$ and some m-dimensional brownian motion W, where the volatility matrix is given by

$$\Sigma(y,x) = diag\big(\sigma_1\sqrt{x_1(y-x_1)}, ..., \sigma_m\sqrt{x_m(y-x_m)}\big),$$

with volatility parameters $\sigma_1, ..., \sigma_m \geq 0$.

Remark 2.28. *The mortality intensity, using the relation* $S_t = Y_t$*, can be expressed in the following way*

$$\mu_t = \gamma' \frac{X_t}{Y_t}.$$

In addition,

$$0 \leq \mu_t \leq \gamma' \underline{1},$$

see [1].

The following proposition gives the conditions of existence and uniqueness of a solution for the SDEs given by (2.31) − (2.32).

Proposition 2.29. *If, for all $i = 1, \dots, m$,*

$$b_i - \sum_{ji} \beta_{ij}^- \geq 0$$
$$\gamma_i + \beta_{ii} + b_i + \sum_{ji} (\gamma_j + \beta_{ij})^+ \leq 0,$$

then, for any initial value (X_0, Y_0) with support E, there exists a unique E-valued solution (Y, X) of (2.31) − (2.32). *It satisfies the boundary non-attainment, for some $i = 1, \dots, m$,*

i) $X_{it} > 0$ for all $t \geq 0$ if $X_{i0} > 0$ and $b_i - \sum_{ji} \beta_{ij}^- \geq \frac{\sigma_i^2}{2}$

ii) $X_{it} < Y_{it}$ for all $t \geq 0$ if $X_{i0} < Y_{i0}$ and $\gamma_i + \beta_{ii} + b_i + \sum_{ji} (\gamma_j + \beta_{ij})^+ \leq -\frac{\sigma_i^2}{2}$

Proof. See proof of Theorem 3.1 of [1]. □

Remark 2.30. *It is important to note that the martingale condition stated in Lemma* 2.15 *is automatically satisfied, this because of the compactness of the state space E (see equation* (2.30)*), which enables us to use the result proved in the abovementioned lemma.*

2.4.2 LHC(1)

Let us consider the Linear Hypercube model with $m = 1$. Then, the dynamics of the process (Y, X) is

$$\begin{aligned} dY_t &= -\gamma X_t dt \\ dX_t &= (bY_t + \beta X_t)dt + \sigma\sqrt{X_t(Y_t - X_t)}dW_t, \end{aligned}$$

$$E = \{(y, x) \in \mathbb{R}^2 : y \in (0, 1] \text{ and } x \in [0, y]\}.$$

We recall that $S_t = Y_t$.

Remark 2.31. *The conditions stated in Proposition* 2.29 *can be rewritten as*

$$b \geq 0, \quad (\gamma + b + \beta) \leq 0.$$

We are interested in the computation of the price of the survival, death benefit and the guaranteed annuity option.
In particular, for the survival and death benefit we will focus on the case where the benefit is a positive constant C.

Survival and Death Benefit

From Sections 2.3.1 and 2.3.2, we need to define a new process $(\mathcal{Y}_2, \mathcal{X}_2)$, where $\mathcal{Y}_2 = H_2(Y) = \begin{bmatrix} Y & Y^2 \end{bmatrix}$ and $\mathcal{X}_2 = H_2^*(Y, X) = \begin{bmatrix} X & X^2 & XY \end{bmatrix}$.

Proposition 2.32. *The dynamics of $(\mathcal{Y}_2, \mathcal{X}_2)$ is*

$$d\mathcal{Y}_{t,2} = -\tilde{\gamma}' \mathcal{X}_{t,2} dt \tag{2.33}$$
$$d\mathcal{X}_{t,2} = (\tilde{b}\mathcal{Y}_{t,2} + \tilde{\beta}\mathcal{X}_{t,2})dt + \tilde{\Sigma}(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2})dW_t, \tag{2.34}$$

where

$$\tilde{\gamma}' = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 0 & 2\gamma \end{bmatrix},$$
$$\tilde{b} = \begin{bmatrix} b & 0 \\ 0 & 0 \\ 0 & b \end{bmatrix},$$
$$\tilde{\beta} = \begin{bmatrix} \beta & 0 & 0 \\ 0 & 2\beta - \sigma^2 & 2b + \sigma^2 \\ 0 & -\gamma & \beta \end{bmatrix},$$
$$\tilde{\Sigma}(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2}) = \begin{bmatrix} \sigma\sqrt{X_t(Y_t - X_t)} \\ 2\sigma X_t\sqrt{X_t(Y_t - X_t)} \\ \sigma Y_t\sqrt{X_t(Y_t - X_t)} \end{bmatrix}.$$

Proof. To prove our statement we can use Theorem 2.14 and we get

i) for $\mathcal{Y}_2$ we have

$$d\mathcal{Y}_{t,2} = \begin{cases} dY_t \\ 2Y_t dY_t \end{cases} = \begin{cases} -\gamma X_t dt \\ -2\gamma Y_t X_t dt \end{cases} = \begin{cases} -\tilde{\gamma}_1' \mathcal{X}_{t,2} dt \\ -\tilde{\gamma}_2' \mathcal{X}_{t,2} dt \end{cases} = -\tilde{\gamma}\mathcal{X}_{t,2} dt.$$

ii) for $\mathcal{X}_2$ we have

$$\begin{aligned} d\mathcal{X}_{t,2} &= \begin{cases} dX_t \\ 2X_t dX_t + d < X >_t \\ Y_t dX_t + X_t dY_t \end{cases} \\ &= \begin{cases} (bY_t + \beta X_t)dt + \sigma\sqrt{X_t(Y_t - X_t)}dW_t \\ (2bX_tY_t + 2\beta X_t^2)dt + 2\sigma X_t\sqrt{X_t(Y_t - X_t)}dW_t + (\sigma^2 X_t Y_t - \sigma^2 X_t^2)dt \\ (bY_t^2 + \beta X_t Y_t)dt + \sigma Y_t\sqrt{X_t(Y_t - X_t)}dW_t - \gamma X_t^2 dt \end{cases} \\ &= \begin{cases} (\tilde{b}_1\mathcal{Y}_{t,2} + \tilde{\beta}_1\mathcal{X}_{t,2})dt + \tilde{\Sigma}_1(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2})dW_t \\ (\tilde{b}_2\mathcal{Y}_{t,2} + \tilde{\beta}_2\mathcal{X}_{t,2})dt + \tilde{\Sigma}_2(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2})dW_t \\ (\tilde{b}_3\mathcal{Y}_{t,2} + \tilde{\beta}_3\mathcal{X}_{t,2})dt + \tilde{\Sigma}_3(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2})dW_t \end{cases} \\ &= (\tilde{b}\mathcal{Y}_{t,2} + \tilde{\beta}\mathcal{X}_{t,2})dt + \tilde{\Sigma}(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2})dW_t. \end{aligned}$$

□

Now, the t-price of the survival benefit is

$$\begin{aligned} SB_t(C,T) &= \frac{C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}\mathbb{E}\big[\alpha(T)Y_T + \omega_1(T)Y_T^2 + \omega_2(T)X_TY_T\big|\mathcal{F}_t\big] \\ &= \frac{C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}a_s'(T)exp\big(\tilde{A}^2(T-t)\big)\begin{bmatrix} \mathcal{Y}_{t,2} \\ \mathcal{X}_{t,2} \end{bmatrix}, \end{aligned}$$

where $a_s'(T) = [\alpha(T)\ \omega_1(T)\ 0\ 0\ \omega_2(T)]$ and $\tilde{A}^2 = \begin{bmatrix} 0 & -\tilde{\gamma} \\ \tilde{b} & \tilde{\beta} \end{bmatrix}$,
and t-price of the death benefit is

$$\begin{aligned} DB_t(C,T) &= \frac{C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}\int_t^T \mathbb{E}\big[\gamma\alpha(u)X_u + \gamma\omega_1(u)X_uY_u + \gamma\omega_2(u)X_u^2\big|\mathcal{F}_t\big]du \\ &= \frac{\gamma C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}\int_t^T a_d'(u)exp\big(\tilde{A}^2(T-t)\big)\begin{bmatrix} \mathcal{Y}_{t,2} \\ \mathcal{X}_{t,2} \end{bmatrix}du, \end{aligned}$$

where $a_d'(u) = [0\ 0\ \alpha(u)\ \omega_2(u)\ \omega_1(u)]$.

Guaranteed annuity option

For what concerns the GAO price, the approximation given by (2.29) is a polynomial of degree $n+1$. Without loss of generality we present the computations for the case of a polynomial of degree n.
So, we need to define a new process $(\mathcal{Y}_n, \mathcal{X}_n)$, where

$$\mathcal{Y}_n = H_n(Y) = \left[Y \; Y^2 \; \dots \; Y^n\right],$$

$$\mathcal{X}_n = H_n^*(Y, X) = \left[X \; X^2 \; XY \; X^3 \; X^2Y \; XY^2 \; \dots \; X^n \; \dots \; XY^{n-1}\right].$$

Proposition 2.33. *The dynamics of $(\mathcal{Y}_n, \mathcal{X}_n)$ is*

$$d\mathcal{Y}_{t,n} = -\tilde{\gamma}' \mathcal{X}_{t,n} \tag{2.35}$$

$$d\mathcal{X}_{t,n} = (\tilde{b}\mathcal{Y}_{t,n} + \tilde{\beta}\mathcal{X}_{t,n})dt + \tilde{\Sigma}(\mathcal{Y}_{t,n}, \mathcal{X}_{t,n})dW_t. \tag{2.36}$$

where $\tilde{\gamma} \in \mathbb{R}^{d^{\mathcal{Y}} \times d^{\mathcal{X}}}$, $\tilde{b} \in \mathbb{R}^{d^{\mathcal{X}} \times d^{\mathcal{Y}}}$, $\tilde{\beta} \in \mathbb{R}^{d^{\mathcal{X}} \times d^{\mathcal{X}}}$ and $\tilde{\Sigma} \in \mathbb{R}^{d^{\mathcal{X}}}$.
We recall that $d^{\mathcal{Y}} = dim(\mathcal{Y}_{t,n})$ and $d^{\mathcal{X}} = dim(\mathcal{X}_{t,n})$.

$$\tilde{\gamma}' = \begin{bmatrix} \gamma & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 2\gamma & \dots & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & \dots & n\gamma & \dots & 0 \end{bmatrix},$$

$$\tilde{b} = \begin{bmatrix} b & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & b \end{bmatrix},$$

$$\tilde{\beta} = \begin{bmatrix} \beta & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2\beta - \sigma^2 & 2b + \sigma^2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -\gamma & \beta & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 3\beta - 3\sigma^2 & 3b + \sigma^2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -\gamma & 2\beta - \sigma^2 & 2b + \sigma^2 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\gamma & \beta & \dots & 0 & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -(n-1)\gamma & \beta \end{bmatrix},$$

$$\tilde{\Sigma}(\mathcal{Y}_{t,n}, \mathcal{X}_{t,n}) = \begin{bmatrix} \sigma\sqrt{X_t(Y_t - X_t)} \\ \vdots \\ Y_t^{n-1}\sigma\sqrt{X_t(Y_t - X_t)} \end{bmatrix}.$$

Proof. The proof of this proposition comes from the computation of the dynamics of the generic component of the process $(\mathcal{Y}_n, \mathcal{X}_n)$.

$$\begin{aligned} dX_t^r Y_t^s &= rX_t^{r-1}Y_t^s dX_t + sX_t^r Y_t^{s-1} dY_t + \frac{1}{2}r(r-1)X_t^{r-2}Y_t^s d < X >_t \\ &= (rbX_t^{r-1}Y_t^{s+1} + r\beta X_t^r Y_t^s)dt + \sigma r X_t^{r-1}Y_t^s\sqrt{X_t(Y_t - X_t)}dW_t - s\gamma Y_t^{s-1}X_t^{r+1}dt + \\ &\quad + \frac{\sigma^2 r(r-1)}{2} Y_t^s X_t^{r-1}(Y_t - X_t)dt \\ &= \left(rbX_t^{r-1}Y_t^{s+1} + r\beta X_t^r Y_t^s - s\gamma Y_t^{s-1}X_t^{r+1} + \frac{\sigma^2 r(r-1)}{2}Y_t^{s+1}X_t^{r-1} - \frac{\sigma^2 r(r-1)}{2}Y_t^s X_t^r\right)dt + \\ &\quad + \sigma r X_t^{r-1}Y_t^s\sqrt{X_t(Y_t - X_t)}dW_t. \end{aligned}$$

The dynamics of $\mathcal{Y}_n$ is found by setting $r = 0$ and $s = 1, \dots, n$, the dynamics of $\mathcal{X}_n$ is determined by setting $r = 1, \dots, n$ and $s = 0, \dots, n$. $\square$

The approximated price of a guaranteed annuity option will be

$$\begin{aligned}
GAO_x(t,T)_m &= \frac{g}{\zeta_t}\mathbb{E}\big[\zeta_T f_m(Y_T,X_T)\big|\mathcal{F}_t\big]\\
&= \frac{g}{\zeta_t}\mathbb{E}\left[\left(\alpha(T)+\omega(T)\begin{bmatrix}Y_T\\X_T\end{bmatrix}\right)f_m(Y_T,X_T)\middle|\mathcal{F}_t\right]\\
&= \frac{g}{\zeta_t}\mathbb{E}\big[a'_g(T)H_{n+1}(Y_T,X_T)\big|\mathcal{F}_t\big]\\
&= \frac{g}{\zeta_t}a'_g(T)exp\big(\tilde{A}^{n+1}(T-t)\big)\begin{bmatrix}\mathcal{Y}_{t,n+1}\\\mathcal{X}_{t,n+1}\end{bmatrix},
\end{aligned}$$

where $a'_g(T)$ is obtained following equation (2.29).

2.4.3 LHCC(2)

Let us now consider the linear hypercube cascade model, introduced in [1], with $m=2$. The dynamics of the process (Y,X) is

$$\begin{aligned}
dY_t &= -\gamma_1\underline{1}X_t dt\\
dX_{1t} &= \kappa_1(\theta_1 X_{2t}-X_{1t})dt+\sigma_1\sqrt{X_{1t}(Y_t-X_{1t})}dW_{1t}\\
dX_{2t} &= \kappa_2(\theta_2 Y_t-X_{2t})dt+\sigma_2\sqrt{X_{2t}(Y_t-X_{2t})}dW_{2t}\\
d<W_1,W_2>_t &= 0,
\end{aligned}$$

for some $\gamma_1\geq 0$ and $\kappa,\theta,\sigma\in\mathbb{R}^2_+$ such that

$$\theta_i\leq 1-\frac{\gamma_1}{\kappa_i}\ for\ i=1,2.$$

The matrices β and b are respectively

$$\begin{aligned}
\beta &= \begin{bmatrix}-\kappa_1 & \kappa_1\theta_1\\ 0 & -\kappa_2\end{bmatrix},\\
b &= \begin{bmatrix}0\\\kappa_2\theta_2\end{bmatrix}.
\end{aligned}$$

In this way conditions stated in Proposition 2.29 are satisfied (see [1]), in other words

$$b_i-\sum_{ji}\beta_j^- = \mathbf{1}_{\{i=m\}}\kappa_m\theta_m = \mathbf{1}_{\{i=m\}}\beta_{mm}\geq 0,$$

$$\gamma_i+\beta_{ii}+b_i+\sum_{ji}(\gamma_j+\beta_{ij})^+ = \gamma_1-\kappa_i+\kappa_i\theta_i = \gamma_1+\beta_{ii}+\mathbf{1}_{i\neq m}\beta_{i,i+1}+\mathbf{1}_{i=m}b_m\leq 0.$$

The advantage of the cascade model is that it allows default intensity values to persistently be close to zero over extended periods of time. It also allows to work with a multidimensional model parsimoniously as the number of free parameters is equal to $3m+1$ (6 in our case) whereas it is equal to $3m+m^2$ for the standard LHC model.

Survival and Death Benefit

As the previous section, we can find a formula for the survival and death benefit using the cascade model. The first thing to do is to define our process $(\mathcal{X}_2,\mathcal{Y}_2)$

$$\begin{aligned}
\mathcal{Y}_2 &= \begin{bmatrix}Y\ Y^2\end{bmatrix},\\
\mathcal{X}_2 &= \begin{bmatrix}X_1\ X_2\ X_1^2\ X_1Y\ X_2^2\ X_2Y\ X_1X_2\end{bmatrix}.
\end{aligned}$$

Proposition 2.34. *The dynamics of* $(\mathcal{Y}_2, \mathcal{X}_2)$ *is*

$$d\mathcal{Y}_{t,2} = -\tilde{\gamma}'\mathcal{X}_{t,2}dt \tag{2.37}$$

$$d\mathcal{X}_{t,2} = (\tilde{b}\mathcal{Y}_{t,2} + \tilde{\beta}\mathcal{X}_{t,2})dt + \tilde{\Sigma}(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2})d\underline{W}_t, \tag{2.38}$$

where

$$\tilde{\gamma}' = \begin{bmatrix} \gamma_1 & \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\gamma_1 & 0 & 2\gamma_1 & 0 \end{bmatrix},$$

$$\tilde{b} = \begin{bmatrix} 0 & 0 \\ \kappa_2\theta_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \kappa_2\theta_2 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{\beta} = \begin{bmatrix} -\kappa_1 & \kappa_1\theta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\kappa_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\kappa_1 - \sigma_1^2 & \sigma_1^2 & 0 & 0 & 2\kappa_1\theta_1 \\ 0 & 0 & -\gamma_1 & -\kappa_1 & 0 & \kappa_1\theta_1 & -\gamma_1 \\ 0 & 0 & 0 & 0 & -2\kappa_2 - \sigma_2^2 & 2\kappa_2\theta_2 + \sigma_2^2 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_1 & -\kappa_2 & -\gamma_1 \\ 0 & 0 & 0 & \kappa_2\theta_2 & \kappa_1\theta_1 & 0 & -\kappa_1 - \kappa_2 \end{bmatrix},$$

$$\tilde{\Sigma}(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2}) = \begin{bmatrix} \sigma_1\sqrt{X_{1t}(Y_t - X_{1t})} & 0 \\ 0 & \sigma_2\sqrt{X_{2t}(Y_t - X_{2t})} \\ \sigma_1 X_{1t}\sqrt{X_{1t}(Y_t - X_{1t})} & 0 \\ \sigma_1 Y_t\sqrt{X_{1t}(Y_t - X_{1t})} & 0 \\ 0 & \sigma_2 X_{2t}\sqrt{X_{2t}(Y_t - X_{2t})} \\ 0 & \sigma_2 Y_t\sqrt{X_{2t}(Y_t - X_{2t})} \\ \sigma_1 X_{2t}\sqrt{X_{1t}(Y_t - X_{1t})} & \sigma_2 X_{1t}\sqrt{X_{2t}(Y_t - X_{2t})} \end{bmatrix},$$

$$d\underline{W}_t = \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix}.$$

Proof. The proof is the same as Proposition 2.32. $\square$

We can now compute the price of the survival benefit at time $t \leq T$,

$$\begin{aligned} SB_t(T) &= \frac{C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}\mathbb{E}\big[\alpha(T)Y_T + \omega_1(T)Y_T^2 + \omega_2(T)X_TY_T\big|\mathcal{F}_t\big] \\ &= \frac{C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}\mathbb{E}\big[\alpha(T)Y_T + \omega_1(T)Y_T^2 + \omega_{21}(T)X_{1T}Y_T + \omega_{22}(T)X_{2T}Y_T\big|\mathcal{F}_t\big] \\ &= \frac{C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}a_s'(T)exp\big(\tilde{A}^2(T-t)\big)\begin{bmatrix} \mathcal{Y}_{t,2} \\ \mathcal{X}_{t,2} \end{bmatrix}, \end{aligned}$$

where $a_s'(T) = [\alpha(T)\ \omega_1(T)\ 0\ 0\ 0\ \omega_{21}(T)\ 0\ \omega_{22}(T)\ 0]$ and $\tilde{A}^2 = \begin{bmatrix} 0 & -\tilde{\gamma} \\ \tilde{b} & \tilde{\beta} \end{bmatrix}$.
The t-price of the death benefit is

$$\begin{aligned} DB_t(T) &= \frac{C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}\int_t^T \mathbb{E}\big[\gamma\alpha(u)X_{1u} + \gamma\alpha(u)X_{2u} + \gamma\omega_1(u)X_{1u}Y_u + \gamma\omega_1(u)X_{2u}Y_u + \gamma\omega_{21}(u)X_{1u}^2 + \gamma\omega_{22}(u)X_{2u}^2\big|\mathcal{F}_t\big]du \\ &= \frac{\gamma C\mathbf{1}_{\{\tau>t\}}}{\zeta_t}\int_t^T a_d'(u)exp\big(A^2(u-t)\big)\begin{bmatrix} \mathcal{Y}_{t,2} \\ \mathcal{X}_{t,2} \end{bmatrix}du, \end{aligned}$$

where $a_d'(u) = [0\ 0\ \alpha(u)\ \alpha(u)\ \omega_{21}(u)\ \omega_1(u)\ \omega_{22}(u)\ \omega_1(u)\ 0]$.

Guaranteed annuity option

For the GAO we can follow the same procedure as before, so we will study the process $(\mathcal{Y}_n, \mathcal{X}_n)$, where

$$\mathcal{Y}_n = \begin{bmatrix} Y \ Y^2 \ \dots \ Y^n \end{bmatrix}$$
$$\mathcal{X}_n = \begin{bmatrix} X_1 \ X_2 \ X_1^2 \ X_1Y \ \dots \ X_1^3 \ X_1^2Y \ X_1Y^2 \ X_2^3 \ X_2^2Y \ X_2Y^2 \ X_1X_2Y \ \dots \ X_1X_2Y^{n-2} \end{bmatrix}.$$

Proposition 2.35. *The dynamics of* $(\mathcal{Y}_n, \mathcal{X}_n)$ *is*

$$d\mathcal{Y}_{t,n} = -\tilde{\gamma}'\mathcal{X}_{t,n}dt \tag{2.39}$$
$$d\mathcal{X}_{t,n} = (\tilde{b}\mathcal{Y}_{t,2} + \tilde{\beta}\mathcal{X}_{t,2})dt + \tilde{\Sigma}(\mathcal{Y}_{t,n}, \mathcal{X}_{t,n})d\underline{W}_t. \tag{2.40}$$

where $\tilde{\gamma} \in \mathbb{R}^{d^{\mathcal{Y}} \times d^{\mathcal{X}}}$, $\tilde{b} \in \mathbb{R}^{d^{\mathcal{X}} \times d^{\mathcal{Y}}}$, $\tilde{\beta} \in \mathbb{R}^{d^{\mathcal{X}} \times d^{\mathcal{X}}}$ *and* $\tilde{\Sigma} \in \mathbb{R}^{d^{\mathcal{X}} \times 2}$.

$$\tilde{\gamma}' = \begin{bmatrix} \gamma_1 & \gamma_1 & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & 2\gamma_1 & 0 & 2\gamma_1 & \dots & 0 & \dots & 0 & \dots \\ \vdots & \dots & \dots & \dots & \ddots & \dots & \ddots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & n\gamma_1 & \dots & n\gamma_1 & \dots \end{bmatrix},$$

$$\tilde{b} = \begin{bmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ \kappa_2\theta_2 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \kappa_2\theta_2 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \vdots \end{bmatrix},$$

$$\tilde{\beta} = \begin{bmatrix} -\kappa_1 & \kappa_1\theta_1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & 0 & 0 \\ 0 & -\kappa_2 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & 0 & 0 \\ & & \ddots & & & & & & & & \vdots \\ \dots & \dots & & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \kappa_2\theta_2 & \dots & \kappa_1\theta_1 & \dots & -(n-2)\gamma_1 & -(n-2)\gamma_1 & -\kappa_1-\kappa_2 & \end{bmatrix},$$

$$\tilde{\Sigma}(\mathcal{Y}_{t,2}, \mathcal{X}_{t,2}) = \begin{bmatrix} \sigma_1\sqrt{X_{1t}(Y_t - X_{1t})} & 0 \\ 0 & \sigma_2\sqrt{X_{2t}(Y_t - X_{2t})} \\ \vdots & \vdots \\ \sigma_1X_{2t}Y_t^{n-2}\sqrt{X_{1t}(Y_t - X_{1t})} & \sigma_2X_{1t}Y_t^{n-2}\sqrt{X_{2t}(Y_t - X_{2t})} \end{bmatrix},$$

$$d\underline{W}_t = \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix}.$$

Proof. To prove this proposition we need to compute the dynamics of the generic component of the process, so we will take $X_1^i X_2^j Y^r$ with i, j and r such that $0 < i + j + r \leq n$,

$$\begin{aligned} d(X_{1t}^i X_{2t}^j Y_t^r) &= iX_{1t}^{i-1} X_{2t}^j Y_t^r dX_{1t} + jX_{1t}^i X_{2t}^{j-1} Y_t^r dX_{2t} + rX_{1t}^i X_{2t}^j Y_t^{r-1} dY_t + \\ &+ \frac{i(i-1)}{2} X_{1t}^{i-2} X_{2t}^j Y_t^r d < X_1 >_t + \frac{j(j-1)}{2} X_{1t}^i X_{2t}^{j-1} Y_t^r d < X_2 >_t \\ &= i\kappa_1(\theta_1 X_{1t}^{i-1} X_{2t}^{j+1} Y_t^r - X_{1t}^i X_{2t}^j Y_t^r)dt + i\sigma_1 X_{1t}^{i-1} X_{2t}^j Y_t^r \sqrt{X_{1t}(Y_t - X_{1t})} dW_{1t} + \\ &+ j\kappa_2(\theta_2 X_{1t}^i X_{2t}^{j-1} Y_t^{r+1} - X_{1t}^i X_{2t}^j Y_t^r)dt + j\sigma_2 X_{1t}^i X_{2t}^{j-1} Y_t^r \sqrt{X_{2t}(Y_t - X_{2t})} dW_{2t} + \\ &- r\gamma_1(X_{1t}^{i+1} X_{2t}^j Y_t^{r-1} + X_{1t}^i X_{2t}^{j+1} Y_t^{r-1})dt + \frac{\sigma_1^2 i(i-1)}{2}(X_{1t}^{i-1} X_{2t}^j Y_t^{r+1} + X_{1t}^i X_{2t}^j Y_t^r)dt + \\ &+ \frac{\sigma_2^2 j(j-1)}{2}(X_{1t}^i X_{2t}^{j-1} Y_t^{r+1} + X_{1t}^i X_{2t}^j Y_t^r)dt \\ &= \Bigg[i\kappa_1(\theta_1 X_{1t}^{i-1} X_{2t}^{j+1} Y_t^r - X_{1t}^i X_{2t}^j Y_t^r) + j\kappa_2(\theta_2 X_{1t}^i X_{2t}^{j-1} Y_t^{r+1} - X_{1t}^i X_{2t}^j Y_t^r) + \\ &- r\gamma_1(X_{1t}^{i+1} X_{2t}^j Y_t^{r-1} + X_{1t}^i X_{2t}^{j+1} Y_t^{r-1}) + \frac{\sigma_1^2 i(i-1)}{2}(X_{1t}^{i-1} X_{2t}^j Y_t^{r+1} + X_{1t}^i X_{2t}^j Y_t^r) + \\ &+ \frac{\sigma_2^2 j(j-1)}{2}(X_{1t}^i X_{2t}^{j-1} Y_t^{r+1} + X_{1t}^i X_{2t}^j Y_t^r) \Bigg] dt + i\sigma_1 X_{1t}^{i-1} X_{2t}^j Y_t^r \sqrt{X_{1t}(Y_t - X_{1t})} dW_{1t} + \\ &+ j\sigma_2 X_{1t}^i X_{2t}^{j-1} Y_t^r \sqrt{X_{2t}(Y_t - X_{2t})} dW_{2t}. \end{aligned}$$

The dynamics of $\mathcal{Y}_n$ can be found by setting $i = j = 0$ and $r = 1, ..., n$ and the one of $\mathcal{X}_n$ can be computed by choosing i, j such that $i + j > 0$.

□

The approximated price of a guaranteed annuity option will be

$$\begin{aligned} GAO_x(t,T)_m &= \frac{g}{\zeta_t} \mathbb{E}\big[\zeta_T f_m(Y_T, X_T) \big| \mathcal{F}_t\big] \\ &= \frac{g}{\zeta_t} \mathbb{E}\left[\left(\alpha(T) + \omega(T) \begin{bmatrix} Y_T \\ X_T \end{bmatrix} \right) f_m(Y_T, X_T) \middle| \mathcal{F}_t \right] \\ &= \frac{g}{\zeta_t} \mathbb{E}\big[a_g' H_{n+1}(Y_T, X_T) \big| \mathcal{F}_t\big] \\ &= \frac{g}{\zeta_t} a_g'(T) \exp\left(\tilde{A}^n (T-t)\right) \begin{bmatrix} \mathcal{Y}_{t,n+1} \\ \mathcal{X}_{t,n+1} \end{bmatrix}, \end{aligned}$$

where $a_g'(T)$ is again obtained following equation (2.29).

2.4.4 Numerical Example

We will present here some numerical experiments concerning the pricing of the aforementioned insurance products in the LHC(1) framework. We will proceed as follows:

i) we will set our model

ii) we will perform an estimation of the prices of the survival, death benefit and the GAO, using the closed formula found in Section 2.4.2

iii) we will carry out a Monte Carlo simulation, in order to show that the price of the GAO found in (2.32) approximates well its real price given by (2.27).

For the specification of the state price density we will follow [19],

$$\zeta_t = exp(-\alpha t) \left(\phi + \psi' \begin{bmatrix} Y_t \\ X_t \end{bmatrix} \right)$$

where α, ϕ and ψ are real valued parameters chosen such that $\zeta_t > 0$ for every $t > 0$.
Then, the set of parameters of our model becomes $(\alpha, \phi, \psi, \gamma, b, \beta)$. According to [1], these parameters

can be estimated by calibration, which consists in finding the values of the parameters which minimise a mean squared error function, or alternatively by performing a quasi-maximum likelihood estimation or a generalised method of moments estimation.
The problem of the estimation of the parameters is outside the scope of this work, this is they are obtained by calibrating the model to the inverse of benchmark portfolio (the MSCI world index) and the longevity index (LLMA index related to the German population) with a sample from January 1970 to January 2013.

For the value of the benefit C and the annual payment g (involved in the GAO price), we set without loss of generality $C = 100$ and $g = 1$.

For the approximation of the payoff concerning the GAO, as it involves, under the assumptions of the LHC(1), the positive part of a polynomial of order 2, we decided to use a polynomial of the same order.
In order to do that we will proceed by using a second order Taylor series around the points $x = 0$ and $y = 0$.
We recall that for a two variables function $f(x, y)$, its Taylor series to second order around the point (a, b) is equal to

$$\begin{aligned} f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + \frac{1}{2}f_{yy}(y-b)^2 + \\ + f_{xy}(a,b)(x-a)(y-b) + o((x-a)(y-b)) \end{aligned}$$

where $f_{.}$ is the first order derivative of f, $f_{..}$ is the second order derivative of f and $o((x-a)(y-b))$ is the Peano form of the remainder.

The approximation of the GAO payoff is performed with $MatLab$ $R2016b$, where we used the built-in function $taylor(f, order, point)$, which takes three input arguments: the function to approximate, the order of approximation and the expansion point. This procedure gives us the following result (for a whole life annuity of 10 years)

$$\left(a_x(T) - \frac{1}{g}\right)^+ \approx 5.17 + 0.98X_T + 3.14Y_T + 1.23X_T^2 + 2.67Y_T^2 + 0.6X_TY_T.$$

Remark 2.36. *Even if the $(\cdot)^+$ function is not differentiable everywhere, this does not cause any problem from a numerical point of view.*

Table (2.11) provides the prices of the survival, the death benefit and the GAO for an individual of age x at time $t = 0$ and for different maturities $T = \{0, 10, 20, 30, 40, 50, 60, 70\}$. Figure (2.1), Figure (2.2) and Figure (2.3) show us the behaviour of this life insurance product prices with respect to the maturity T.

Maturity T	$SB_0(T)$	$DB_0(T)$	$GAO_x(0,T)_m$
0	100	0	7.8494
1	91.6605	4.9101	5.8915
10	61.4419	36.4830	3.7564
20	37.2664	61.7419	2.2785
30	22.6032	77.0622	1.3820
40	13.7095	86.3544	0.8383
50	8.3153	91.9904	0.5084
60	5.3020	95.4089	0.3084
70	3.0590	97.4822	0.1870

Table 2.1: Price of the survival, death benefit and guaranteed annuity option

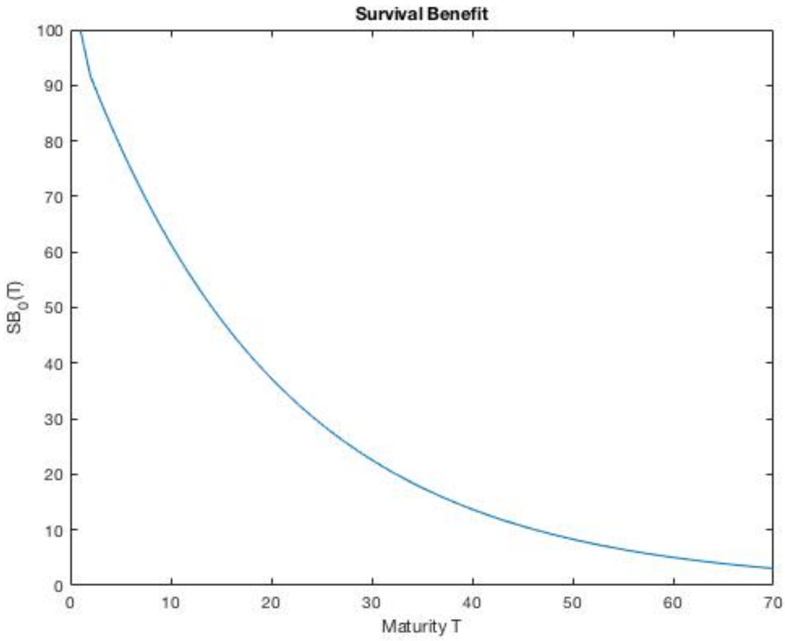

Figure 2.1: SB price as a function of the maturity

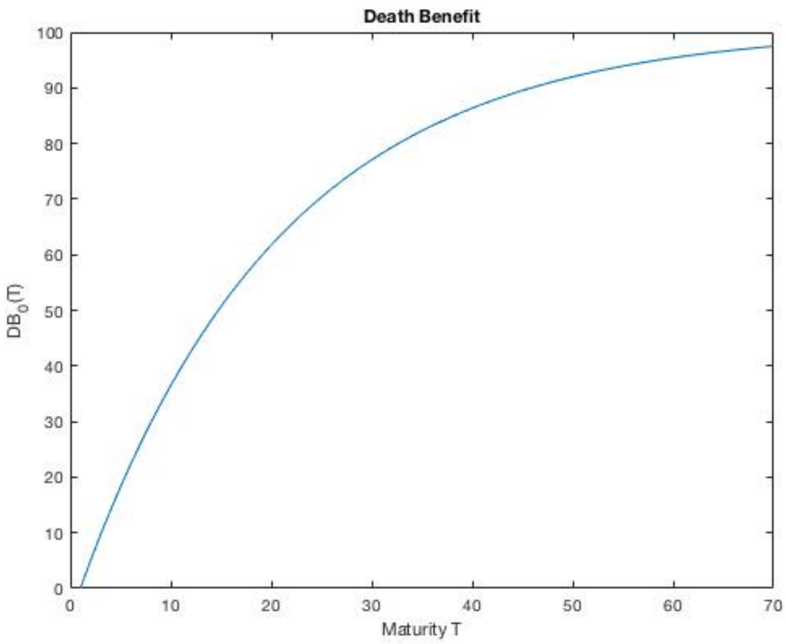

Figure 2.2: DB price as a function of the maturity

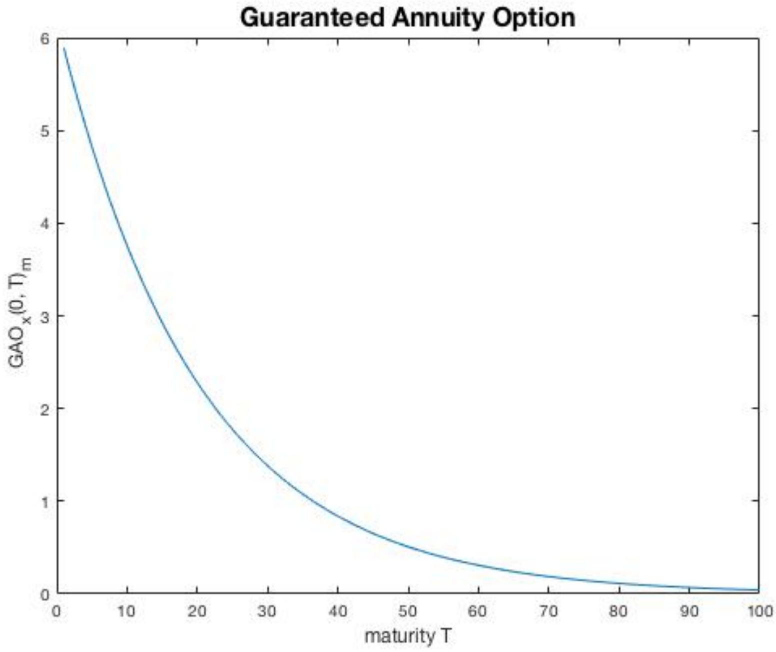

Figure 2.3: GAO price as a function of the maturity

We can split our comments in three parts, one for each insurance product we evaluated

i) concerning the survival benefit we can notice, observing the second column of table (2.1) and figure (2.1), that it is a decreasing function of the maturity, which is quite trivial, because as time goes by, the probability of surviving decreases, so does the price.

ii) the same reasoning, but reversed, can be done about the death benefit. Looking at the third column of the table (2.1) and figure (2.2), we notice that it is an increasing function of the time T. As the maturity rolls away from the initial date $t = 0$, the probability of surviving decreases, which has as a consequence the fact that the price of this benefit increases over time.

iii) finally, observing the last column of table (2.1) and figure (2.3), we can notice that the price of the guaranteed annuity option decreases with respect to the term of this contract. This can be explained by the fact that this option contains the sum of survival benefits, so, as the survival benefit is a decreasing function of T, the GAO as well will have the same behaviour.

Even if these comments are quite trivial, this means that the model we built, prices correctly our insurance products, at least the survival and death benefit, as we used their exact closed formula for their pricing.

Monte Carlo Simulation We can now carry out a Monte Carlo simulation, in order to check if the approximated formula given by (2.30), gives us a good approximation of the real price of the GAO.
To do that we have to deal with the discretization of our stochastic process (Y, X), given by $(2.31)-(2.32)$. In an attempt to achieve this goal, we have to be careful with the diffusion part of X, which involves a square root, which has to contain positive values only. In order to overcome this problem, there are several possibilities, also presented in [33], which consist in forcing the value of the discretized process to be positive. One choice consists in taking the absolute value of the process who contains the square root, the process X, then in this case, the Euler scheme would be written as

$$\begin{aligned} Y_{t+h} &= Y_t - \gamma h X_t \\ X_{t+h} &= |X_t + (bY_t + \beta X_t)h + \sigma\sqrt{X_t(Y_t - X_t)}B| \end{aligned}$$

where h is the time step of our discretization and B is a normal distribution with 0 mean and a standard deviation equal to $\sqrt{h}$.

The convergence of this scheme is assured by [4] and [7] who proved the weak and strong convergence of such a scheme in a general setting.
We choose a time step $h = 0.1$.

Table (2.2) shows the results of our study for a number of simulation $N = \{1000, 10000\}$ and the absolute error computed with respect the results provided in table 2.1.

Maturity T	N=1000	ϵ_{1000}	CI 95%	N=10000	ϵ_{10000}	CI 95%
0	8.1462	0.2968	(7.7683, 8.5241)	8.1459	0.2965	(7.7789, 8.5129)
1	6.1284	0.2369	(5.6631, 6.5935)	6.1282	0.2367	(5.6775, 6.5789)
10	3.9075	0.1515	(3.7321, 4.0829)	3.9075	0.1515	(3.7400, 4.0750)
20	2.3701	0.0916	(2.2578, 2.483)	2.3700	0.0915	(2.2580, 2.482)
30	1.4375	0.0555	(1.2987, 1.5763)	1.4375	0.0555	(1.2989, 1.5761)
40	0.8719	0.0337	(0.6506, 1.0932)	0.8719	0.0337	(0.6566, 1.0872)
50	0.5288	0.0204	(0.3309, 0.7267)	0.5288	0.0204	(0.3334, 0.7242)
60	0.3207	0.0123	(0.1804, 0.4610)	0.3205	0.0121	(0.1819, 0.4591)
70	0.1945	0.0075	(0.0724, 0.3166)	0.1943	0.0073	(0.0771, 0.3115)

Table 2.2: Monte Carlo simulation with confidence intervals at 95% and absolute error of the guaranteed annuity option price

The first thing we can notice, comparing the last column of table (2.1) and table (2.2), is that the approximated formula for the price of the guaranteed annuity option underestimates the price found with the Monte Carlo method, and this for every maturity T.
We can also see that ϵ, which represents the absolute error between the approximated and the Monte Carlo prices is consistent and it is a decreasing function of the maturity T. In addition the value of ϵ seems to be independent from the number of sumulations N.
This means that our approximation does not provide a good estimation of the price of a GAO, this can be due to the fact that maybe we are using a polynomial with a low degree (the order two in this experiment). So, we can try to see what happens if we approximate the payoff of the GAO with a higher degree polynomial. According to [1], the price approximation stabilises rapidly such that a polynomial of degree 10 appears to be accurate in order to provide a good approximation of our payoff.
In this case our approximated function, always found using a Taylor approximation, will contain 66 terms, as we deal with a ten degree polynomial of two real positive variables x and y.
Table (2.3) presents, for different maturities T, the approximated price of the GAO option when using a ten degree polynomial.

Maturity T	$GAO_x(t,T)_m$	ϵ_{1000}	ϵ_{10000}
0	8.1453	0.0009	0.0006
1	6.1287	0.0003	0.0005
10	3.9067	0.0008	0.0008
20	2.3696	0.0005	0.0004
30	1.4381	0.0006	0.0006
40	0.8726	0.0007	0.0007
50	0.5283	0.0005	0.0005
60	0.3198	0.0009	0.0009
70	0.1949	0.0004	0.0004

Table 2.3: Approximated price of a GAO using a ten degree polynomial with the absolute error

Remark 2.37. *The value of ϵ has been computed with respect to the Monte Carlo prices found in table* (2.2).

As we can see from this table, the absolute error seems to be significantly lower than before, which means that this approximation works better than the previous one. In addition, we can remark that the error does not seem to be dependent on the maturity of the insurance contract, as we remarked in the last experiment with the two order polynomial.
This means that a polynomial of order 10 is able to well approximate the payoff of a guaranteed annuity option in a linear rational framework.

Sensitive Analysis In this paragraph we will carry out a sensitive analysis by perturbing, one at a time, the parameters that affect the survival probability involved in the approximated GAO price: (γ, b, β). We focused on the price of a guaranteed annuity option with maturity $T = 10$.
Figure (2.4), (2.5) and (2.6) shows the sensitivity of $GAO_x(0, 10)$ with respect to the parameters γ, b and β.

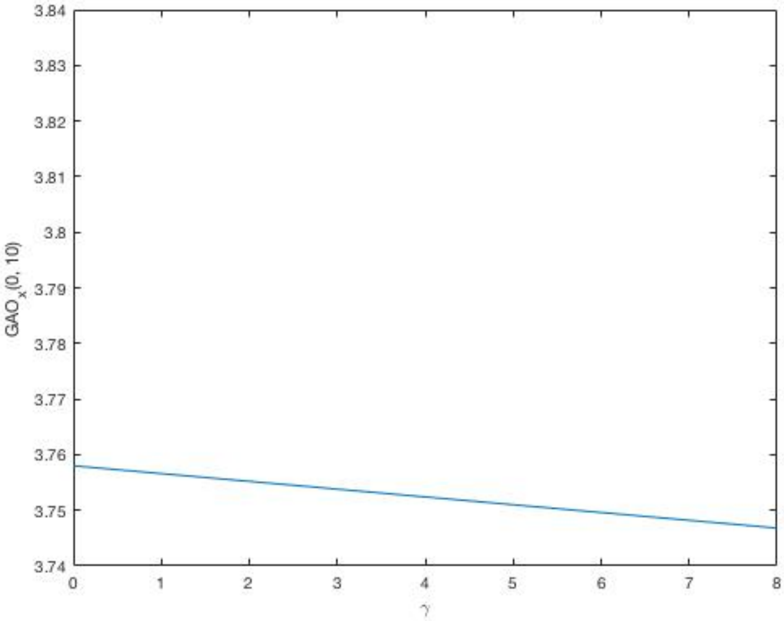

Figure 2.4: GAO price as a function of γ

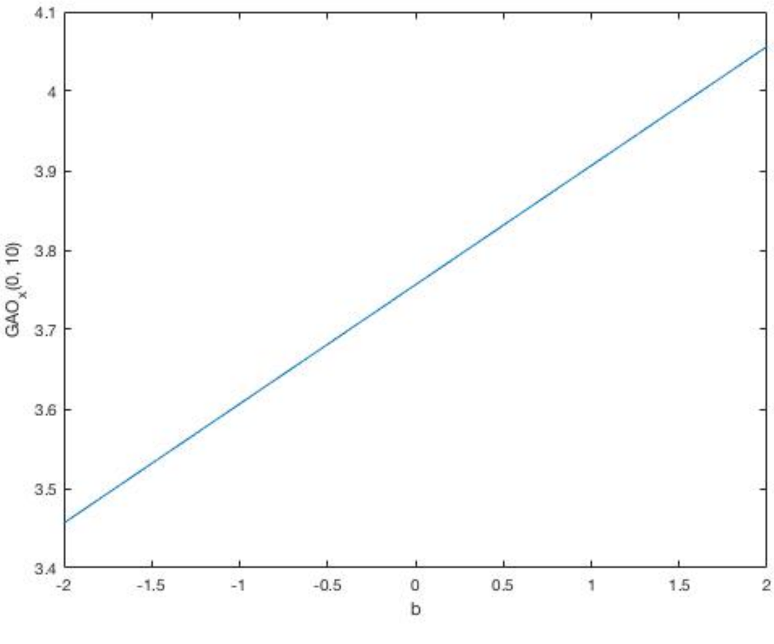

Figure 2.5: GAO price as a function of b

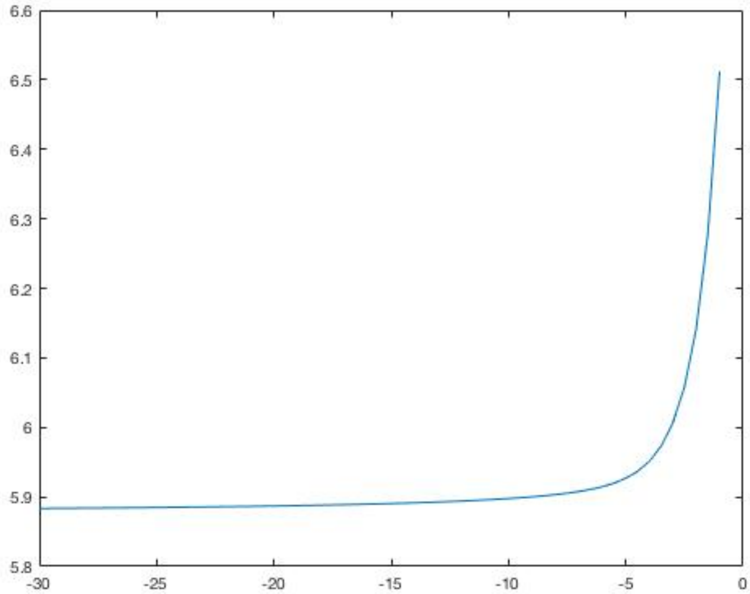

Figure 2.6: GAO price as a function of β

In Figure (2.4) we show the impact of the parameters γ on the price of the GAO. We can notice that as γ increases, the value of the insurance contracts decreases, but very slowly. This is reasonable, as we can see in Remark 2.28, the value of the mortality intensity is an increasing function of γ, so the bigger μ_t is, the smaller will be the survival probability.
The range of the parameter γ has be chosen in order to satisfy the conditions stated in Remark 2.31, as the other parameters (b and β) are fixed.
Figures (2.5) and (2.6) show the sensibility of the guaranteed annuity option price with respect to the parameters b and β. As we can see, the price of the GAO is an increasing function for both of these parameters, which means that they reduce the survival probability as their value increases. We can observe that for the b parameter, the relationship looks like to be linear, whereas for β, it seems to be like a reciprocal function, with a vertical asymptote for $\beta = -(b+\gamma)$. This is a predictable result, as from Remark 2.31 and the fact that γ must be positive, the only values β can take have to be negative, and at most equal to $-(b+\gamma)$.

Conclusion

We worked in a linear rational framework in order to price a family of insurance liabilities. To achieve this goal, we focus on the pricing of the quantities which are building blocks for these products, as well other class of life insurance conracts which are functions of these building blocks.
One of the main features of this framework is that under the hypothesis that the benefit is a constant value (which, despite its simplicity, covers a set of contracts that are actually traded on the market) or a polynomial function of a linear drift process, we are always able to calculate the exact price of the survival and death benefit, thanks to the result found in Theorem 2.14. This is also possible in a linear rational framework where the coefficients of the mortality intensity are not anymore constant, but time dependent. This was motivated by the fact that in practice our model should be able to replicate the observed trend in mortality, which cannot always be done if we consider models with constant coefficients.

The most important feature of this work is that, even when considering an insurance product whose payoff is not necessarily a linear function, we are capable, under the condition that our state space is compact, to approximate this function by a polynomial, which allows us to benefit once again of the result of Theorem 2.14.
This is very important because, instead of treating a complex function, which can involve a very large number of terms (see for instance the whole life annuity (2.25)), we treat a polynomial, which is easier to handle in our setting.
This advantage of the model is then illustrated in a numerical example at the end of this work. We showed that the choice of the degree of the polynomial which approximates the payoff function is a crucial step when pricing a GAO: a low degree, such as two, cannot guarantee a good approximation, if compared to the Monte Carlo price, its absolute error being a decreasing function of the maturity of the contract and it does not depend on the number of simulations we have performed.
Conversely, if we increase the order of the polynomial which approximates the payoff function, the approximation starts to be more accurate as the absolute error decreases and it is no more a function of the maturity of the guaranteed annuity option.

Part II

Economic Scenario Generator

Introduction

BPI France Financement

The Banque publique d'investissement or Bpifrance is a french institution who finances the enterprises. Its aim is to support the french firms, specially the small and the mid size enterprises, but also the big ones, when they have a dimension such that they are important for the national economy or the employment.
Bpifrance supports the french enterprises from the beginning to the quotation in Stox exchange, proposing different solutions:

i) co-financing the other banks to support different projects.

ii) funding of innovation: Bpifrance supports innovating processes through financial aids, or in the form of equity participation.

iii) seizure of new markets in France or in foreign countries: Bpifrance supports export projects (development, implantation) with UBIFRANCE and Coface, partners for Bpifrance Export.

iv) investments in own funds: Bpifrance invests, alongside public and private actors, in investement capital funds, which invest in SMEs.

v) collateral for the other banks: Bpifrance provides a guarantee according to the financial projects to encourage the SMEs during the more risky phases.

Guarantee Activity

Bpifrance provides banks with a guarantee from 40% to 70% of the loan value, depending on the financial projects, to encourage them to finance SMEs, in the most risky phases. The banks, benefiting from the guarantee, pay a guarantee premium (commission) which is calculated according to the purpose of the project and the amount of financing. The commission can be collected at the starting date of the loan or collected, over time. In case of default, the payment is interrupted.

Several guarantee funds are backed by the different coverages offered by the Bpifrance guarantee. They are classified in National Funds (Creation, Transmission, Development, for example) and Regional. Several flows debit or credit a guarantee fund. Indeed, a guarantee fund is credited by:

i) The fees paid by the banks for the Bpifrance guarantee (it only covers part of the Bpifrance risks).

ii) The Government's endowment to cover losses with a probability of 70%.

iii) The financial products that are resources resulting from the management of the assets of the guarantee fund.

Conversely, a guarantee fund is also debited by:

i) Possible compensation when the guarantee is involved.

ii) Costs resulting from the delay between the guarantee and the compensation

Economic Scenario Generator

An ESG is a mathematical tool that allows us to produce simulations of the joint behaviour of financial and economic variables.
Economic Scenario Generators are used by financial institutions to assess the impact of such scenarios on the firm's asset and liabilities.

According to [16], an economic scenario generator is integral to an enterprise risk management approach and is a critical component of the ensemble of models that analyse the external risks to an organisation. The ESG can be seen as the basis for projecting economic and capital market scenarios for use in financial risk management applications.
Two common applications, also presented in [16] and [39], are:

i) market consistent valuation for pricing complex financial derivatives and insurance contracts with embedded options. This kind of implementation involves mathematical relationships within and among financial instruments, whereas the forward-looking expectation of economic variables is less concerned.
In this case the ESG has to be able to reproduce the observable prices of the traded derivatives, in order to determine comparable prices for derivatives instruments and insurance contracts with embedded options that are not traded but that require market valuation. In this case the generator must have some strict mathematical properties : risk neutral measure and absence of arbitrage condition. In this case the calibration of the model depends on the pricing date and on the set of financial derivatives available. The validation associated to this calibration is based on how the model reproduces the market value of all the derivatives used to perform the calibration.

ii) real world models for risk management for calculating regulatory capital and rating agency requirements. This application concerns the forward-looking expectation of economic variables and their potential influence on capital and solvency.
In this case the generator has to be able to produce the possible future path of a set of economic variables. It requires explicit view of how the economy will develop in the future and also a significant amount of expert judgement to state the likelihood of the scenarios generated. Real world models are in general parameterized and calibrated to be consistent with the historical dynamics of the variables, although it is possible to parametrize the model in order to match the current consensus expectations about the future path of the economic variables.

Other, less developed, applications of ESG are related to the insurance industry and in particular

i) Life insurance: application of ESG are focused on the interaction between different factors, such as interest rate changes, annuity payout and investment performance on assets. Because of the complexity of these interactions over an extended time horizon, an economic scenario generator would help to understand the likelihood of different scenarios and the range of potential outcomes. Typical applications in this field are: life liability valuation and stress testing.

ii) Pension Funds: similarly to the case of the life insurance, the ESG could have different applications in the pension industry: liability-driven investments (investment strategy based on the cash flows needed to fund future liabilities) and pension risk transfer. The test of different strategies of both aforementioned aspects can be done using an ESG.

It is important to note that the use of ESG, as stated in [39], is included or considered in possible responses to evolving regulatory rules and principles, keeping in mind that the use of ESG is still emerging. This applications are used to test the solvency and liquidity and the risk exposure of a given bank. The framework, where the economic scenario generators are included, can be found in the principles of ComFrame (a common framework for international insurance supervision) and in the EIOPA rules (the European Insurance and Occupancy Pensions Authority). With respect to the risk management it is shown that ESG are more effective to determine the required allowance. Nevertheless the use of this tool is only allowed but not required.

In order to build an ESG, it is important to specify what are the minimum requirements in order to create a comprehensive tool:

i) it must be able to capture the critical features of the historical data.

ii) it must be composed by a set of models that is sufficient to capture the most important financial and economic factors of a given institution.

iii) it must meet the requirements of the regulators and provides enough information for model validation

iv) it must produce extreme but also plausible outcomes

v) it must bw computationally efficient and numerically stable

We usually have, always according to [39], two classes of generators:

i) composite: we will first describe each asset class and then aggregate them to get an overall description of our economy. For example: we assume for the shares a Black-Scholes model and for the short rate a Vasicek and then we think about the correlation between these two assets.

ii) integrated or cascade: starting from an explanatory variable, we will describe each asset class as a function of this variable. For example the inflation.

Composite models are the most used because it is only necessary to introduce the correlations between the assets, the point of weakness of this approach is that it does not take into account that there are other variables that may affect other asset classes.
Integrated models have been created to deal with this weakness, by describing the factor from which the other elements of the model derive.

The advantages and disadvantages of both models can be summarised in the following scheme

A) Composite models

   i) Advantages: independent modelization

   ii) Disadvantages: correlation structure determined from history and we exclude arbitrage opportunities, these models do not take into account macroeconomic factors that may affect the price of the asset

B) Integrated models

   i) Advantages: taking into account macroeconomic factors

   ii) Disadvantages: we have to be careful about the number of parameters, an high number of parameters could lead to estimation problem or overfitting of the model, it means that there are more parameters than can be justified by the data

The aim of the Quantitative Engineering Domain is to study other financial sources of the guarantee activity and innovation less reliant on public resources in the medium and long term: financial products resulting from an optimal management of assets can be a source of financing. In this context, the Quantitative Engineering Domain is working on the development of quantitative methods of asset allocation. The desire is to set up an asset allocation tool with several modules: construction of economic scenarios, search for optimal allocation and analysis of allocations and production of indicators.
The aim of this work is to develop an economic scenario generator module which will consist in the choice of a diffusion model for equity indices and inflation, its calibration and validation.
The first thing to do will be to present the most frequently used models, concerning interest rates, equity indices and macroeconomic factors (inflation in our case), that can found in a standard ESG of any investment bank.
The second step is to study, by a principal component analysis, the present portfolio of Bpifrance in order to find which components explain the most of its variability and select on the market the risk factors that allows us to replicate them. This step is then concluded by an univariate and multivariate statistical analysis of the risk factors that have been chosen, in order to select the most appropriate diffusion model among the ones presented at the beginning. The study is then completed by the estimation of the parameters of the selected models (for the equity indices and the inflation) and by their validation based on different tools such as confidence intervals and qq-plot.

Chapter 3

Model Selection

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$. $\mathbb{P}$ is the historical measure.

3.1 Interest Rate Modeling

3.1.1 Libor Market Model

Introduced by [9] it is a pricing model used to price interest rate derivative products such as swaptions, caps, floors, etc. The quantities that are modeled are a family of forward rates (LIBORs) which have the advantage of being directly observed in the market and whose volatility is naturally linked to traded contracts. Each forward rate is modeled by a log-normal process under its forward measure.
To adapt this in the Real World universe, it will be necessary to integrate a risk premium.
We will set

i) $\sigma_k(t) = ae^{-c(\tau_k - t)}$,

ii) $\rho_{k,l}dt = d < W_k(t), W_l(t) >$, where $W_{.}(\cdot)$ is a standard brownian motion,

iii) $F_k(t) = \frac{1}{\tau_k}(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1)$ is the simply compounded forward interest rate, where $P(t, T_{.})$ is the t-price of a zero coupon bond which pays 1 at time T,

iv) λ is the constant risk premium that changes the drift part without modification of the volatility,

v) $\tau_j = T_j - T_{j-1}$.

$$\begin{aligned} dF_k(t) &= \sigma_k(t)F_k(t)\sum_j \frac{\tau_j F_j(t)\sigma_j(t)\rho_{j,k}}{1+\tau_j F_j(t)}dt + \sigma_k(t)F_k(t)dW_k(t) \\ &= \sigma_k(t)F_k(t)\sum_j \frac{\tau_j F_j(t)\sigma_j(t)\rho_{j,k}}{1+\tau_j F_j(t))}dt + \sigma_k(t)F_k(t)dW_k^{\mathbb{P}}(t) + \sigma_k(t)F_k(t)\lambda dt, \end{aligned} \tag{3.1}$$

Remark 3.1. *The last inequality follows by a change of measure, we can pass from the forward measure to the real word measure thanks to Girsanov theorem (see* [22]*), $dW_k(t) = dW_k^{\mathbb{P}}(t) + \lambda dt$*

So, there are 3 groups of parameters to estimate:

i) the parameters of the volatility (a, d),

ii) the risk premium λ,

iii) the correlations $\rho_{j,k}$.

Remark 3.2. *It is important to note that in this second part of the thesis τ_j represents the length of the interval $[T_{j-1}, T_j]$, in the first part τ was a stopping time, i.e. a random variable.*

Estimation

In order to estimate the parameters, we need the history of the monthly forward rates (to obtain the different monthly data we must also make interpolations of the yield curves that are available on the market). The steps to do are:

1) estimation of inter-forward instantaneous correlations based on a hypothesis of normality

$$\left(\log(\frac{F_i(t\Delta)}{F_i(t)})\right)_t \sim N(\mu, V)$$

$$\hat{\rho_{i,j}} = \frac{\hat{Cov}_{i,j}}{\hat{V}_i\hat{V}_j}.$$

2) estimation of volatility parameters and risk premium as follows

   a) study of the law of $\epsilon_t^{\Delta} = \sum_{k=1}^{M} ln(\frac{F_k(t\Delta)}{F_k(t)})$,

   b) log-likelihood associated with $(\epsilon_{t_0 i\Delta}^{\Delta})_{i\in[0,N-1]}$, which depends on the different unknown parameters (correlation parameters fixed at point 1), we will note this function as $l(a, d, \lambda)$,

   c) maximising the log-likelihood function in (a, d, λ)

$$\max_{(a,d,\lambda)} l(a, d, \lambda).$$

Simulation

To make simulations it is necessary to know how to diffuse the interest rate with the chosen model

i) discretization of the forward rate dynamics

$$F_k(t + \Delta t) = F_k(t)e^{(\mu_k(t)\sigma_k(t)+\sigma_k(t)\lambda-\frac{\sigma_k(t)^2}{2})\Delta+\sigma_k(t)\sqrt{\Delta}\epsilon_k(t)},$$

with $\mu_k(t) = F_k(t)\sum_j \frac{\tau_j F_j(t)\sigma_j(t)\rho_{j,k}}{1+\tau_j F_j(t)}$ and $\epsilon_k(t) \sim N(0, 1)$,

ii) simulation of the random variables,

iii) simulation of the interest rate from the discretization carried out.

3.1.2 Principal Component Analysis Vasicek Model

The Principal component analysis Vasicek Model consists in finding the three factors (level, slope and curvature), that explain the movements of the yields curve, by performing a principal component analysis (see [34]).
Before presenting the model, we will synthesise the Principal Components Analysis (PCA in the following) procedure.

The PCA was for the first time introduced in [38] and then developed in [30].
It is is a statistical tool which consists in transforming a large set of possibly correlated variables into a new smaller set of uncorrelated variables which can explain a large part of the variance of our phenomenon.
PCA is mathematically defined using the spectral decomposition theorem of Linear Algebra, which states that any real symmetric matrix Q can be written as $Q = ALA'$, where L is a diagonal matrix whose value in the main diagonal are the eigenvalues of Q, and A is an orthogonal matrix (such that $A' = A^{-1}$) whose columns are the normalised eigenvectors of Q (see [17]).

We will consider the interest rate to model as

$$R(t, m) = \frac{1}{P(t, m)^{1/m}} - 1, \tag{3.2}$$

where $P(t, m)$ is the price in t of a zero coupon bond paying 1 at maturity m.
We will set:

i) $(R(t_0 + k\Delta, m))_k$ the time series of historical observations,

ii) t_0 the initial date,

iii) Δ the time step,

iv) T the size,

v) M the maximum number of maturities on the interest rate curve.

Let

$$LR(k,m) = \log\left(\frac{R(t_0 + (k+1)\Delta, m)}{R(t_0 + k\Delta, m)}\right),$$

$$\mu(m) = \frac{1}{T}\sum_{k=1}^{T} LR(k,m),$$

$$\sigma(m) = \sqrt{\frac{1}{T}\sum_{k=1}^{T}\big(LR(k,m) - \mu(m)\big)^2},$$

be respectively the log-ratio, its mean and its standard deviation.
Following Section 3.4 of [17], the idea is to perform a PCA on the history of the normalised log-ratios of the interest rates in order to identify the three usual components of the interest rate risk (level, slope and curvature) and then model them according to the discrete Vasicek model, which is

$$Y_n(k) = Y_n(k-1) + \theta_n\big(\mu_n - Y_n(k-1)\big)dt + \sigma_n\sqrt{\Delta}\epsilon_n(k), \tag{3.3}$$

where the subscript $n \in \{1,2,3\}$ indicates the first three factors issued from the PCA.
We calculate the normalised log-ratio $X(k,m) = \frac{LR(k,m)-\mu(m)}{\sigma(m)}$ and we construct the matrix of observations of standardised log-ratios X. Then, we compute the covariance matrix S associated to X and on which we will apply the PCA

$$S(m,n) = \frac{1}{T}\sum_{k=1}^{T} X(k,m)X(k,n).$$

Finally, provided that S is diagonalisable (see conditions on [29]), we diagonalise S: that is, a diagonal matrix D and a change-of-basis matrix P such that

$$P'DP = D.$$

Remark 3.3. *It is important to notice that, according to* [17]*, the distribution of the random variables $LR(k,m)$ is unknown, the only assumption on this distribution is that those variables are identically distributed with mean $\mu = (\mu(m))_{m>0}$ and covariance matrix S.*

Estimation

The estimation of the parameters of the Vasicek model is done in two steps. We set the level of mean-reversion equal to zero for all the components and then we estimate the remaining parameters by maximum likelihood. Let $(Y_n(k))_k$ be the vector of the observations of the variable $n = \{1,2,3\}$, then the log-likelihood associated with this vector is

$$l(\theta_n, \sigma_n, Y_n(k)) = \sum_{k=2}^{T}\Big[-\frac{1}{2}ln(2\pi\Delta) - \frac{1}{2}ln(\sigma_n^2) - \frac{1}{2\Delta\sigma_n^2}(Y_n(k) - (1-\theta_n\Delta)Y_n(k-1))^2\Big], \tag{3.4}$$

that gives us the following maximum likelihood estimators

$$\hat{\theta}_n = \frac{1}{\Delta}\left(1 - \frac{\sum_{k=2}^{T})Y_n(k-1)Y_n(k)}{\sum_{k=2}^{T} Y_n(k-1)^2}\right), \tag{3.5}$$

$$\hat{\sigma}_n = \frac{1}{(T-1)\Delta}\sum_{k=2}^{T}\big(Y_n(k) - (1-\hat{\theta}_n\Delta)Y_n(k-1)\big)^2. \tag{3.6}$$

Simulation

For the simulation of the interest rates we proceed as follows

i) diffusion on a horizon of H years with step Δ,

ii) diffusion of the Zero coupon:

a) diffusion of the main components via the relation

$$Y_n\left(\frac{k}{p}\right) = (1-\hat{\mu}_n\Delta)Y_n\left(\frac{k-1}{p}\right) + \hat{\sigma}_n\sqrt{\Delta}\epsilon_n\left(\frac{k}{p}\right),$$

b) reconstruction of the log-ratios via

$$X\left(\frac{k}{p}, m\right) = \sum_{n=1}^{3} Y_n\left(\frac{k}{p}\right)P_{(n)}(m),$$
$$LR\left(\frac{k}{p}, m\right) = \sigma(m)X\left(\frac{k}{p}, m\right),$$

where $P_{(n)}$ is the n-th column of the change-of-basis matrix P.

3.2 Modelisation of shares and stocks indexes

For the modelling of equity indices, it is important to use models that are able to take into account the main stylized facts that we can observe in the market (variable volatility, autocorrelation, jumps, ...), see [12]. So we consider three models: the Merton model, the GARCH model and the GARCH jumps model.

3.2.1 Merton Model

In this section we will follow [25], so let

$$R(t) = ln\left(\frac{S(t+1)}{S(t)}\right) - r(t), \tag{3.7}$$

$$Y(t) = \sum_{k=1}^{N(t)}(U_k - 1), \tag{3.8}$$

be respectively the log-return rescaled from the risk free rate $r(t)$ and a compounded Poisson process with intensity λ, where U_k is a random variable such that $\log(U_k) \sim N(m, \delta^2)$, with $m \in \mathbb{R}$ and $\delta > 0$. Let

i) $(W(t))_{t>0}$ be a Brownian motion,

ii) $\bar{\mu} = e^{m+\delta^2/2}$,

iii) $\tilde{S}(t) = S(t)e^{-\int_0^t r(s)ds}$,

iv) μ be the risk premium,

v) $\epsilon(t) \sim N(0, 1)$.

In this model we suppose that the real world dynamics of the equity index is,

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = (\mu - \lambda\bar{\mu})dt + \sigma dW(t) + dY(t), \tag{3.9}$$

that gives us, after discretisation, with time step $\Delta t = 1$,

$$R(t) = (\mu - \lambda\bar{\mu}) + \sigma\epsilon(t) + U\Delta N(t), \tag{3.10}$$

where

i) $N(t)$ and $\Delta N(t)$ are respectively a Poisson process and a Poisson random variable with intensity λ,

ii) U is a random variables such that $U \sim N(m, \delta^2)$.

In addition, $N(t), \Delta N(t)$ and U are mutually independent.
For the parameters we can distinguish two possible cases: the case $m \neq 0$ and the case $m = 0$, which allows to have more stable calibration and simulation results.

Estimation and Calibration

It is divided into two parts:

1) estimation of the parameters by the method of the moments: it consists in equalising the theoretical moments of our model with the empirical moments (computed on the data), and solves this system as a function of the parameters

$$\begin{cases} \mu_{emp} = \mu - \lambda\bar{\mu} \\ \sigma_{emp} = \sqrt{\sigma^2 + \delta^2\lambda(1+\lambda)} \\ \gamma_{emp} = 0 \\ \kappa_{emp} = \frac{3\sigma^2+\lambda[3\delta^4+21\delta^4\lambda+6\delta^2\sigma^2+18\delta^4\lambda^2+6\sigma^2\delta^4\lambda+3\delta^4\lambda^3]}{(\sigma^2+\delta^2\lambda(1+\lambda))^2} \end{cases},$$

where the left-hand side of the equation contains the empirical moments computed on the data (the mean, the variance, the skewness and the kurtosis), and on the right-hand side we have the corresponding theoretical moments computed under the discretised Merton model.
However, in the case where the number of parameters is high, as suggested in [16], their value can be calculated by minimising a well-chosen error function,

$$f(\mu, \sigma, \lambda, \sigma_u) = \left(\frac{\mu_{th} - \mu_{emp}}{\mu_{emp}}\right)^2 + \left(\frac{\sigma_{th} - \sigma_{emp}}{\sigma_{emp}}\right)^2 + \left(\frac{\gamma_{th} - \gamma_{emp}}{\gamma_{emp}}\right)^2 + \left(\frac{\kappa_{th} - \kappa_{emp}}{\kappa_{emp}}\right)^2,$$

where μ_{th}, σ_{th}, γ_{th} and κ_{th} represent the theoretical mean, variance, skewness and kurtosis, computed according to the Merton model.

2) estimation of the parameters by maximum likelihood. The likelihood function is computed by summing, with respect to all possible realisation of m, the conditioned densities which will have a weight determined by the probability density function of the Poisson. This function is then maximised with respect to the parameters using, as initialisation, the values of 1)

$$\max_{\mu,\sigma,\lambda,\sigma_u} L(R_i, \mu, \sigma, \lambda, \delta) = \prod_{i=1}^{n} \frac{e^{-\lambda}}{\sqrt{2\pi}} \left[\sum_{m=0}^{\infty} \frac{\lambda^m}{m!\sqrt{\sigma^2 + (m\delta)^2}} e^{-\frac{(R_i-\mu+m\bar{\mu})^2}{2(\sigma^2+(m\delta)^2)}}\right]. \tag{3.11}$$

One way to solve this maximisation problem is to truncate the sum in (3.11) at a level η such that the probability of having a number of jumps above this level is less than a value ξ small enough. This means that we will not consider in (3.11) all the term of the summation that does not have an impact on the likelihood function, because they are close to zero.

3.2.2 GARCH $(1, 1)$ model

Following [6], let

$$R_t = \log\left[\frac{S((t+1)\Delta)}{S(t\Delta)} P(t\Delta, (t+1)\Delta)\right], \tag{3.12}$$

be the log-return netted of the risk free rate (with Δ the time step of the history).
This quantity is modeled by a GARCH (1,1) (the choice of this model finds its justification in its simplicity

and in the fact that in practice it is difficult to find something that could do better, according to [26]), so

$$\begin{cases} R_t & = \lambda + \sqrt{h(t)}\epsilon(t) \\ h(t) & = \omega + \alpha(R_{t-1} - \lambda)^2 + \beta h(t-1) \\ & = \omega + \alpha h(t-1)\epsilon(t-1)^2 + \beta h(t-1) \end{cases}, \tag{3.13}$$

with

i) λ the constant risk premium,

ii) $\epsilon(t)$ iid random variables with law $N(0,1)$,

iii) $\omega,\ \alpha$ and β real positive parameters.

In this way the parameters to be estimated become $(\lambda, \omega, \alpha, \beta)$.

Assumption 3.4. *We will assume that $\alpha + \beta < 1$ and $\omega > 0$.*

Estimation

It is done in two steps

1) estimation of the risk premium by the sample mean

$$\hat{\lambda} = \frac{1}{T}\sum_{t=1}^{T} R_t, \tag{3.14}$$

2) estimation of the other parameters by maximum likelihood

$$l(\omega, \alpha, \beta, R_t) = \sum_{t=2}^{T}\Big(-\frac{1}{2}ln(\hat{h}(t)) - \frac{1}{2}\frac{(R_t - \hat{\lambda})^2}{\hat{h}(t)}\Big), \tag{3.15}$$

with $\hat{h}(2) = (R_1 - \hat{\lambda})^2$ and $\hat{h}(t) = \omega + \alpha(R_{t-1} - \hat{\lambda})^2 + \beta\hat{h}(t-1)$ (3.15) is obtained by the normality of R_t given the pas observations.

Mean Reversion

Proposition 3.5. *If we have Assumption 3.4, then the variance of the log-returns converges to a long-term variance*

$$E[h(t)] \xrightarrow{t\to\infty} = \sigma_\infty = \frac{\omega}{1 - \alpha - \beta}$$

Proof. See proof Theorem 1 of [7]. □

The GARCH (1,1) takes into account the tendency of the variance to mean revert to an long-term average level.

Simulation

Let H be the diffusion horizon and Δ the time step, we can divide the simulation into 4 steps:

1) simulation of the random variables involved in the calculations,

2) prediction of the variance h,

3) reconstruction of the log-returns,

4) diffusion of the stocks index (iteratively),

3.2.3 GARCH $(1, 1)$ model with jumps

The purpose of this model, who has been for the first time introduced in [32] and also used in [41], is to try to take into account the presence of serial dependence in the time series of the returns, and the fact that the volatility is a quantity who changes overtime.
In order to do this [32] introduce a jump component in the GARCH $(1, 1)$ model. Let

$$R(t) = ln\left[\frac{S((t+1)\Delta)}{S(t\Delta)}P(t\Delta, (t+1)\Delta)\right],$$

be the log-return netted of the risk free rate (with Δ the time step of the history).
We will model this quantity as follows

$$\begin{cases} R(t) = \lambda + \sqrt{h(t)}\epsilon_1(t) + \epsilon_2(t) \\ h(t) = \omega + \alpha\epsilon(t-1)^2 + \beta h(t-1), \end{cases} \tag{3.16}$$

where

i) $\lambda,\ \epsilon_1(t),\ \omega,\ \alpha$ and β are defined as section 3.2.2,

ii) $\epsilon_2(t)$ is a random variable such that $\mathrm{E}[\epsilon_2(t)] = 0$. This random variable represents the "jump innovation" and can be represented as follows

$$\epsilon_2(t) = \sum_{k=1}^{n(t)} Y_k(t),$$

with $n(t)$ and $Y_k(t)$ random variables independent from $\epsilon_1(t)$.

iii) $\epsilon(t) = \sqrt{h(t)}\epsilon_1(t) + \epsilon_2(t)$

The distribution of jumps is assumed to be a Poisson of parameter γ. Let n_t be the number of jumps between t and $t-1$, then

$$\mathbb{P}(n_t = j) = \frac{e^{-\gamma}\gamma^j}{j!}.$$

The parameter γ can be considered as a constant or time-dependent.
In addition, the magnitude of the jumps $(Y_k(t))$ is assumed to be a Normal centered variable of variance δ^2 for every k.
For the sake of simplicity, we will look at the case of constant parameters over time.

Estimation

There are two stages:

1) estimation of the parameter λ

$$\hat{\lambda} = \frac{1}{T}\sum_{t=1}^{T} R(t),$$

2) estimation by maximum likelihood. We know that R_t, given the realisation of $n(t) = j$, is a Gaussian variable, so we can easily calculate its likelihood, which is given as the product of the conditional density of R_t. The likelihood is then calculated by summing, with respect to all the possible values of $n(t) = j$, the conditional densities which will have a weight determined by the probability function of the Poisson distribution. So, we will maximise the following function

$$\max_{\omega,\alpha,\beta,\gamma,\delta} L(R_t, \omega, \alpha, \beta, \gamma, \delta) = \prod_{t=2}^{T} \frac{e^{-\gamma}}{\sqrt{2\pi}}\left[\sum_{j=0}^{\infty} \frac{\gamma^j}{j!\sqrt{\hat{h}(t) + j\delta^2}} e^{-\frac{(R_t-\hat{\lambda})^2}{2(\hat{h}(t)+j\delta^2)}}\right], \tag{3.17}$$

with $\hat{h}(t) = \omega + \alpha(R_{t-1} - \hat{\lambda})^2 + \beta\hat{h}(t-1)$

As before, we truncate the sum in (3.17) at a level η such that the probability of having a number of jumps above this level is less than a value ξ small enough.

3.2.4 Advantages and Disadvantages of each model

	Jumps	Volatility Clustering	Estimation	Skewness
Merton Model	✓	X	✓	X
GARCH Model	X	✓	✓	X
GARCH Model with jumps	✓	✓	✓	X

This table synthesise the main advantages and disadvantages of the three models we have studied for the modelling of the stock prices.
We can see that there no exists a perfect model, in the sense that it allows to take into account the different aspects we want to replicate: the jumps in the stock price distribution, the volatility clustering, smooth estimation and negative skewness.
Looking for the presence of jumps, only the GARCH model fails in replicating this aspect.
Concerning the volatility clustering (the fact that large/small changes in the stock prices are followed by large/small changes, either signs), the Merton model is the only one (among the models we have studied) that is not able to capture this feature.
When it comes to model the negative skewness that we observe in the real data, we observe that no model is able to treat this, as we always work with normal distribution or Poisson mixture of normal distribution.
Conversely, all the models we treated allows to estimate easily their parameters, this because their likelihood function are simple to be computed, as we deal with gaussian random variables and Poisson mixture of gaussian random variables.

3.3 Macro-economic variables

The aim is to build an economic scenario generator model that is able to represent the different asset classes consistently in a medium-term perspective. We must therefore pay attention to other, more macroeconomic, variables (particularly inflation) which impact the level of premiums and services. An asset model must, on the one hand, incorporate short-term fluctuations in the value of assets and, on the other hand, be consistent in the long term with the macroeconomic balances linking inflation, the interest rates and the growth rate of the economy. In the long term, it is considered that there is a close link between the nominal interest rate, the inflation rate and the real interest rate, as also suggested by [20] who proposed the following equation, also called the Fisher equation,

$$Nominal\ Interest\ Rate = \mathbb{E}[Inflation\ Rate] + Real\ Interest\ Rate. \tag{3.18}$$

At the same time, there is a close long-term link between the real interest rate and the growth rate of the economy. The real interest rate can not be much different from the growth rate of the economy except to bring arbitrages between financial activity and actual activity, on the one hand, between investment in one country and investment in other countries, on the other hand. We can write

$$Long\ Term\ Real\ Interest\ Rate = Long\ Term\ Real\ Growth\ Rate. \tag{3.19}$$

3.3.1 Modelling Inflation

We will discuss here a possible way of modelling inflation, following [16], which is done through the BEIR (break-even inflation rate), which is the expected inflation, which reflects the market's expectations on the future inflation. Let:

i) $Q(t)$ be the consumer price index in t, observable on the market,

ii) $P_r(t,T)$ be the real price of a zero coupon in t delivering in T a currency unit indexed on inflation ($ln(\frac{Q(T)}{Q(t)}) = I(T)$), unobservable on the market,

iii) $P_n(t,T)$ the price of a nominal zero coupon in t for the maturity T, deduced from the yield curve,

We will consider the following contract between two counterparties (A and B):

a) in T, A pays B a cash-flow equal to $(1+K)^T$, which is worth $P_n(0,T)(1+K)^T$ in $t=0$,

b) en T, B pays A a flow equal to $I(T)$, which is worth $P_r(0,T)$ in $t=0$.

So, on the market, for each fixed maturity T, the value that is quoted is the value of K such that the contract has a zero value in 0. Let $K = i(0,T)$ this value, which we will call break-even inflation rate of maturity T (BEIR), we have that

$$P_n(0,T)(1+i(0,T))^T = P_r(0,T).$$

So modelling the real rates comes to model the BEIR rates.

Diffusion of the BEIR

This model is based on the modelling of the absolute variations of break-even inflation rates: $(i(t+1,m)-i(t,m))_t$, where m is the maturity of the BEIR rate. The principle is to apply a PCA on the history of absolute centered BEIR rates.
We note:

i) $(i(t_0+k\Delta,m))_k$ the initial history,

ii) t_0 the initial date of the history,

iii) Δ the step of the history,

iv) T the size of the history,

v) M the maximum number of maturities of the BEIR yield curve.

We compute

$$\begin{aligned}
\Delta i(k,m) &= i(t_0+\Delta(K+1),m) - i(t_0+\Delta k,m),\\
\mu(m) &= \frac{1}{T}\sum_{k=1}^{T} VA(k,m),\\
\sigma(m) &= \sqrt{\frac{1}{T}\sum_{k=1}^{T}\big(\Delta i(k,m)-\mu(m)\big)^2},
\end{aligned}$$

and we will apply a PCA (as in the case of rate modelling).
The first 3 components will then be modelled, estimated and simulated according to a Vasicek model.

Chapter 4

Analysis of the assets

4.1 Introduction

The aim of the internship is to work on the development of an economic scenario generation module and thus the proposal of diffusion models for interest rates and shares, the calibration of these models, their validation and implementation. The desire is to set up an asset allocation tool with several modules: construction of economic scenarios, search for an optimal allocation and analysis of the allocations and production of indicators.
The objective is therefore to propose a ESG that is able to satisfactorily simulate the assets of which it is composed and that can improve the allocation of Bpifrance. The approach can be synthesised according to the following steps:

i) selection of the variables that we want to project in the ESG

ii) check that the variables chosen make it possible to describe the evolution of the portfolio assets of Bpifrance

iii) statistical study of our variables to understand what are the stylized facts to be taken into account when choosing a diffusion model to simulate them: the study consists in the calculation of several indicators (mean, variance, asymmetry, kurtosis, auto-correlations, Pareto alpha and queues dependence index) and in the graphical representation of their distribution

iv) choice of a model for each component of our ESG. In this part it is fundamental to understand how our ESG will be structured (integrated or composite), in order to choose the most adapted models

v) calibration and implementation of the selected models. The calibration of the models will be done on the basis of the historical data, thus in the real world universe

vi) to finish our work, it might be interesting to apply our ESG to an asset allocation problem

4.2 Bpifrance's portfolio

The purpose of this paragraph is to identify which components of Bpifrance's portfolio explain for most the volatility of the portfolio, so that they can be compared with the assets that we want to project in our ESG.
We present here the assets of Bpifrance's portfolio, as far as the "equity" part is concerned

i) MSCI EMU Cons Staples - Tot Return

ii) MSCI EMU Health Care - Tot Return

iii) MSCI EMU Utilities - Tot Return

iv) MSCI EMU Real Estate - Tot Return

v) MSCI Europe Growth Enterprise - Net Return

vi) MSCI Europe Value Enterprise - Net Return

Based on historical data from 2001 to 2018, a PCA has been implemented on these 6 assets. As it can be seen, the first 3 components explain for more than 90% of the variance, so we will take these 3 variables to calculate the correlations with the assets we selected.

Components	Eigenvalues	Difference	Proportion	Cumulation
1	4.5918	4.0252	0.7653	0.7653
2	0.5666	0.2387	0.0944	0.8597
3	0.32793	0.0786	0.0546	0.9144
4	0.2493	0.037	0.0415	0.9559
5	0.2122	0.1599	0.0354	0.9913
6	0.0523		0.0087	1

Table 4.1: Principal components: eigenvalues, difference, proportion of variance they explain and the cumulated explained variability

The $Difference$ column contains the difference if two consecutive eigenvalues. The $Proportion$ column is computed by dividing the eigenvalues by their summation, which in our case is 6. This represent the percentage of variance explained by every component.
The $Cumulation$ column represents the percentage of variance explained by the first x components. For our study we used weekly returns.

4.3 Choice

After determining the main components of BPI's portfolio, we will calculate the correlations with the asset returns, which are summarised in the following list

i) Eurostoxx 50 and Cac 40 for the Euro

ii) Ftse Emerging for the emerging countries

iii) 1 year and 10 years OAT for bond, where OAT is french acronym for Term Bonds.

iv) Eonia for the monetary

v) Oil and exchange rate Euro-Dollar

vi) Euro Inflation and France Inflation as macro-economic factor

The computation of the correlations gives the following matrix, the data set consists of the weekly returns of the different asset classes, computed from 2001 to 2018, except for the inflation, for which we have only monthly data.

ρ	Prin1	Prin2	Prin3
Eurostoxx50	0.952**	-0.0646**	-0.0974**
Cac40	0.9541**	-0.0544**	-0.0562**
Ftse E	0.5919**	0.1077**	-0.0121
Oat 1 an	0.0926**	0.0533**	-0.0045
Oat 10 ans	0.1348**	-0.0169	-0.013
Eonia	0.0467**	-0.0201	0.0162
Oil	0.1958**	0.0113	-0.05
US to Eur	0.0491**	0.1869**	-0.1547**
Inflation	0.0344	0.0044	-0.0315*
Inflation France	-0.0491*	0.023	-0.0204*

Table 4.2: Correlation matrix between the asset returns and the first three components

A value marked by ** indicates that it is extremely significant, that is, its p-value is less than 0.01. Values marked by * indicate that they are significant, $0.01 \leq$ p-value ≤ 0.05. On the other hand, all other values that have no mark are not significant, their p-value is strictly greater than 10%.

It can be deduced from the table that the first component has a very strong and very significant correlation with the three equity indices that we have chosen, so the first component can be seen as the market factor. The second component has a rather interesting and very significant correlation with the dollar-euro exchange rate, with a value of 18.69%. Likewise with the third component, but with a lower value. We present here the evolution of the historical series of the three equity indices, compared to the first component, and the historical series of the exchange rate compared to the second component.

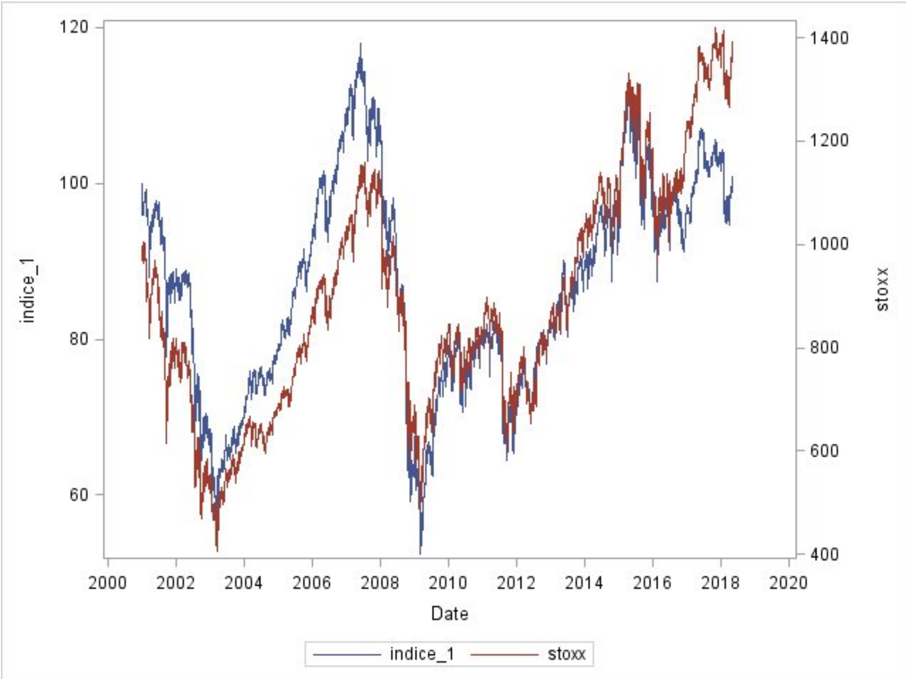

Figure 4.1: Historical series of the first component and the Eurostoxx 50

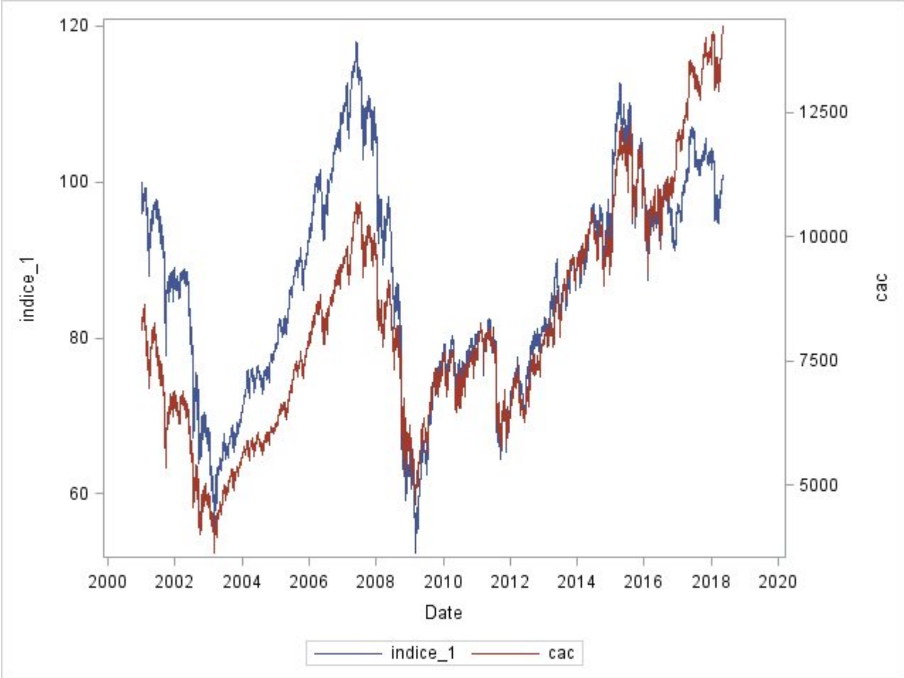

Figure 4.2: Historical series of the first component and the Cac 40

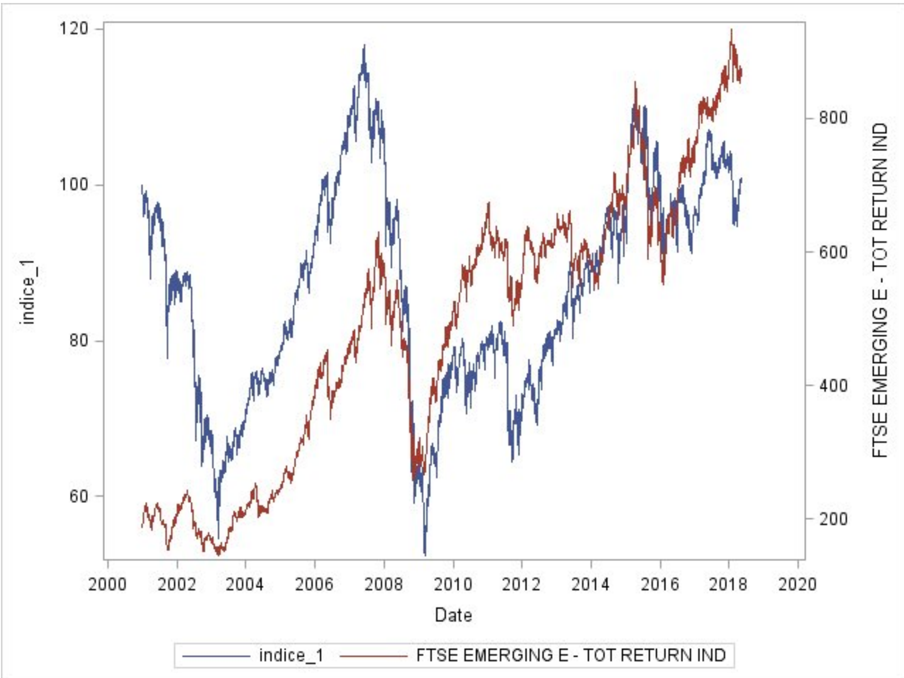

Figure 4.3: Historical series of the first component and the Ftse Emerging

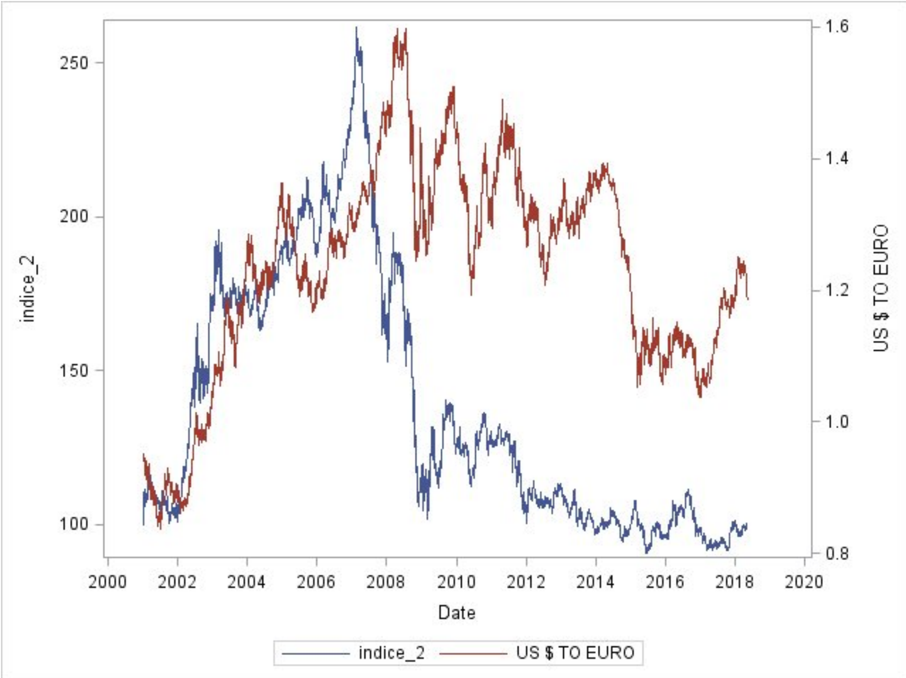

Figure 4.4: Historical series of the second component and the US/EUR change rate

As it can be seen, the relation between the first component and the equity indices is quite obvious, in fact, except for some translations (note: by construction, a component of PCA has a mean zero), the behaviour of the indices, compared to the component, is always the same: same periods of growth and decay.

For the second component and the exchange rate we can say that their behaviour, even if there is a considerable difference (especially since 2008), has similarities: periods of increasing of the second component are followed by periods of increasing of the exchange rate, and similarly for periods of decay.

So we can conclude that the choice of our assets is consistent with the choice of the assets of Bpifrance.

4.4 Statistical study of the chosen variables

In this section we present the results of the statistical analysis that we have done on the assets that we selected for the implementation of our ESG. For each asset we present the main indicators and graphs that are believed to be the most important for the choice of the diffusion models. The analysis is divided into two parts:

i) in the first part we will do a univariate study, i.e. we take and consider each asset separately

ii) in the second part we will do a joint analysis of the variables, in particular the computation of the matrix of correlations and tail dependence indices.

For both analysis we will follow [12] and compute the indices suggested in there.

4.4.1 Univariate Analysis

It consists in the presentation of a table with the indicators that could be calculated and in the graphical representation of the data in different forms (series, histogram and qq-plot). Remarks:

1) the weekly return series were considered for all our computations (from 2001 to 2018)

2) for the calculation of Pareto's alpha, which is a measure of heaviness of the tails of the distribution and it corresponds to the parameter of a Pareto distribution, we will use the following procedure:

we will calculate the quantile of order 10% of the empirical distribution, then, considering the tail of the distribution from this quantile and supposing that it follows a Pareto law, we will estimate, with a method of maximum likelihood, the parameter of this law, which we will call alpha.

3) for all rates (including inflation), the notion of return was simply the change in the rate

The presentation of the results and graphs is made in Appendix A.

Conclusions

On the basis of what we observe, the conclusion, the most immediate that can be drawn, is that the assumption of Normality of returns is not possible for each asset class: the tails of the distribution are too thick, which is demonstrated by the values of the kurtosis index and by the values of the Pareto alpha index.
Actually a value of the kurtosis greater than 3 is synonymous of a distribution with thick tails. For the Pareto alpha is the opposite, actually the less is the value the more thick are the tails of the distribution we are studying.

Secondly, if one observes the evolution over time (especially every year) of volatility, one can observe a non-constant volatility phenomenon and thus a stochastic volatility.
We show here the evolution of the volatility of the Eurostoxx 50 index, the Eonia rate and the 1 year Oat rate.

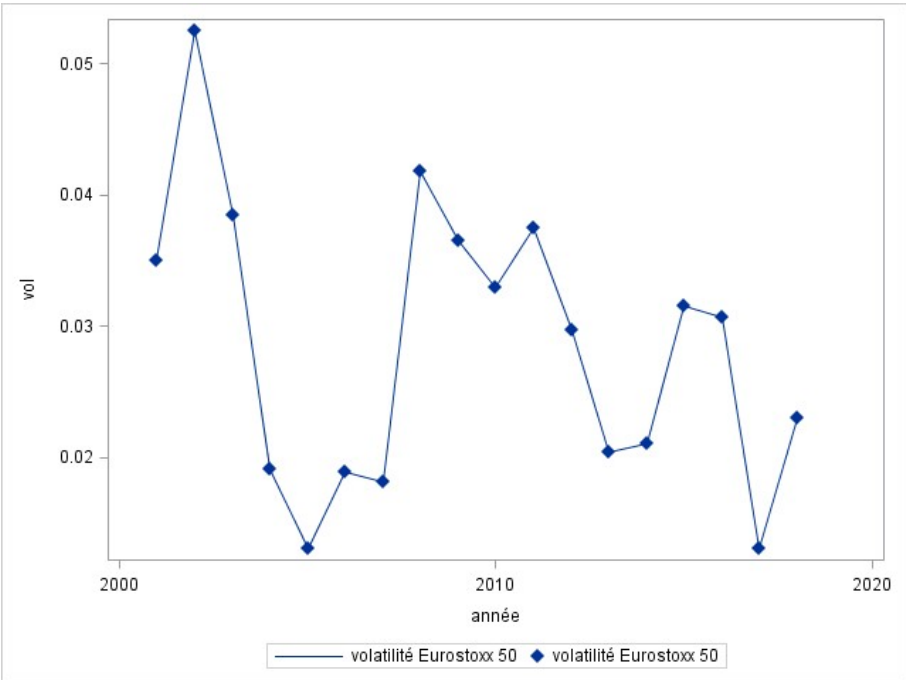

Figure 4.5: volatility of Eurostoxx 50

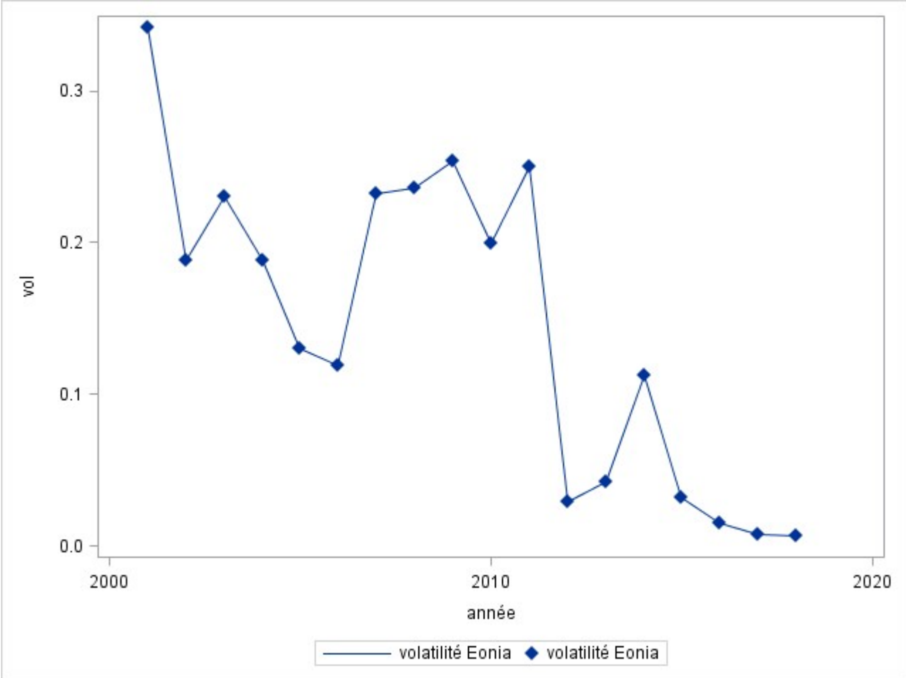

Figure 4.6: volatility of Eonia index

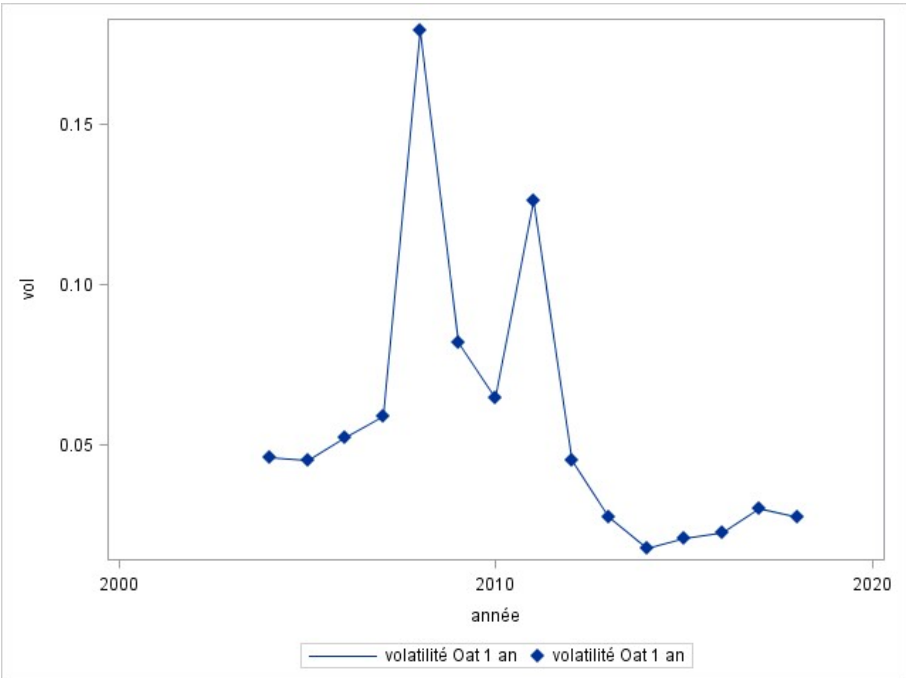

Figure 4.7: volatility of Oat 1 y

Note: we can see how the volatility, in the case of the Eurostoxx 50 and 1 year Oat, increases surprisingly during the period of the crisis of 2008.

4.4.2 Joint or Multivariate Analysis

We will present here the results of the joint analysis of the assets, using indicators suggested in [12]. The results are summarised as follows:

i) a correlation matrix of the asset returns: for each correlation we will indicate its significance: a value marked by ** indicates that it is extremely significant (p-value less than 0.01) on the other hand, a value marked by * means that it is significant (p-value less than 0.05).

ii) a matrix where for each pair of assets we calculate the tail dependency index:

$$\mathbb{P}\big(X < VaR_\alpha(X)|Y < VaR_\alpha(Y)\big),$$

where $VaR_\alpha(\cdot)$ represents the Value at Risk at the level alpha.
This statistics allows to measure the comovement in the tails of the distributions. It is important to compute this index because it happens that random variable that shows no correlation can exhibit tail dependence in extreme deviations (see [27]).

Regarding the point i), the correlation matrix, we will present three different matrices:

1) the Pearson Correlation Matrix: it is a measure of the linear correlation between two variables X and Y. Using Cauchy-Schwarts inequality one can show that this value belongs to the interval $[-1, +1]$, where

   a) $+1$ indicates the perfect positive linear correlation (if X increases then Y increases as well)
   b) 0 means that there are no linear correlation between X and Y
   c) -1 stands for the perfect negative linear correlation (if X increases, then Y decreases).

   It can be defined by the following formula

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y},$$

   where σ_{XY} is the covariance between the variable X and Y, and $\sigma_.$ is the standard deviation.

2) Kendall rank correlation coefficient matrix: the Kendall rank correlation coefficient measures the rank correlation between two variables. The correlation will be high when the observations have similar ranks and conversely when they have different ranks.
   Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be a set of observations of the joint variables X and Y such that the values of (x_i) and (y_i) are unique. The pairs of observations (x_i, y_i) and (x_j, y_j) are said to be concordant if $x_i < x_j$ and $y_i < y_j$ or if $x_i > x_j$ and $y_i > y_j$. They are said to be discordant if $x_i < x_j$ and $y_i > y_j$ or if $x_i > x_j$ and $y_i < y_j$. In the case where $x_i = x_j$ or $y_i = y_j$, the pair is neither concordant nor discordant.
   So, we define the Kendall rank correlation coefficient as:

$$\tau = \frac{matching\ pairs\ -\ pairs\ discordant}{n(n-1)/2}.$$

3) Spearman correlation matrix: Spearman correlation is a measure of non-parametric dependence between two variables. We are interested in this indicator when two variables seem to be correlated but with this correlation not being of affine type. It consists in finding a coefficient of correlation between the ranks of the values of the variables. She considers how the relationship between the two variables can be described by a monotonic function. If there are no repeated data, a perfect Spearman correlation of $+1$ or -1 is obtained when one of the variables is a perfect monotone function of the other. For two variables X and Y, we define the Spearman correlation as follows:

$$s = \frac{\sigma_{rang_X, rang_Y}}{\sigma_{rang_X} \sigma_{rang_Y}}.$$

Those matrices (Pearson, Kendall and Spearman) are estimated using functions already implemented on SAS, which give us directly the matrices with the values of the correlations together with the associated p-value, which allows us to say if the value is significant or not.
For what concerns the Tail dependence index, we did not found any already implemented function available on SAS, so we had to code it ourselves, using the equivalent representation of the conditional probability

$$\mathbb{P}\big(X < VaR_\alpha(X)|Y < VaR_\alpha(Y)\big) = \frac{\mathbb{P}\big(X < VaR\alpha(X) \cap Y < VaR_\alpha(Y)\big)}{\mathbb{P}(Y < VaR_\alpha(Y))}.$$

ρ	Eurostoxx 50	Cac 40	Ftse E	1 year Oat	10 years Oat	Eonia	Oil	US to Eur
Eurostoxx50	1	0.9813**	0.6689**	0.2161**	0.2499**	0.0349	0.2227**	0.1022**
Cac 40	0.9813**	1	0.6811**	0.2251**	0.2512**	0.0254	0.2305**	0.0945**
Ftse E	0.6689**	0.6811**	1	0.2418**	0.1814**	0.0025	0.31028**	-0.0691*
1 year Oat	0.2161**	0.2251**	0.2418**	1	0.4922**	0.0212	0.1923**	0.2026**
10 years Oat	0.2499**	0.2512**	0.1814**	0.4922**	1	0.0279	0.1123**	0.0596
Eonia	0.0349	0.0254	0.0025	0.0212	0.0279	1	0.0445	-0.0444
Oil	0.2227**	0.2305**	0.3103**	0.1923**	0.1123**	0.0445	1	-0.0232
US to Eur	0.1022**	0.0945**	-0.0691*	0.2026**	0.00596	-0.0444	-0.232	1

Table 4.3: Pearson correlation matrix

ρ	Eurostoxx 50	Cac 40	Ftse E	1 year Oat	10 years Oat	Eonia	Oil	US to Eur
Eurostoxx50	1	0.8678**	0.4698**	0.0845**	0.1636**	0.0346	0.1433**	0.0212
Cac 40	0.8678**	1	0.4798**	0.0851**	0.1664**	0.0365	0.1506**	0.0136
Ftse Es	0.4698**	0.4798**	1	0.0594*	0.1213**	0.0171	0.1762**	-0.1015**
1 year Oat	0.08451**	0.0851**	0.0594*	1	0.3733**	0.026	0.0581*	0.1183**
10 years Oat	0.1636**	0.1664**	0.1213**	0.3733**	1	0.0611**	0.0812**	0.02098
Eonia	0.0346	0.0365	0.0171	0.026	0.0611**	1	0.01595	-0.0273
Oil	0.1433**	0.1505**	0.1762**	0.0581*	0.0812**	0.01595	1	-0.0054
US to Eur	0.02119	0.0136	-0.1015**	0.1183**	0.02098	-0.0273	-0.0054	1

Table 4.4: Kendall rank coefficient correlation matrix

ρ	Eurostoxx 50	Cac 40	Ftse E	1 year Oat	10 years Oat	Eonia	Oil	US to Eur
Eurostoxx50	1	0.9738**	0.6470**	0.1216**	0.2351**	0.0502	0.2112**	0.0353
Cac 40	0.9738**	1	0.6597**	0.1231**	0.2374**	0.0526	0.2209**	0.0248
Ftse E	0.6470**	0.6597**	1	0.0854*	0.1749**	0.0222	0.2573**	-0.1461**
1 year Oat	0.1216**	0.1231**	0.0854*	1	0.5043**	0.0376	0.08424*	0.1639**
10 years Oat	0.2351**	0.2374**	0.1749**	0.5043**	1	0.08807**	0.0119**	0.0317
Eonia	0.0502	0.0526	0.0222	0.0376	0.0881**	1	0.0233	-0.0388
Oil	0.2112**	0.2209**	0.2573**	0.08424*	0.1192**	0.0233	1	-0.0078
US to Eur	0.0353	0.0248	-0.1461*	0.1639**	0.0317	-0.0388	-0.0078	1

Table 4.5: Spearman correlation matrix

ρ	Eurostoxx 50	Cac 40	Ftse E	1 year Oat	10 years Oat	Eonia	Oil	US to Eur
Eurostoxx50	1	0.8631**	0.4945**	0.28**	0.2418**	0.1099**	0.2088**	0.2637**
Cac 40	0.8631**	1	0.5165**	0.3194**	0.2527**	0.0879*	0.2308**	0.2747**
Ftse E	0.4945**	0.5187**	1	0.2639**	0.1978**	0.1209**	0.2418**	0.1758**
1 year Oat	0.2876**	0.3151**	0.2603**	1	0.4306**	0.2222**	0.1667**	0.2917**
10 years Oat	0.2418**	0.25**	0.2065**	0.4267**	1	0.1392**	0.1429**	0.2308**
Eonia	0.1099**	0.0879*	0.1209**	0.2133**	0.1319**	1	0.0769	0.1209**
Oil	0.2088**	0.2283**	0.2391**	0.1733**	0.1413**	0.0769	1	0.1099**
US to Eur	0.2937**	0.2826**	0.1739**	0.2933**	0.2391**	0.1209**	0.1304**	1

Table 4.6: Tail dependence matrix

As we did for the variance, in the univariate analysis, we can analyze the evolution of correlations over time and see how they change. We present here the graph which shows the evolution of the correlation (according to Pearson) between:

i) Eurostoxx 50 and Ftse Émergents

ii) Cac 40 and 10 years Oat

iii) Inflation and Inflation France

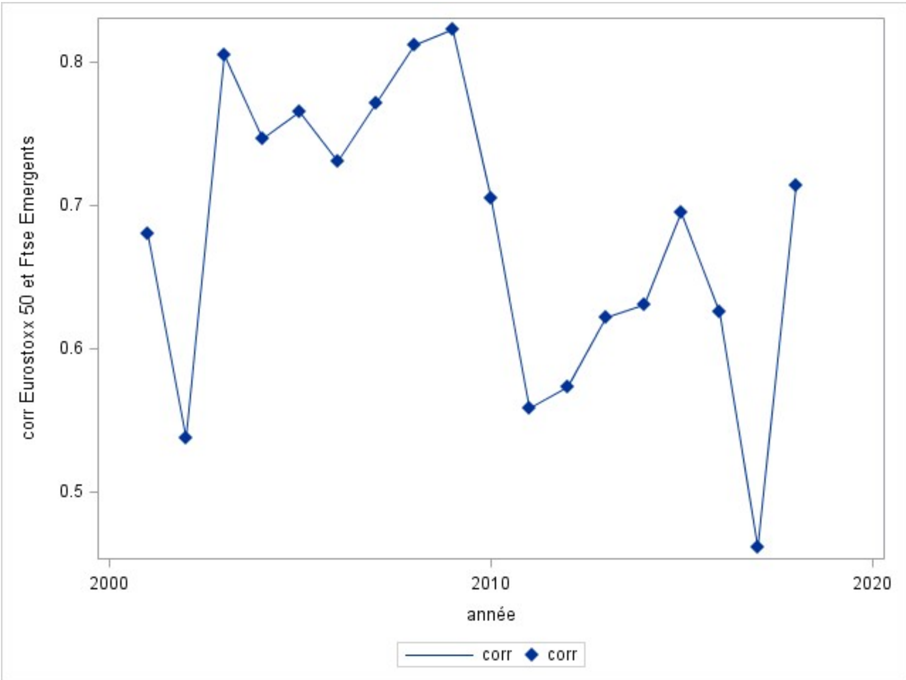

Figure 4.8: Evolution of the correlation between Eurostoxx and Ftsee Emerging

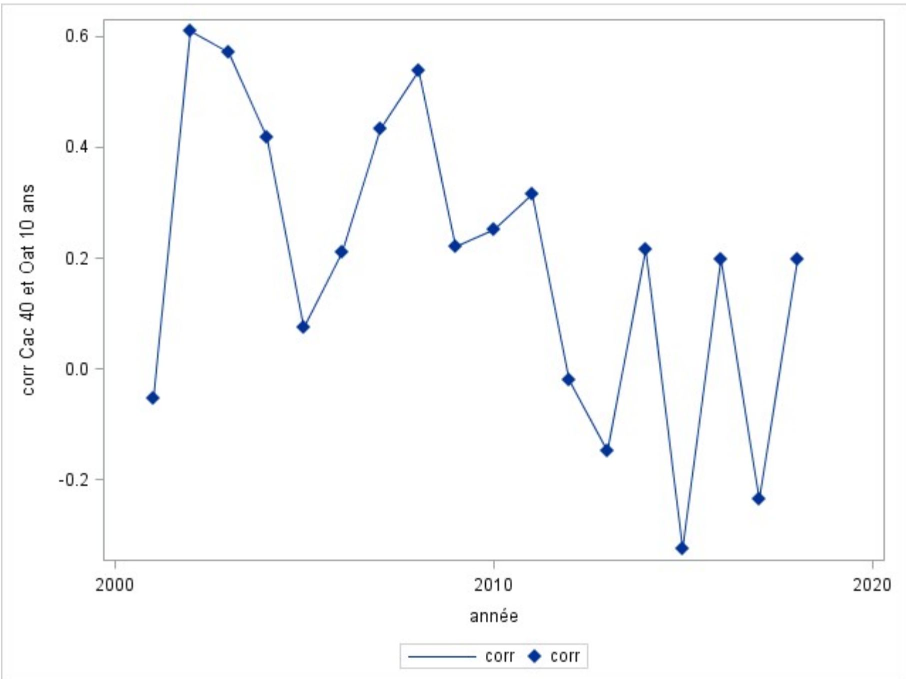

Figure 4.9: Evolution of the correlation between Cac 40 and Oat 10 y

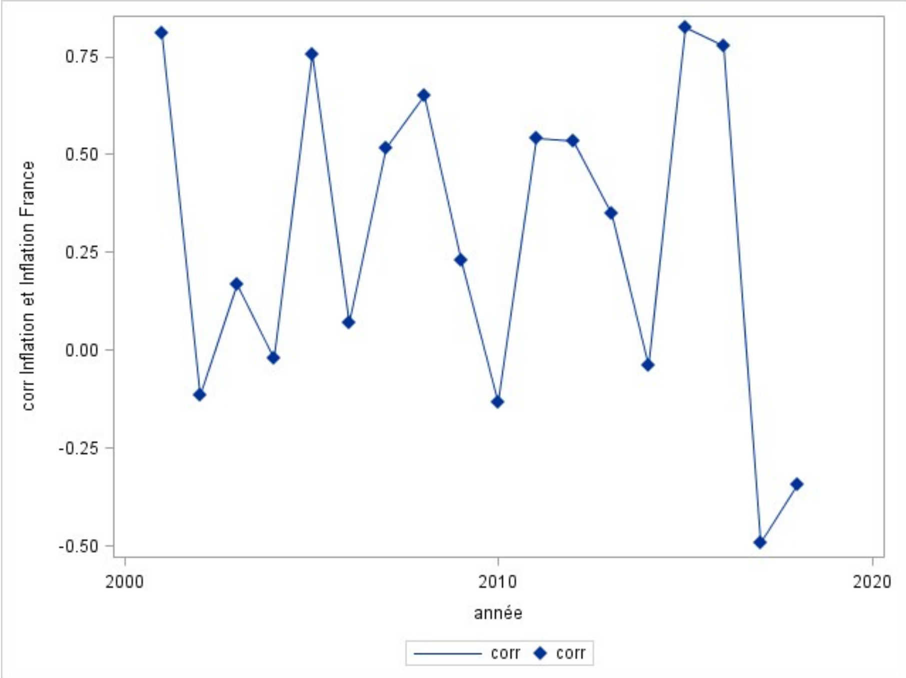

Figure 4.10: Evolution of the correlation between Inflation and Inflation France

Conclusions

By observing the table of correlations we can see that:

i) the Eurostoxx 50 return shows a correlation with respect to the Cac 40 return which attains values between 0.8671 and 0.9813 which is almost the perfect positive correlation. The same goes if we compare the Eurostoxx 50 return with the Ftse Emerging return and the Cac 40 return with the Ftse Emerging return, but with more lower values which by the way do not fall under the threshold 0.45. Those are reasonable results, because for instance the Eurostoxx 50 index contains some of the enterprises contained in the Cac 40 index, and viceversa.

ii) we have a non negligible positive correlation between the Oat 1 year and the Oat 10 years returns (especially if we look at the Pearson and Spearman matrices), which is also quite logic, the interest rate for 1 year zero coupon bond is connected with the interest rate for 10 years zero coupon bond.

iii) we can also see that all the three equity indices returns are positive correlated (even if the values are not so high) with the Oil return.

If we look at the Tail index matrix, we can see that there is a strong dependence between the tails of the distribution of the equity indices returns, which means that an extreme event of a random variable which represents an equity return can be followed with an high probability by another extreme event of another random variable which represents another equity returns.

As in the univariate case, we can see that the correlation (at least according to Pearson) is not constant over time (in the years, in our case). This effect will have to be taken into account when modelling the dependency structure between the assets in our ESG.

In conclusion, when we model multiple asset classes at the same time we need to take into account the fact that this asset classes are correlated and the fact also that this correlation may vary in time. In the next section we will focus on the modelling of the equity stock returns of one asset, and then we will consider also the inflation to give an example of joint analysis.

Chapter 5

Estimation

In this chapter we will present the procedures and results of the estimations of our models.
We will start by modelling and estimate the equity indices returns distribution. To do this we will suggest three possible choices:

1) modelization of the equity indices returns by a GARCH (1, 1)

2) modelization of the equity indices returns by a GARCH (1, 1) with jumps

3) modelization of the equity indices returns by a Merton model

Then, we will try to integrate the inflation in our models, to do that we will propose a cascade structure: we will model the inflation by using the following scheme

$$\begin{cases} I(t) = I_m + a_Q(I(t-1) - I_m) + \sigma_Q \epsilon_Q(t) \\ R(t) = \omega_R I(t) + \lambda + \sqrt{h(t)}\epsilon_1(t) + \epsilon_2(t) \\ h(t) = \omega + \alpha\epsilon(t-1)^2 + \beta h(t-1), \end{cases} \tag{5.1}$$

where I_m is the long term inflation, a_q is the mean reversion level, σ_Q is the standard deviation, $\epsilon_Q(t)$ is a standard normal random variable and ω_R is the weight of the inflation in the return of the equity indices. The other parameters and variables are defined as in chapter 3.

Remark 5.1. *All the equity indices returns are weekly.*
Concerning the inflation, we have only monthly data in our database, to overcome this problem and have weekly data, we supposed that for every week of the month i, except the last week (for which the data is available), the inflation is equal to the one of the month $i-1$.

5.1 Equity share calibration

5.1.1 Garch (1, 1)

We proceed to the estimation following the method proposed in chapter 3, so a subset of the parameters will be estimated by the sample mean (for λ), and the others by solving the following problem

$$\begin{aligned} \max_{\omega,\alpha,\beta} l(\omega,\alpha,\beta,R_t) = \max_{\omega,\alpha,\beta} & \sum_{t=2}^{T} \Big(-\frac{1}{2}\log(\hat{h}(t)) - \frac{1}{2}\frac{(R_t-\hat{\lambda})^2}{\hat{h}(t)} \Big) \\ & s.t.\ \alpha + \beta < 1 \\ & \quad \hat{h}(t) = \omega + \alpha(R(t-1) - \hat{\lambda})^2 + \beta\hat{h}(t-1) \\ & \quad \omega = \sigma^2(1-\alpha-\beta). \end{aligned} \tag{5.2}$$

Remark 5.2. *To initialise our algorithm, we choose $h(1) = \hat{\sigma}^2$, i.e. the historical variance of the returns.*

Results

We present here the results of our estimation, using the equity indices returns of: Eurostoxx 50, Cac 40 and Ftse Emerging (size of the sample = 910)

Parameters	Eurostoxx 50	Cac 40	Ftse Emerging
$\hat{\lambda}$	0.0004	0.0006	0.0017
$\hat{\omega}$	0.00003	0.0000283	0.0000378
$\hat{\alpha}$	0.1422	0.1464	0.0898
$\hat{\beta}$	0.8258	0.8222	0.8669
log-likelihood	2825.7787	2854.2794	2820.4207

Table 5.1: Estimation of the parameters for the GARCH (1, 1)

Analyse and quality of the estimation

We need now to check that the results we obtained are actually good results, i.e. we need to control that the estimators we used verify some properties:

i) the estimator is unbiased (at least asymptotically): in other words for fairly large samples, the value of the estimator converges to the true value of the parameter

ii) the estimator is consistent (at least asymptotically): i.e. for samples large enough, the variability of the estimator decreases to zero.

To check this we will use a Monte Carlo method:

i) we will simulate a large number of samples, from a Garch $(1,1)$ with the parameters we estimated, of the same size as the one used to estimate the parameters and using the estimated parameters

ii) for any simulated sample, the parameters are estimated again using the same method as before

iii) the mean and the variance of all the estimations made are calculated, and the results are observed

For a number of simulations of 1000, using the parameters estimated before, we obtain, for the three equity indices, the following results (for the parameters ω, α and β):

Eurostoxx 50 Parameters	Mean	Standard deviation
$\hat{\omega}$	0.0000364	0.0000226
$\hat{\alpha}$	0.1386	0.0265
$\hat{\beta}$	0.8177	0.0479

Table 5.2: Results of the simulation for the Eurostoxx 50

Cac 40 Parameters	Mean	Standard deviation
$\hat{\omega}$	0.0000339	0.0000339
$\hat{\alpha}$	0.1433	0.0267
$\hat{\beta}$	0.8255	0.0343

Table 5.3: Results of the simulation for the Cac 40

Ftse Emerging Parameters	Mean	Standard deviation
$\hat{\omega}$	0.0000925	0.0001446
$\hat{\alpha}$	0.0929	0.0253
$\hat{\beta}$	0.7925	0.1918

Table 5.4: Results of the simulation for the Ftse Emerging

We can immediately see that the results of the simulations do not coincide with the results of the estimation, it may be due to the fact that the size of each sample is limited to 910, whereas the convergence is ensured for a size that tends to infinite.
In any case, the properties we wanted are assured by the theory (see [21]): let $\hat{\theta}$ be an estimator obtained by maximum of likelihood, then

$$\hat{\theta} \xrightarrow{d} N\big(\theta, RC(\theta)\big),$$

where $RC(\theta)$ is the Rao-Cramer lower bound.
For our data, we obtain the following results

Parameters	Eurostoxx 50	Cac 40	Ftse Emerging
$RC(\omega)$	0.000009	0.000008	0.000009
$RC(\alpha)$	0.0286	0.0285	0.02
$RC(\beta)$	0.0313	0.0309	0.0234

Table 5.5: Rao Cramer lower bound matrix

Knowing these results, we can calculate, for each parameter, three confidence intervals (at 95%): a confidence interval using as a variance that one which derives from the simulations, another interval using the variance given by the lower bound of the Rao-Cramer inequality and another interval based on the quantiles of the parameter distribution (derived by simulations). In particular

$$CI\ simulations\ (\theta) = \hat{\theta} \pm z_{0.975}\hat{\sigma}^{\hat{\theta}},$$
$$CI\ RC\ (\theta) = \hat{\theta} \pm z_{0.975}RC(\theta),$$
$$CI\ quantile\ (\theta) = (\hat{q}^{\theta}_{0.025}; \hat{q}^{\theta}_{0.075}),$$

where z_α and q_α are respectively the quantile of a standard centred normal distribution and the empirical quantile of the distribution at level α.

Eurostoxx 50 Parameters	sim CI	RC CI	quantile CI
ω	(-0.000014296; 0.000074296)*	(0.00001236; 0.00004764)	(0.0000165; 0.0000722)
α	(0.0903; 0.1941)	(0.0861; 0.1983)	(0.0869; 0.191)
β	(0.7319; 0.9197)	(0.7645; 0.8871)	(0.743; 0.886)

Table 5.6: CI for the parameters of the Eurostoxx 50 distribution

Cac 40 Parameters	sim CI	RC CI	quantile CI
ω	(-0.000038144; 0.000094744)*	(0.00001262; 0.00004398)	(0.000016; 0.0000666)
α	(0.0941; 0.1987)	(0.0905; 0.2023)	(0.0932; 0.1964)
β	(0.755; 0.8894)	(0.7616; 0.8828)	(0.7452; 0.8753)

Table 5.7: CI for the parameters of the Cac 40 distribution

Ftse Emerging Parameters	sim CI	RC CI	quantile CI
ω	(-0.000245616; 0.000321216)*	(0.00002016; 0.00005544)	(0.0000161; 0.000578)
α	(0.0402; 0.1394)	(0.0506; 0.129)	(0.0478; 0.1478)
β	(0.4908; 1.2428)	(0.821; 0.9128)	(0.1979; 0.9198)

Table 5.8: CI for the parameters of the Ftse Emerging distribution

* = non significant parameters at level 95%.
This allows us to say that the three parameters, ω, α and β, are significant at level 95% (the only case of non significance is given by ω for the confidence interval issued by simulation).

Residual and simulation analysis

To check that the model we chose is actually a good model, we could study the distribution of the residuals which should be, under our hypothesis, a standard normal

$$\epsilon(t) = \frac{R(t) - \lambda}{\sqrt{h(t)}} \sim N(0,1).$$

On the basis of our data, we have the following results

Indicators	Eurostoxx 50	Cac 40	Ftse Emerging
mean	0.0015	0.0043	-0.002
standard deviation	1.0093	1.0126	0.9947
skewness	-0.5402	-0.4976	-0.8913
kurtosis	1.4638	1.4453	3.1404

Table 5.9: Computation of the mean, standard deviation, skewness and kurtosis of the centered reduced distribution of the equity indices returns

We can notice that the error is presented as a variable of zero mean and a unitary variance, however the presence of a negative asymmetry and a slight kurtosis (except the case of the Ftse Emerging), does not allow us to state that the error follows a reduced normal centred law and, consequently, that the GARCH model is an appropriate model.
Another way to verify this, could be to compare the empirical distribution of the equity index, with the simulated distribution of the same index by a qq-plot, which gives us

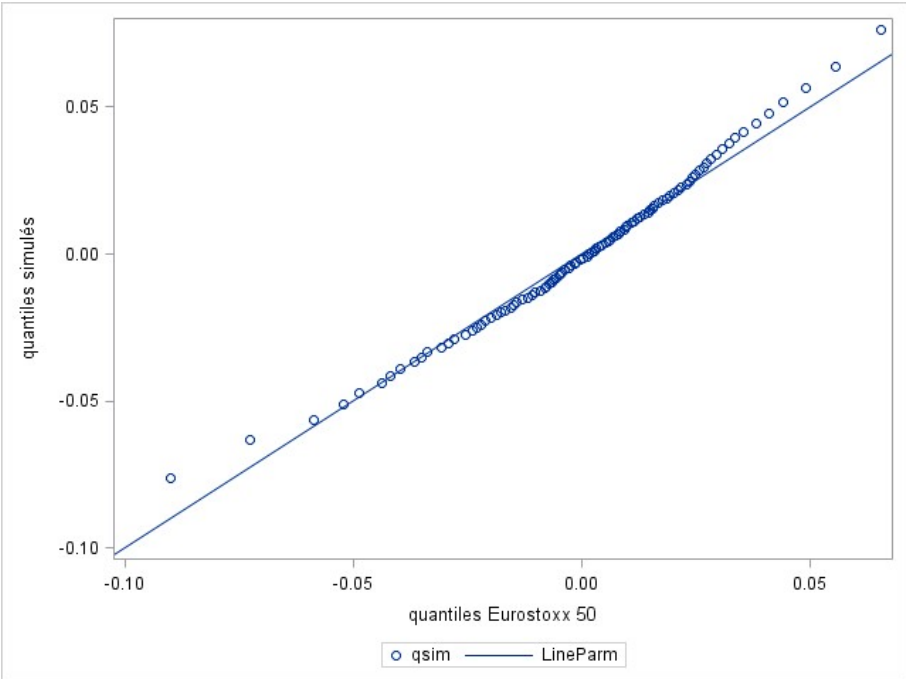

Figure 5.1: qq-plot Eurostoxx 50

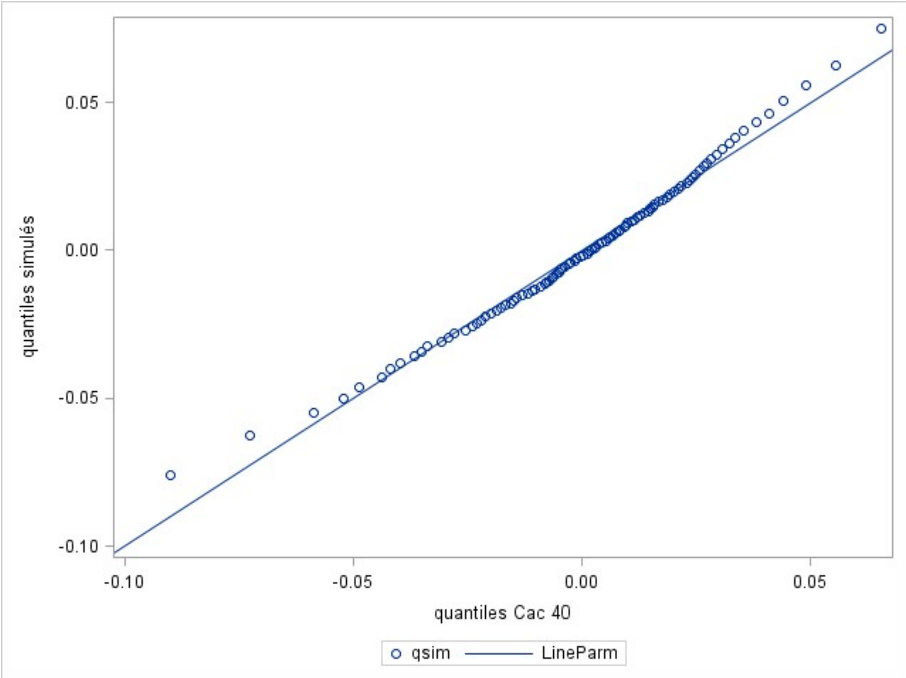

Figure 5.2: qq-plot Cac 40

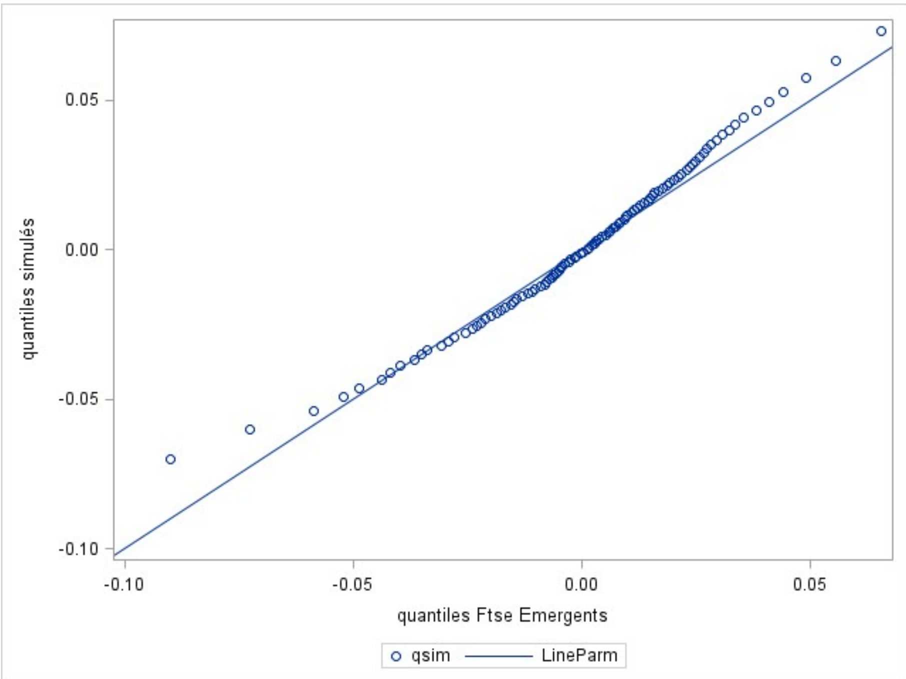

Figure 5.3: qq-plot Ftse Emerging

It may be noted that, for each equity index, the two distributions (empirical and simulated) have almost the same values, except for extreme events: the simulated distribution has lower extreme values than the empirical distribution.

Verification of the implementation

To verify that our method works correctly, we can also look at what happens to the estimation of the parameters, when we make a simulation of a large size sample (100000 in our case), i.e. we want to check

if the estimators are asymptotically unbiased or not. If the estimators of our parameters are biased we will not find the same values that we found in table (5.1).
For the Eurostoxx 50 we get the following result

$$\hat{\lambda} \xrightarrow{n\to\infty} 0.0004$$
$$\hat{\omega} \xrightarrow{n\to\infty} 0.00003$$
$$\hat{\alpha} \xrightarrow{n\to\infty} 0.1441$$
$$\hat{\beta} \xrightarrow{n\to\infty} 0.8229.$$

As we can see, the value of the parameters this time is closer to the one obtained from the estimation we did on the initial sample.

5.1.2 GARCH (1, 1) with jumps

The estimation is performed by the same method proposed in chapter 3, so the risk premium (λ) will be estimated by the sample mean, whereas the other parameters will be estimated by solving the following problem

$$\max_{\omega,\alpha,\beta,\gamma,\delta} l(R_t,\omega,\alpha,\beta,\gamma,\delta) = \sum_{t=2}^{T} \log\left(e^{-\gamma}\left[\sum_{j=0}^{\infty} \frac{\gamma^j}{j!\sqrt{\hat{h}(t)+j\delta^2}} e^{-\frac{(R_t-\hat{\lambda})^2}{2(\hat{h}(t)+j\delta^2)}}\right]\right) \tag{5.3}$$
$$s.t.\ \alpha+\beta<1$$
$$\hat{h}(t) = \omega + \alpha(R(t-1)-\hat{\lambda})^2 + \beta\hat{h}(t-1)$$
$$\sigma^2 = \frac{\omega}{(1-\alpha-\beta)} + \gamma\delta^2.$$

Remark 5.3. *As we have an infinite sum (Poisson part), we will truncate this sum at the level η such that $\mathbb{P}(n(t)>\eta)\approx 0$ (in our optimisation algorithm we fix $\eta = 30$)*

Results

We present here the results of our estimation for two different initialisation of the optimisation algorithm (for the Eurostoxx 50)

Parameters	first initialisation	second initialisation
$\hat{\lambda}$	0.0004	0.0004
$\hat{\omega}$	3.8448E-7	9.32E-6
$\hat{\alpha}$	0.0009	0.0007
$\hat{\beta}$	0.6177	0.231
$\hat{\gamma}$	0.0009	0.0005
$\hat{\delta}$	1.0018	1.3524
log-likelihood	8629.8277	8648.4861

Table 5.10: Estimation of the parameters with two different initialisation

First initialisation: $(0.0004, 0.00004, 0.2, 0.7, 0.01, 1)$.
Second initialisation: $(0.0004, 0.00001, 0.15, 0.7, 0.1, 0.5)$.
For the parameters λ, ω, α and β we tried to use, as initialisation, some values close to the ones we obtained for the Garch (1, 1).

We can notice that the results of the estimation are dependent on the initialisation of the algorithm.

We can see also that the repartition of the volatility of the model is entirely concentrated on the jumps part. As it is shown in the following table

	First initialisation	Second initialisation
% volatility GARCH	0.0379	0.0259
% volatility jumps	0.9621	0.9741

Table 5.11: Repartition of the volatility of the model between the GARCH and the volatility parts

The repartition of the volatility has been done in the following way: we computed the volatility of the GARCH $(1,1)$ part of the model, using the parameters estimated above, then we divided this value by the historical volatility, which represents the total volatility. The volatility of the jump part has been calculated by subtracting 1 to the value aforementioned.
This allows us to see the inconsistency of the results of the estimation.
Indeed, as the probability of having jumps is really low, and, as their intensity is high, it implies that the process will tend to have values very close to the average, except having some very high values for a limited number of times.

All this may be due to the fact that the number of parameters is too high, so a possible solution could be to move to an "expert" approach, i.e. we set the value of one or more parameters to reduce the size of the problem.

In the following we will focus on the parameter that controls the number of jumps between each period, the parameter γ, and we will propose two possible values: $\gamma = 0.01$ and $\gamma = 0.05$. The results of the new estimations are as follows

Parameters	first initialisation	second initialisation
$\hat{\lambda}$	0.0004	0.0004
$\hat{\omega}$	8.235E-5	9.3151E-6
$\hat{\alpha}$	0.0008	0.0007
$\hat{\beta}$	0.5143	0.2318
$\hat{\gamma}$	0.01	0.05
$\hat{\delta}$	0.3044	0.1361
log-likelihood	8640.595	8611.7231

Table 5.12: Estimation of the parameters when two of them are fixed, with two different initialisation

First initialisation: $(0.0004, 0.00004, 0.2, 0.7, 0.01, 1)$.
Second initialisation: $(0.0004, 0.00001, 0.15, 0.7, 0.05, 1)$.
The results are not satisfactory, even fixing the value of λ, they look like to be dependent on the initialisation.

5.1.3 Merton

Since the GARCH $(1,1)$ jump model is not the optimal choice, we will propose another model for equity indices: the Merton model, which is no other than a Black-Scholes model with a jump component.
We will study the case $m = 0$ for stability of the estimation and simulations.
The estimation is performed by the maximum likelihood method (as proposed in the previous sections), thus solving the following optimisation problem

$$\begin{aligned} \max_{\mu,\sigma,\lambda,\delta} L(R_i, \mu, \sigma, \lambda, \delta) = \prod_{i=1}^{n} \frac{e^{-\lambda}}{\sqrt{2\pi}} \Big[\sum_{m=0}^{\infty} \frac{\lambda^m}{m!\sqrt{\sigma^2 + (m\delta)^2}} e^{-\frac{(R_i - \mu + m\bar{\mu})^2}{2(\sigma^2 + (m\delta)^2)}} \Big] \quad (5.4) \\ s.t.\ E[R(t)] = \mu - \lambda\bar{\mu} \\ Var(R(t)) = \sigma^2 + \delta^2\lambda(1+\lambda) \\ Kurt(R(t)) = \frac{3\sigma^2 + \lambda[3\delta^4 + 21\delta^4\lambda + 6\delta^2\sigma^2 + 18\delta^4\lambda^2 + 6\sigma^2\delta^4\lambda + 3\delta^4\lambda^3]}{(\sigma^2 + \delta^2\lambda(1+\lambda))^2}. \end{aligned}$$

Remark 5.4. *As stated in Remark* 5.3 *we truncate the summation at a level* η *s.t.* $\mathbb{P}(\Delta N(t) > \eta) \approx 0$ *(*$\eta = 30$*).*

Results

We present here the results of the estimation performed on the three equity indices we studied before

Parameters	Eurostoxx 50	Cac 40	Ftse Emerging
$\hat{\mu}$	0.0005	0.0007	0.0018
$\hat{\sigma}$	0.0003	0.0003	0.0003
$\hat{\lambda}$	3.6576	3.0419	2.6816
$\hat{\delta}$	0.0074	0.0086	0.0094
log-likelihood	8258.2556	8335.3132	10554.4502

Table 5.13: Estimation of the parameters for the Merton model

We can see how the value of λ is high. Actually, it tells us that the average number of jumps between each period is between 2.6816 and 3.6576. But this is consistent with the fact that the volatility of the intensity of the jumps is low (it goes from 0.0074 to 0.0094). This means that even if, for each realisation of our random variable, the probability of having jumps is high, this effect is moderated by the fact that their intensities will be (in most cases) low (this is the same argument of to the previous model).
A table showing the repartition of the variability of the model, between variability of the jump component and variability of the so-called Black-Scholes component can be presented.

	Eurostoxx 50	Cac 40	Ftse Émergents
% variability BS	0.0063	0.0001	0.0001
% variability jumps	0.9937	0.99999	0.9999

Table 5.14: Repartition of the volatility of the model between the Black Scholes and jumps parts

As we can see, almost 100% of the variability of the model is entirely concentrated in the jumps part.

Analysis and quality of the estimation

Like we did before, we are gonna study, by simulations, the properties of the estimations we have done (for all the parameters).

Eurostoxx 50 Parameters	Expectation	Standard deviation
$\hat{\mu}$	0.0004	0.001
$\hat{\sigma}$	0.0001	0.0001
$\hat{\lambda}$	4.4292	1.5299
$\hat{\delta}$	0.0071	0.0034

Table 5.15: Results of the simulation for the Eurostoxx 50

Cac 40 Parameters	Expectation	Standard deviation
$\hat{\mu}$	0.0007	0.0016
$\hat{\sigma}$	0.0003	0.0011
$\hat{\lambda}$	3.4173	1.1499
$\hat{\delta}$	0.0084	0.0032

Table 5.16: Results of the simulation for the Cac 40

Ftse Emerging Parameters	Expectation	Standard deviation
$\hat{\mu}$	0.0014	0.0025
$\hat{\sigma}$	0.0043	0.0318
$\hat{\lambda}$	3.2898	1.3218
$\hat{\delta}$	0.0091	0.005

Table 5.17: Results of the simulation for the Ftse Emerging

It can be noticed that the results of the simulations show a bias with respect to the estimations made on the historical data. In addition, the results of the simulations have a very high standard deviation, especially for the λ parameter.

We can also check the significance of the parameters (we had some problems with the Rao-Cramer lower bound): CI based on the simulations and CI based on the quantiles of the empirical distribution of parameters.
These are the results

Eurostoxx 50 Parameters	sim CI	quantile CI
$\hat{\mu}$	(-0.0016 ; 0.0025)	(-0.0015 ; 0.0024)
$\hat{\sigma}$	(0 ; 0.0005)*	(0 ; 0.0005)**
$\hat{\lambda}$	(0.659 ; 6.6556)	(1.9658 ; 7.7406)
$\hat{\delta}$	(0.0007 ; 0.0141)	(0.0037 ; 0.0134)

Table 5.18: CI for the parameters of the Eurostoxx 50 distribution

Cac 40 Parameters	sim CI	quantile CI
$\hat{\mu}$	(-0.0024 ; 0.0038)	(-0.0013 ; 0.0032)
$\hat{\sigma}$	(0 ; 0.0025)*	(0 ; 0.0006)**
$\hat{\lambda}$	(0.7881 ; 5.2957)	(1.3386 ; 6.0907)
$\hat{\delta}$	(0.0023 ; 0.0149)	(0.0035 ; 0.0172)

Table 5.19: CI for the parameters of the Cac 40 distribution

Ftse Emerging Parameters	sim CI	quantile CI
$\hat{\mu}$	(-0.0031 ; 0.0067)	(-0.0005 ; 0.0039)
$\hat{\sigma}$	(0 ; 0.0626)*	(1.2348E-9 ; 0.0007)
$\hat{\lambda}$	0.0893 ; 5.2707)	(0.8601 ; 6.0054)
$\hat{\delta}$	(0 ; 0.0192)*	(0.00002 ; 0.0254)

Table 5.20: CI for the parameters of the Ftse Emerging distribution

* = correction of the confidence interval due to the positivity constraint on the parameters.
** = the confidence interval touches the lower bound of the constraint of positivity.

As one can see, the estimated parameters would seem significant, the only doubt is given by the parameter σ. It would be interesting to see what would happen to our model if we remove this parameter and therefore the Gaussian component.

Analysis of the simulations

To verify that our model fits well with the historical data, we could present two types of graphs (for each equity index): a qq-plot (to compare the empirical distribution and the simulated one) and an example of simulated trajectory (to see if the model replicates the index). For the qq-plot, we get

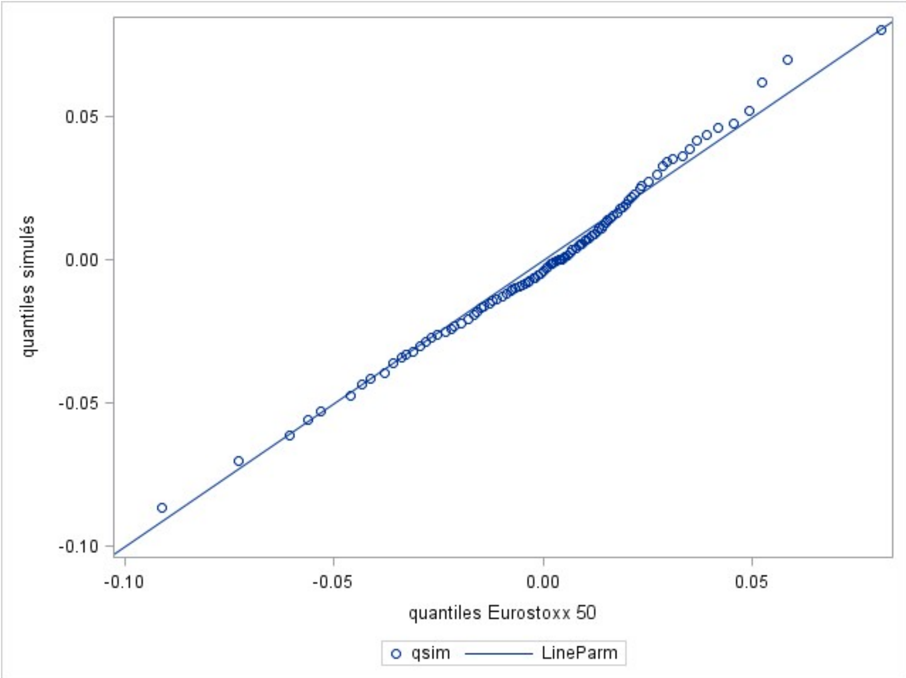

Figure 5.4: qq-plot Eurostoxx 50

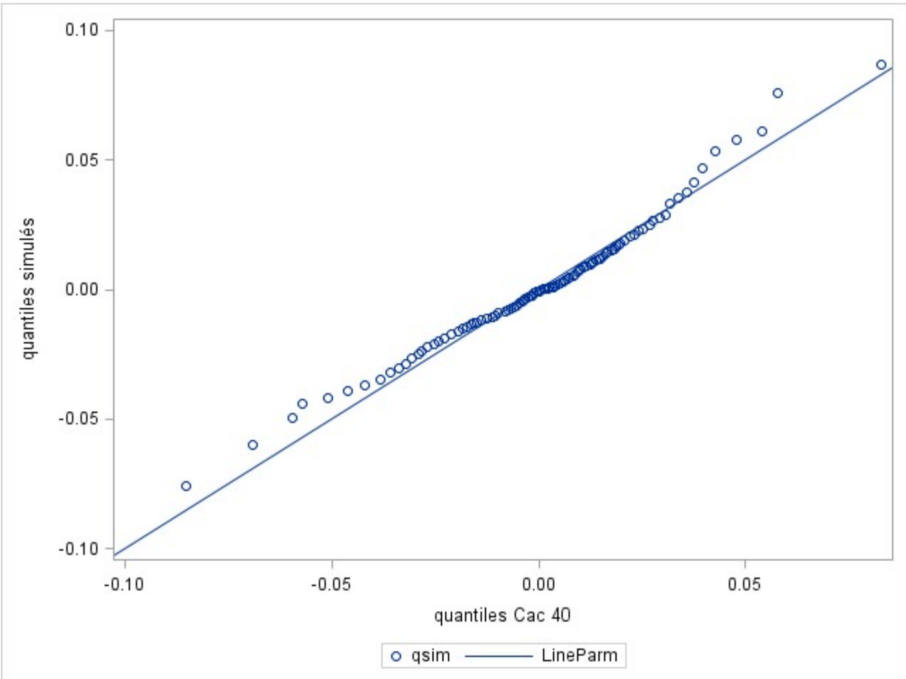

Figure 5.5: qq-plot Cac 40

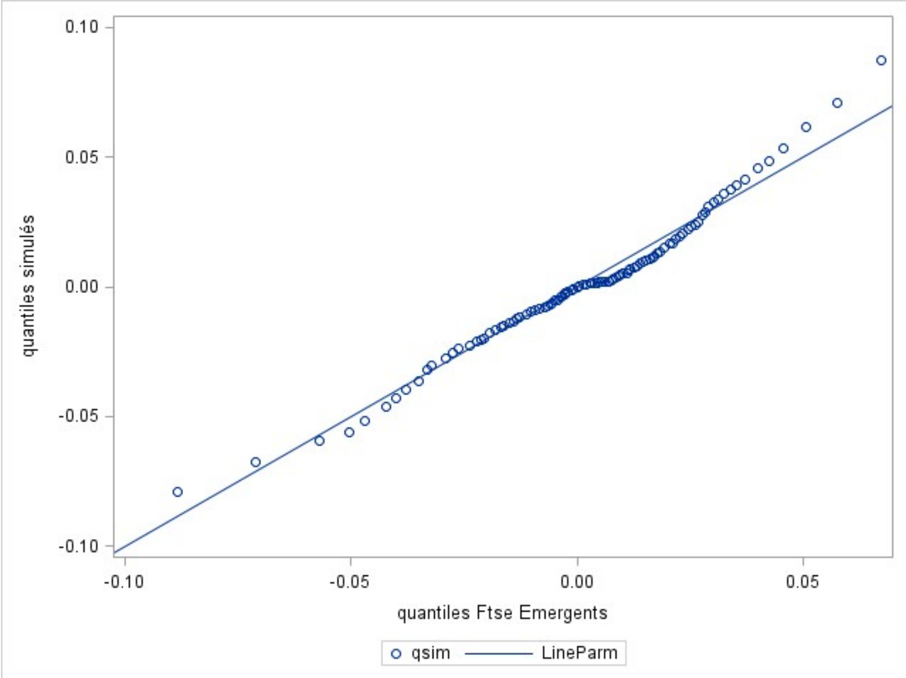

Figure 5.6: qq-plot Ftse Emerging

As for the Eurostoxx 50, the two distributions seem to stick together, except for some extreme values. With the Cac 40, we see that the two distributions have different values at the queues. The same with the Ftse Emerging, where we can see a double trend.
This is a result that can be also found in the literature: Levy models fit very well the data (see [13]).

Verification of the implementation

As we did previously, we check if the estimators are asymptotically unbiased or not (when we stretch the size of a sample to a very high value, 100000 in our case). For the Eurostoxx 50 we get the following result

$$\begin{aligned}
\hat{\mu} &\xrightarrow{n\to\infty} 0.0004 \\
\hat{\sigma} &\xrightarrow{n\to\infty} 0.000008 \\
\hat{\lambda} &\xrightarrow{n\to\infty} 3.6519 \\
\hat{\delta} &\xrightarrow{n\to\infty} 0.0074.
\end{aligned}$$

As we can see, the value of the parameters is very close to the one of the estimations we did on our initial sample (expect for the parameter σ).

New specification of the model

As we have seen, it seems that the parameter σ (which represent the volatility of the Black Scholes part of our model) is not significant, so it might be of interest to analyse what would happen if we remove this parameter from our model

$$R(t) = (\mu - \lambda\bar{\mu}) + U(t)\Delta N(t), \tag{5.5}$$

where the other parameters and variables are the same as before, The estimation problem can be rewritten as follow

$$\max_{\mu,\lambda,\sigma_u} L(R_i,\mu,\lambda,\delta) = \prod_{i=1}^{n} \frac{e^{-\lambda}}{\sqrt{2\pi}} \left[\sum_{m=0}^{\infty} \frac{\lambda^m}{m!\sqrt{(m\delta)^2}} e^{-\frac{(R_i-\mu+m\bar{\mu})^2}{2((m\delta)^2)}} \right] \tag{5.6}$$
$$s.t.\ E[R(t)] = \mu - \lambda\bar{\mu}$$
$$Var(R(t)) = \sqrt{\delta^2\lambda(1+\lambda)}$$
$$Kurt(R(t)) = \frac{\lambda[3\delta^4 + 21\delta^4\lambda + 18\delta^4\lambda^2 + 3\delta^4\lambda^3]}{(\delta^2\lambda(1+\lambda))^2}.$$

Estimation and simulation

The results of the estimation are then

Parameters	Eurostoxx 50	Cac 40	Ftse Emerging
$\hat{\mu}$	0.0005	0.0007	0.0018
$\hat{\lambda}$	3.3262	2.6569	2.4247
$\hat{\delta}$	0.0081	0.0096	0.0102

Table 5.21: Estimation of the parameters for the Merton model without σ

One can notice how the results do not vary so much compared to the case $\sigma = 0$. So we could conclude that in the case of the Merton model, the σ component is not significant. As a result, the volatility of the process is entirely explained by the volatility of the jumps.

Remark 5.5. *Before to proceed with the last section it is important to note that based on our data, the Merton model performs better than the GARCH* $(1,1)$. *This, even if it seems surprising at first glance, it is a reasonable result. If we look at* [24], *we notice that the use of GARCH models has become more persistent in order to overcome one of the main disadvantages of the Black-Scholes model: the constant volatility. By the way, in this work we used a variant of the Black-Scholes model, the so called Merton jump diffusion, which introduced a jump component in the dynamics of the stock prices in order to make the volatility stochastic.*
This means that in this case, our data, fit well and better a Merton jump diffusion instead of a GARCH $(1,1)$ *model, but it cannot be taken as a general result.*

5.2 Integrated models

We now proceed here assuming that the returns of the equity indices also depend on macroeconomic factors, in our case the inflation. We will assume the dynamics of inflation and, from this dynamics, we will build the one of the equity indices (see [16]).
In the following we will propose a possible scheme:

$$\begin{cases} I(t) = I_m + a_Q(I(t-1) - I_m) + \sigma_Q\epsilon_Q(t) \\ R(t) = \omega_R I(t) + \lambda + \sqrt{h(t)}\epsilon(t) \\ h(t) = \omega + \alpha\epsilon(t-1)^2 + \beta h(t-1), \end{cases} \tag{5.7}$$

We will go through the following steps:

i) estimation of the inflation from the historical series,

ii) estimation of the equity indices models conditioned on the results of point i).

5.2.1 Inflation

The first thing to do is the estimation of the chosen model for the inflation.
We know that $I(t) = I_m + a_Q(I(t-1) - I_m) + \sigma_Q\epsilon_Q(t)$, where the parameters are the same of the previous

section. So, we can easily deduce the log-likelihood associated to this model,

$$l(I(t), a_Q, \sigma_Q) = \sum_{t=2}^{T} \log\left(\frac{1}{\sqrt{\sigma_Q 2\pi}} e^{-\frac{(I(t)-Im-a_Q(I(t-1)-Im))^2}{2\sigma_Q^2}}\right).$$

From this point we have two possible choices: the first one consists in fixing the parameter Im, which represents the long term inflation, by an expert approach. So we have to estimate the two remaining parameters that can be explicitly computed by easy computations

$$\hat{a_Q} = \frac{\sum_{t=2}^{T}\left[I(t)(I(t-1)-Im)\right] - Im\sum_{t=2}^{T}\left[I(t-1)-Im\right]}{\sum_{t=2}^{T}\left[I(t-1)-Im\right]^2},$$

$$\hat{\sigma_Q} = \sqrt{\frac{1}{T-1}\sum_{t=2}^{T}\left[I(t)-Im-\hat{a_Q}(I(t-1)-Im)\right]^2}.$$

The second choice consists in considering Im as an unknown parameter to be estimated. In this case the estimation we have to solve the following problem

$$\max_{a_Q,\sigma_Q,Im} l(I(t), a_Q, \sigma_Q, Im) = \sum_{t=2}^{T} \log\left(\frac{1}{\sqrt{\sigma_Q 2\pi}} e^{-\frac{(I(t)-Im-a_Q(I(t-1)-Im))^2}{2\sigma_Q^2}}\right). \tag{5.8}$$

We will chose the expert approach, by choosing as value for Im the sample mean of the process $I(t)$.

Results of the estimation

For the seasonally adjusted monthly European Inflation data, the following calibrations are obtained ($Im = 0.01653$)

Parameters	Estimations
$\hat{a_Q}$	0.9586
$\hat{\sigma_Q}$	0.0025

Table 5.22: Estimation of the parameters of the Inflation

Analysis and quality of the estimation

We are going to check the asymptotically properties of the estimators.
For n=1000 simulations of the same size (2099), we obtain the following values

Parameters	Expectation	Standard deviation
$\hat{a_Q}$	0.9381	0.0273
$\hat{\sigma_Q}$	0.0025	0.0001

Table 5.23: Results of the simulation for the Inflation

The values are satisfactory, except the volatility of a_Q is quite high, but it may be due to the fact that the size of each sample is not so big.

As always, confidence intervals are presented to verify the significance of the estimated parameters

Parameters	Inflation
$RC(a_Q)$	0.018
$RC(\sigma_Q)$	0.0001

Table 5.24: Rao Cramer lower bound matrix

The table at the top contains the values of the lower bound of the Rao-Cramer inequality for each variable.

Parameters	sim CI	RC CI	Quantile CI
$\hat{a_Q}$	(0.9051 ; 1.0121)	(0.9233 ; 0.9939)	(0.8846 ; 0.9761)
$\hat{\sigma_Q}$	(0.0023 ; 0.0027)	(0.0023 0.0027)	(0.0023 ; 0.0027)

Table 5.25: CI for the parameters of the Inflation distribution

According to the CI, we can state that the parameters are all significant with the level of significance we have fixed (95%).

Analysis of the simulations

It remains to control that the model chosen for inflation allows to reproduce the historical data. To do this we will use, as we did in the previous paragraphs, the qq-plot and the representation of a simulated trajectory.

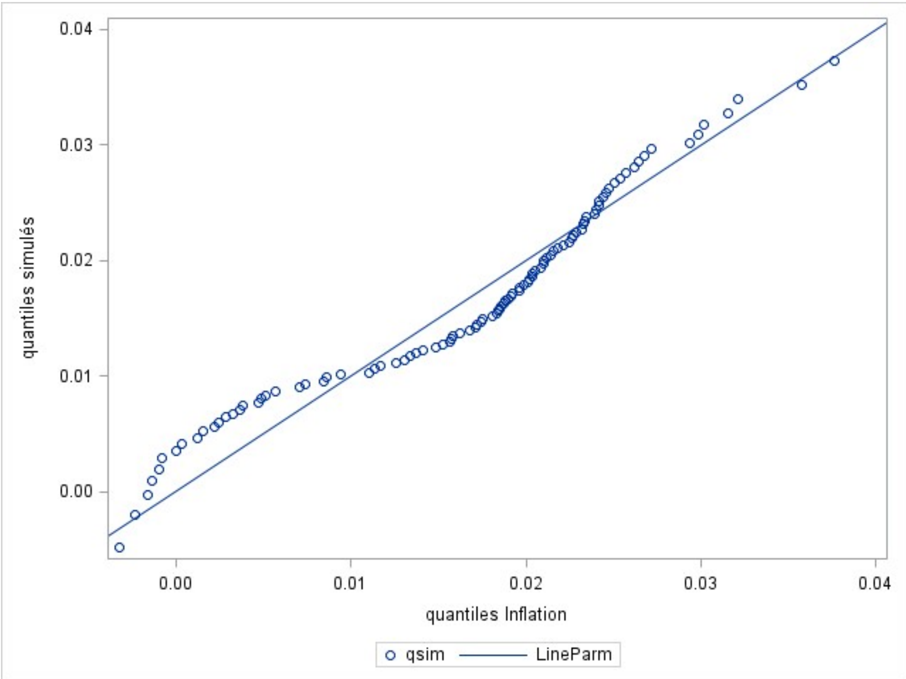

Figure 5.7: qq-plot Inflation

The simulated quantiles does not correspond to the empirical quantiles of the Inflation, which means that the hypothesis of normality for the inflation is not suitable.

5.2.2 Equity indices: GARCH $(1, 1)$ model

We now suppose schema 5.10. To calibrate the model we will use the maximum likelihood method, which leads us to solve the following problem

$$\max_{R,\lambda,\omega,\alpha,\beta} l(\omega,\alpha,\beta,R_t) = \max_{\omega_R,\lambda,\omega,\alpha,\beta} \sum_{t=2}^{T} \Big(-\frac{1}{2}\log(\hat{h}(t)) - \frac{1}{2}\frac{(R_t - \omega_R I(t) - \lambda)^2}{\hat{h}(t)} \Big) \tag{5.9}$$
$$s.t.\ \alpha + \beta < 1$$
$$\hat{h}(t) = \omega + \alpha(R(t-1) - \hat{\lambda})^2 + \beta\hat{h}(t-1)$$
$$\omega = \sigma^2(1-\alpha-\beta).$$

This comes from the fact that $R(t)|I(t)$ follows a Garch (1, 1) model.

Using the data we have, we obtain the following estimations

Parameters	Eurostoxx 50	Cac 40	Ftse Emerging
$\hat{\omega_R}$	-0.0453	-0.0361	-0.037
$\hat{\lambda}$	0.003	0.0031	0.0032
$\hat{\omega}$	0.00003	0.00003	0.00004
$\hat{\alpha}$	0.1546	0.1579	0.092
$\hat{\beta}$	0.8106	0.8084	0.8624
log-likelihood	2824.3964	2846.2015	2812.9943

Table 5.26: Estimation of the parameters for the equity indices returns

It should be noted that, compared to the results of estimation on the simple Garch (1, 1) model, there are no major differences except for the λ parameter which seems to compensate for the introduction of inflation.

Analysis and quality of the estimation

For every equity index, we will perform some simulations to check the usals properties of the estimators

Eurostoxx 50		
Parameters	Expectation	Standard deviation
$\hat{\omega_R}$	-0.045	0.0313
$\hat{\lambda}$	0.0029	0.0009
$\hat{\omega}$	0.00004	0.00002
$\hat{\alpha}$	0.1548	0.031
$\hat{\beta}$	0.7987	0.0408

Table 5.27: Results of the simulation for the Eurostoxx 50

Cac 40		
Parameters	Expectation	Standard deviation
$\hat{\omega_R}$	-0.0337	0.03
$\hat{\lambda}$	0.0031	0.0009
$\hat{\omega}$	0.00003	0.00001
$\hat{\alpha}$	0.157	0.0308
$\hat{\beta}$	0.8038	0.0389

Table 5.28: Results of the simulation for the Cac 40

Ftse Emerging		
Parameters	Expectation	Standard deviation
$\hat{\omega_R}$	-0.0363	0.0342
$\hat{\lambda}$	0.0031	0.001
$\hat{\omega}$	0.00007	0.00009
$\hat{\alpha}$	0.0937	0.028
$\hat{\beta}$	0.8261	0.1503

Table 5.29: Results of the simulation for the Ftse Emerging

The values are quite satisfactory, which tend to coincide with the estimation made, although there is probably a bias due to the limited size of the samples.

We can always calculate confidence intervals, with the following table of the lower bounds of Rao-Cramer inequality

Parameters	Eurostoxx 50	Cac 40	Ftse Emerging
$RC(\omega_R)$	0.0297	0.0288	0.032
$RC(\lambda)$	0.0009	0.0009	0.001
$RC(\omega)$	9.0701E-6	8.4435E-6	0.0001
$RC(\alpha)$	0.0297	0.0295	0.0255
$RC(\beta)$	0.0323	0.0318	0.0239

Table 5.30: Rao Cramer lower bound matrix

Eurostoxx 50 Parameters	sim CI	RC CI	quantile CI
$\hat{\omega_R}$	(-0.1066 ; 0.016)*	(-0.1035 ; 0.0129)*	(-0.114 ; 0.104)*
$\hat{\lambda}$	(0.0012 ; 0.0048)	(0.0012 ; 0.0048)	(0.0014 ; 0.0048)
$\hat{\omega}$	(3.3E-4 ; 0.0006)	(0.00002 ; 0.00005)	(0.00002 ; 0.00007)
$\hat{\alpha}$	0.0938 ; 0.2153)	(0.0964 ; 0.2128)	(0.1073 ; 0.2138)
$\hat{\beta}$	(0.7473 ; 0.8739)	(0.7306 ; 0.8906)	(0.7144 ; 0.8616)

Table 5.31: CI for the parameters of the Eurostoxx 50 distribution

Cac 40 Parameters	sim CI	RC CI	quantile CI
$\hat{\omega_R}$	(-0.0949 ; 0.0227)*	(-0.0925 ; 0.0203)*	(-0.0757 ; 0.0246)*
$\hat{\lambda}$	0.0013 ; 0.0049)	(0.0013 ; 0.0049)	(0.0014 ; 0.0045)
$\hat{\omega}$	(4.7E-4 ; 0.00006)	(0.00001 ; 0.00005)	(0.00001 ; 0.00006)
$\hat{\alpha}$	(0.0975 ; 0.2183)	(0.1 ; 0.2157)	(0.0988 ; 0.2189)
$\hat{\beta}$	(0.7322 ; 0.8846)	(0.7461 ; 0.8707)	(0.7066 ; 0.8753)

Table 5.32: CI for the parameters of the Cac 40 distribution

Ftse Emerging Parameters	sim CI	RC CI	quantile CI
$\hat{\omega_R}$	(-0.104 ; 0.03)*	(-0.0987 ; 0.0247)*	(-0.0819 ; 0.0175)*
$\hat{\lambda}$	(0.0012 ; 0.0052)	(0.0012 ; 0.0052)	(0.0011 ; 0.0049)
$\hat{\omega}$	(0.00002 ; 0.00006)	(-0.0001 ; 0.0002)*	(0.00002 ; 0.0004)
$\hat{\alpha}$	(0.0371 ; 0.1469)	(0.0522 ; 0.1318)	(0.0455 ; 0.1519)
$\hat{\beta}$	(0.5678 ; 1.157)	(0.8156 ; 0.9092)	(0.2541 ; 0.9222)

Table 5.33: CI for the parameters of the Ftse Emerging distribution

It looks like that the parameters are significant (at 95% level), except for the parameter ω_R (*).

Analysis of the simulations

We present and comment the qq-plot and some exemples of the trajectory for every equity index.

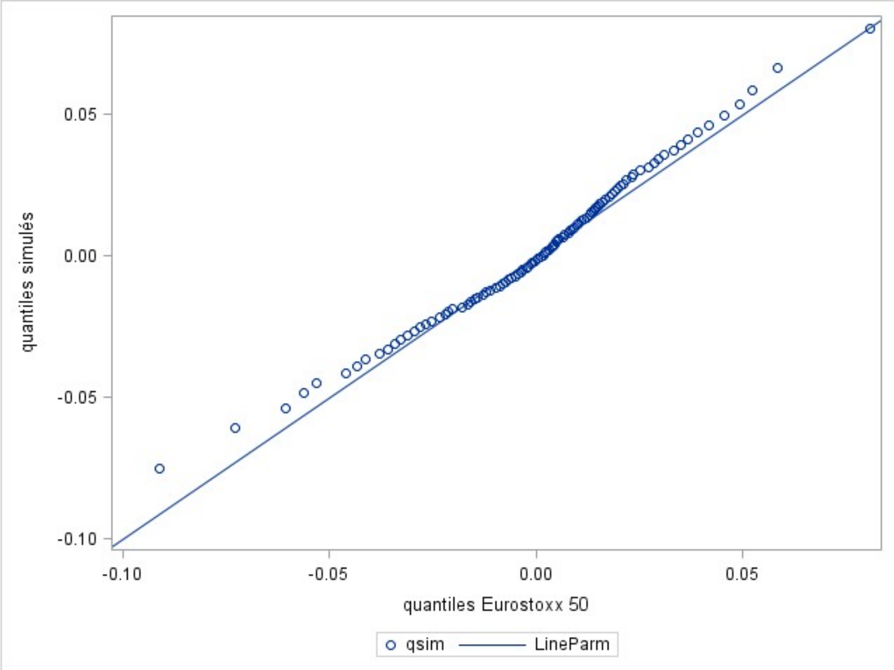

Figure 5.8: qq-plot Eurostoxx 50

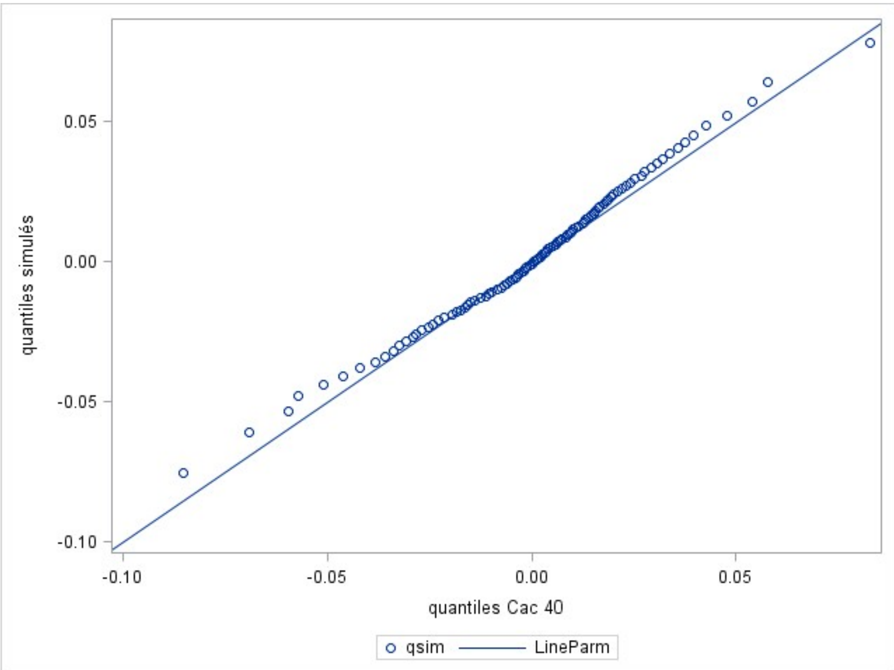

Figure 5.9: qq-plot Cac 40

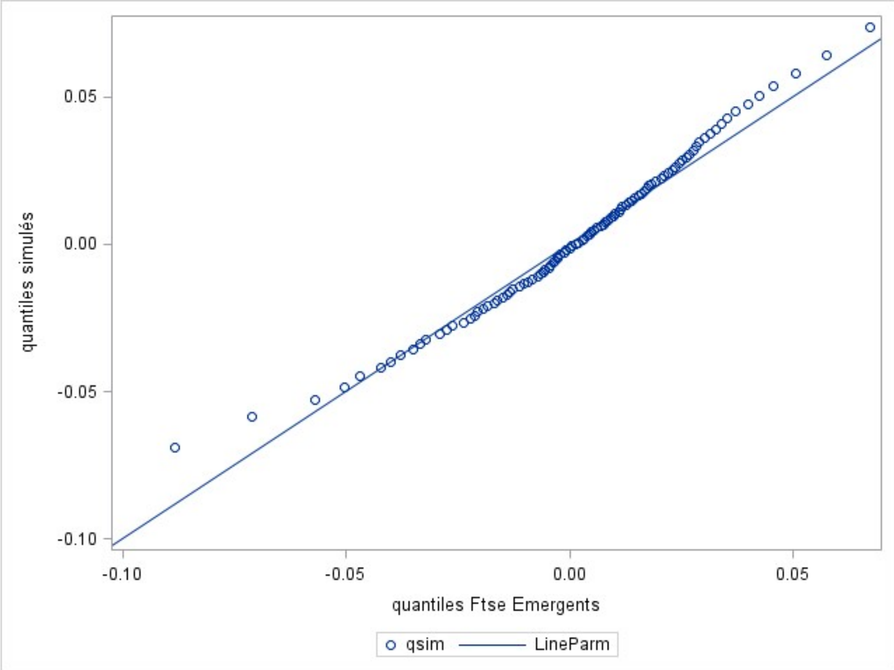

Figure 5.10: qq-plot Ftse Emerging

The empirical distribution of every equity index seems, except for some extreme value, to be similar to the simulated one.
We can notice a difference in the tails, especially for the negative part.

We could therefore conclude that this model (5.10) makes possible to explain the returns of equity indices by assuming a kind of link with inflation, and therefore a macro factor. It seems that the Garch $(1,1)$ model previously presented was not much improved (in fact the ω_R parameter is not significant (level fixed at 95%).

Conclusion

In this part of the thesis we were interested in the construction of a generator of economic scenarios.

The basic idea is to choose a certain number of financial assets that we would like to project in the ESG.
In order to do that, we need to perform a univariate and multivariate statistical analysis, which will help us to understand the stylized facts that will have to be faced during the construction of our generator.

The construction takes place through the choice of a diffusion model for each asset, taking into account the relations between them. To face this, we have two choices: a composite ESG or a cascade ESG. We decided to work with a cascade model that explains the returns of equity indices through a macroeconomic factor: the inflation.
The results do not show huge differences between modelling from a macro factor or a modelling without. In any case, cascade models could lead to improvements because they take into account factors that other diffusion models do not, which can be the inflation, the GDP, the debt-to-GDP ratio and so on.

Here, we have only dealt with the case of a composite model with equity indices. It would be interesting to develop this ESG, taking into account other existing asset classes (interest rates, exchange rates, commodities, etc.).

Appendix A

Univariate Analysis

We will present here the univariate analysis that we performed on the assets we decided to use in order to replicate the portfolio of Bpifrance. For each asset (equity index, interest rate, oil, change rate and inflation) we provide three different outputs: a table with the most important indicators we can compute from the empirical distribution (mean, standard deviation, skewness, kurtosis, alpha of Pareto and autocorrelation); a plot of the empirical distribution of the asset compared to a normal distribution, built using the empirical mean and the standard deviation of the financial instrument; a qq-plot, where we compare the quantiles of the empirical distribution and the ones of the normal distribution represented in the plot before.

Cac 40

Mean	0.0006
Standard deviation	0.03
Skewness	-0.2789
Excess Kurtosis	3.8145
alpha of Pareto	2.223
autocorrelation (lag 1)	-0.1161**
autocorrelation (lag 2)	-0.0435
autocorrelation (lag 3)	0.0701*
autocorrelation (lag 4)	-0.0111
autocorrelation (lag 5)	0.0138
autocorrelation squared (lag 1)	0.2644**
autocorrelation squared (lag 2)	0.1343**
autocorrelation squared (lag 3)	0.2527**
autocorrelation squared (lag 4)	0.1196**
autocorrelation squared (lag 5)	0.1041**

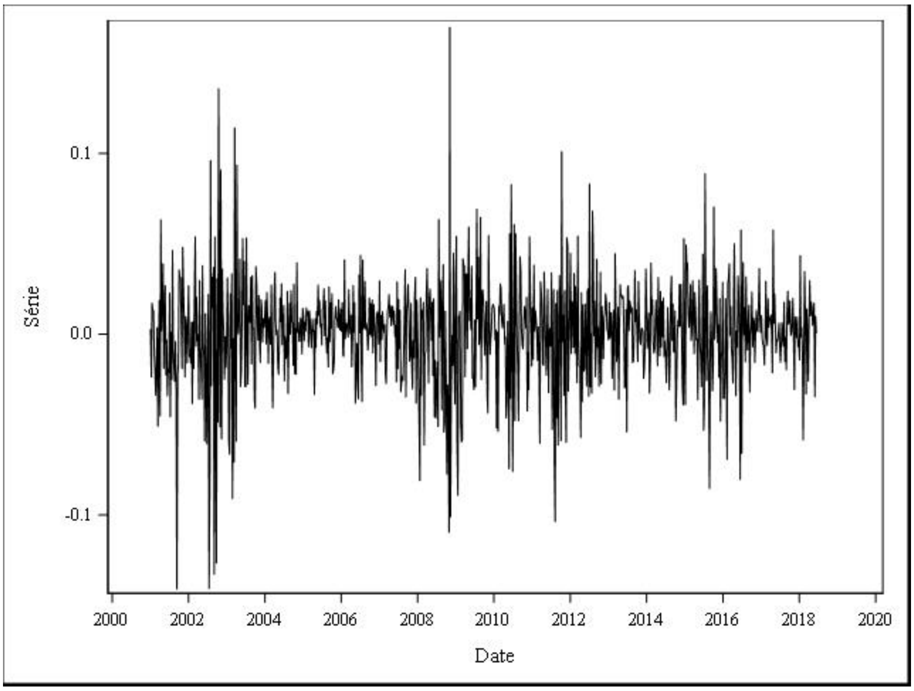

Figure A.1: Historical series of the Cac 40 weekly returns

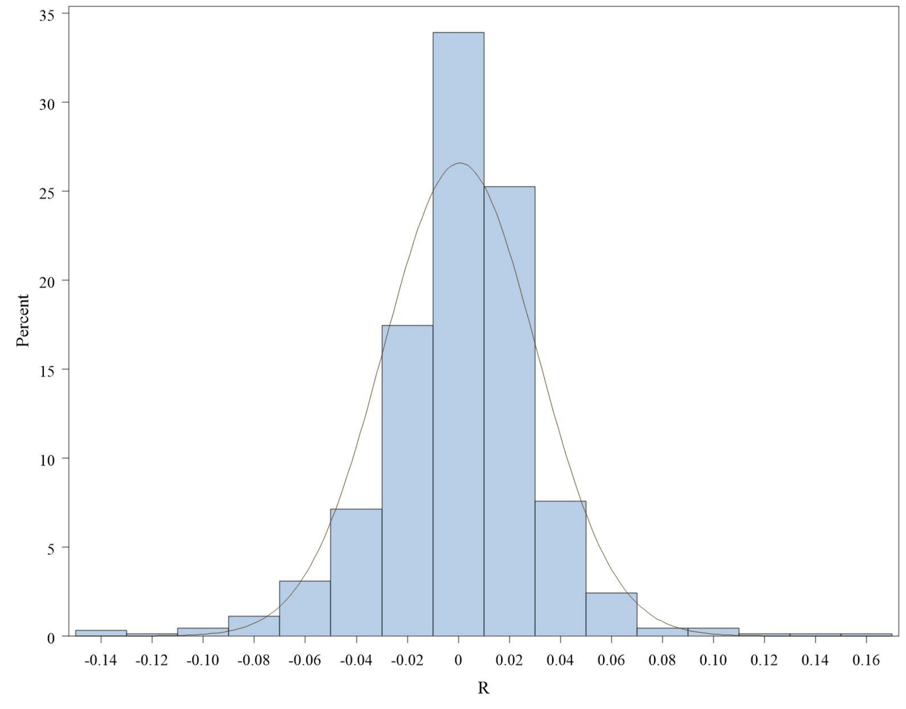

Figure A.2: Empirical distribution of the Cac 40 return vs Normal distribution

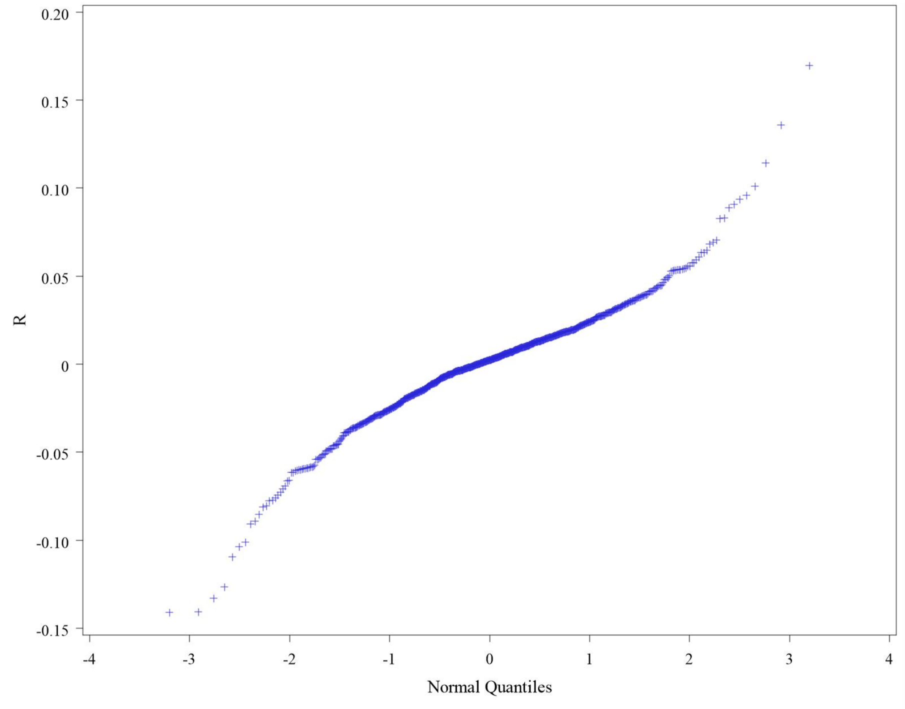

Figure A.3: qq-plot empirical distribution vs Normal distribution

Eurostoxx 50

Mean	0.0004
Standard deviation	0.03064
Skewness	-0.3942
Excess Kurtosis	3.1426
alpha of Pareto	2.2931
autocorrelation (lag 1)	-0.1033**
autocorrelation (lag 2)	-0.0355
autocorrelation (lag 3)	0.0524
autocorrelation (lag 4)	-0.0022
autocorrelation (lag 5)	0.0254
autocorrelation squared (lag 1)	0.2448**
autocorrelation squared (lag 2)	0.1462**
autocorrelation squared (lag 3)	0.2913**
autocorrelation squared (lag 4)	0.1276**
autocorrelation squared (lag 5)	0.11**

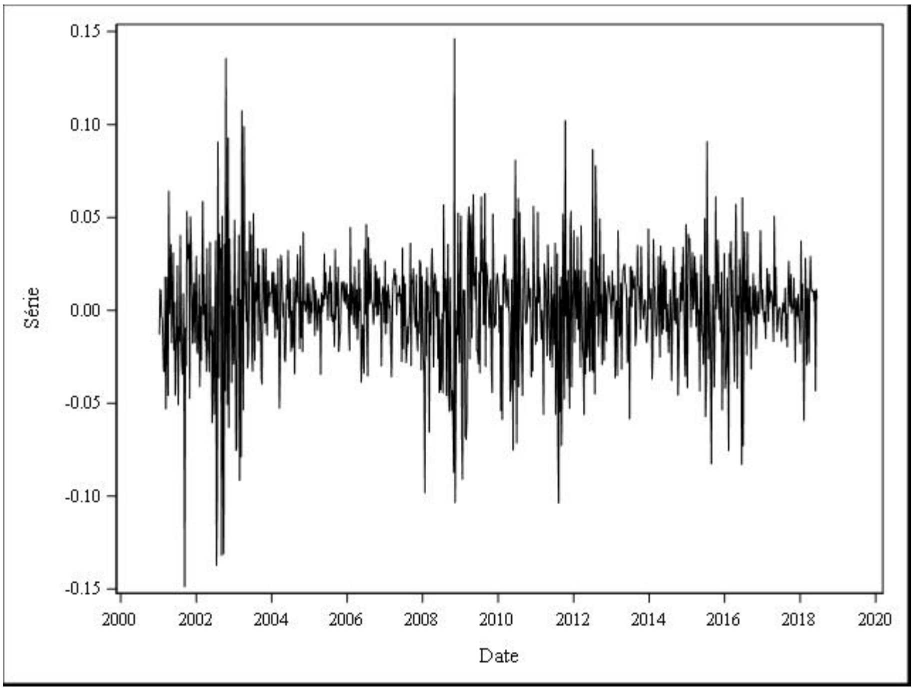

Figure A.4: historical series of the Eurostoxx weekly return

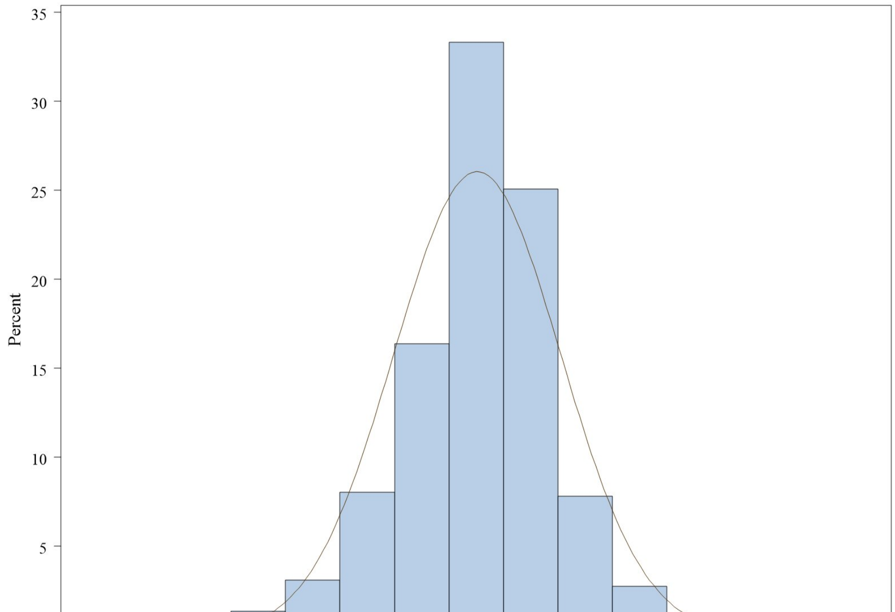

Figure A.5: Empirical distribution of the Eurostoxx 50 weekly return vs Normal distribution

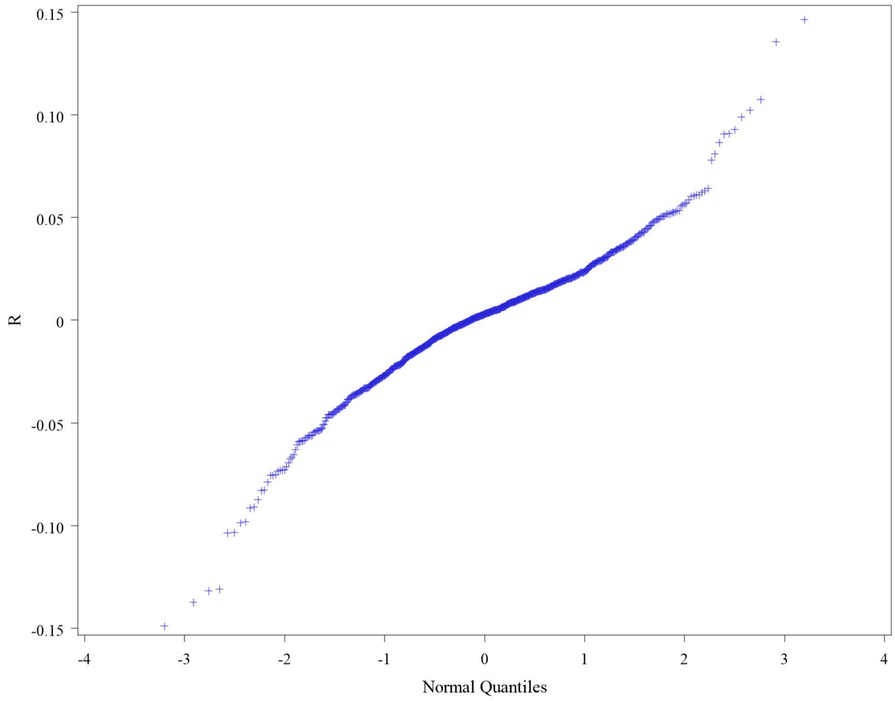

Figure A.6: qq-plot empirical distribution vs Normal distribution

Ftse Emerging

Mean	0.0017
Standard deviation	0.02953
Skewness	-0.5038
Excess Kurtosis	4.121
alpha of Pareto	2.2833
autocorrelation (lag 1)	0.0226
autocorrelation (lag 2)	-0.0449
autocorrelation (lag 3)	0.0633
autocorrelation (lag 4)	0.0162
autocorrelation (lag 5)	0.0604
autocorrelation squared (lag 1)	0.2346**
autocorrelation squared (lag 2)	0.2612**
autocorrelation squared (lag 3)	0.1902**
autocorrelation squared (lag 4)	0.258**
autocorrelation squared (lag 5)	0.0603

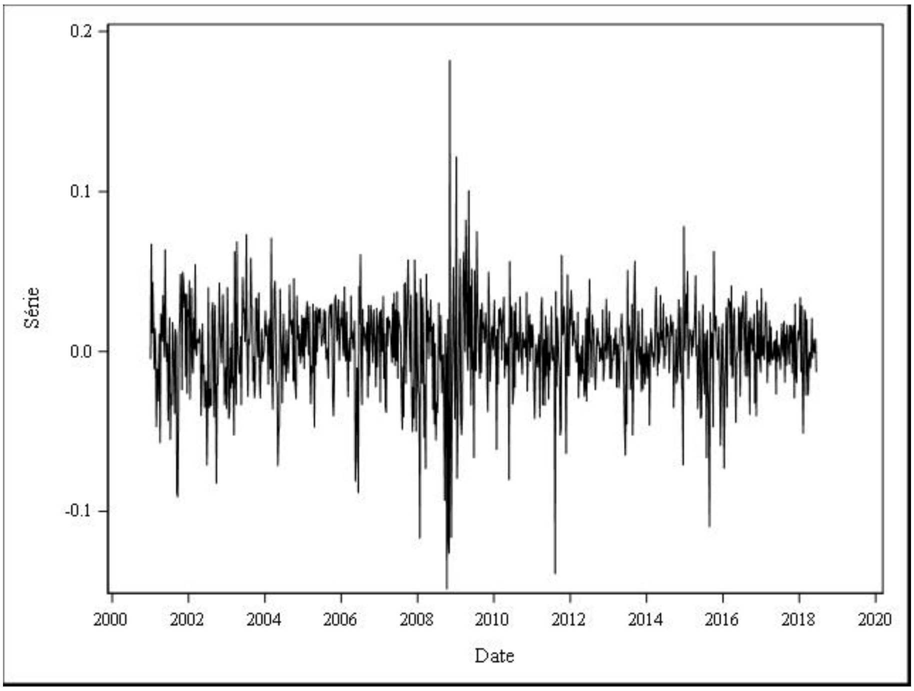

Figure A.7: Historical series of the Ftse Emerging weekly returns

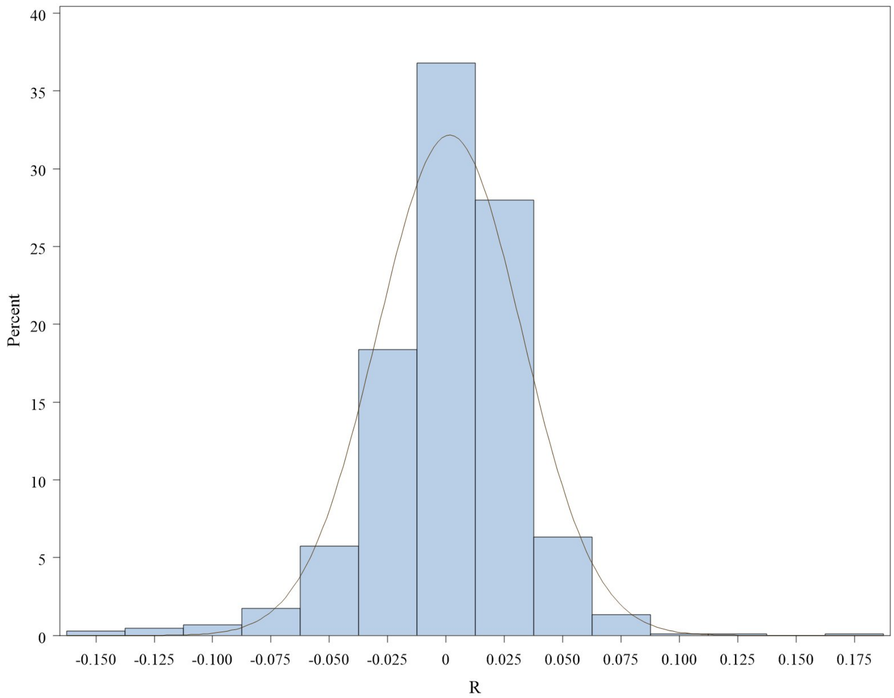

Figure A.8: Empirical distribution of the Ftse Emerging return vs Normal distribution

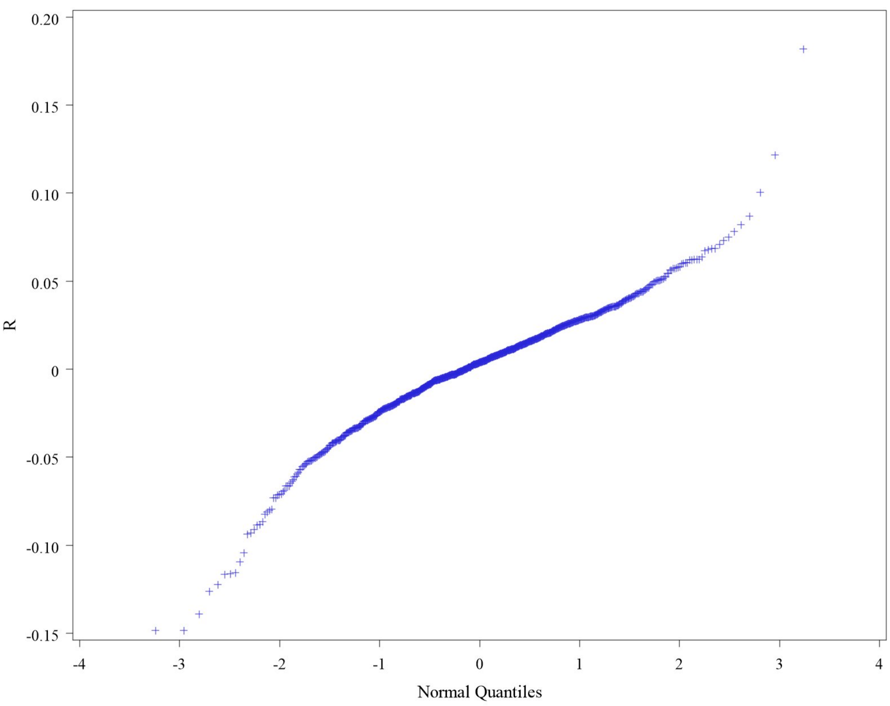

Figure A.9: qq-plot empirical distribution vs Normal distribution

Looking the tables concerning the equity indices we can observe at least four stylised facts:

i) absence of autocorrelation, which means that we can't forecast the future returns using the past data.

ii) a high excess kurtosis (value greater than 0), which indicates the fact that the events situated in the tails of the distribution have a non negligible probability of happening.

iii) volatility clustering, large changes in the returns are followed by large changes, either signs. It is observable from the historical series of the weekly returns.

iv) slight negative skewness, which means that the probability of having negative values is bigger than the one of having positive values, this can be seen from the empirical distribution of the equity indices.

Eonia

Mean	-0.0057
Standard deviation	0.1789
Skewness	-0.402
Excess Kurtosis	8.6017
alpha of Pareto	1.1017
autocorrelation (lag 1)	-0.3708**
autocorrelation (lag 2)	-0.0445
autocorrelation (lag 3)	0.0247
autocorrelation (lag 4)	-0.029
autocorrelation (lag 5)	0.0333
autocorrelation squared (lag 1)	0.3419**
autocorrelation squared (lag 2)	-0.0106
autocorrelation squared (lag 3)	0.0579
autocorrelation squared (lag 4)	0.0917**
autocorrelation squared (lag 5)	0.0569

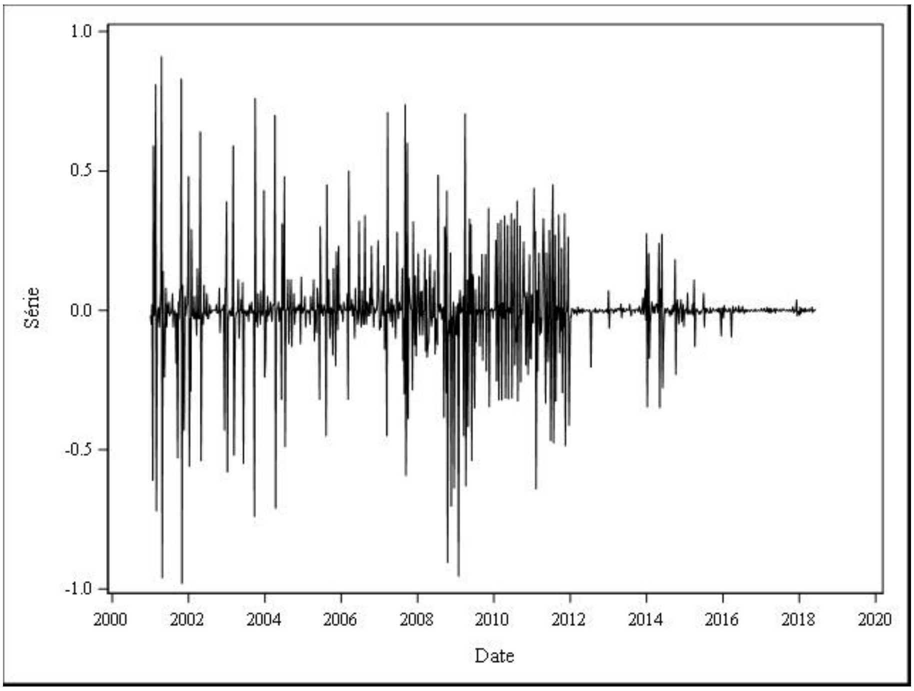

Figure A.10: Historical series of the Eonia weekly return

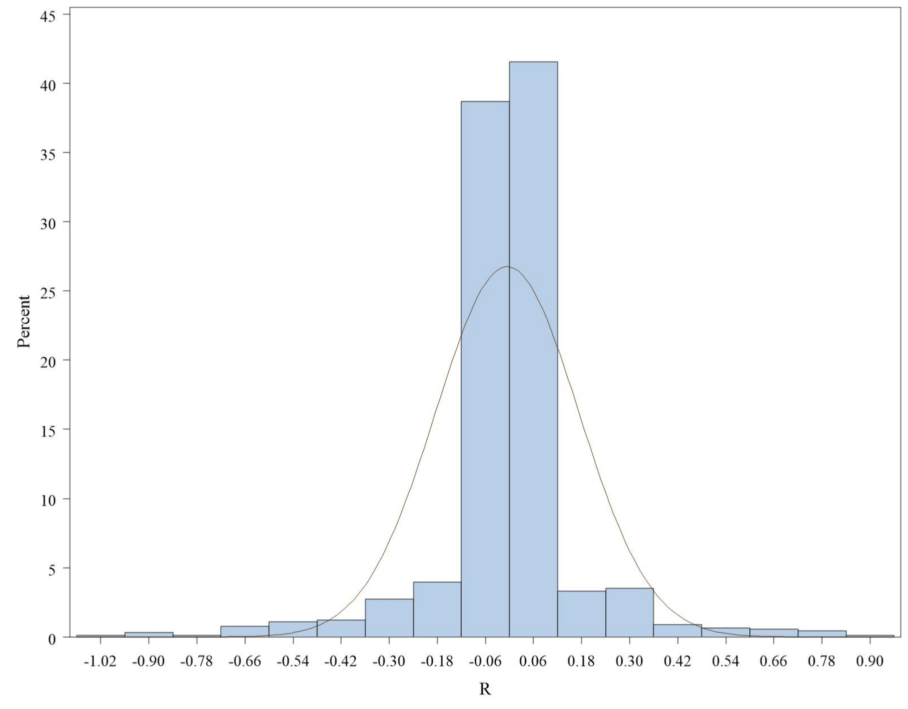

Figure A.11: Empirical distribution of the Eonia return vs Normal distribution

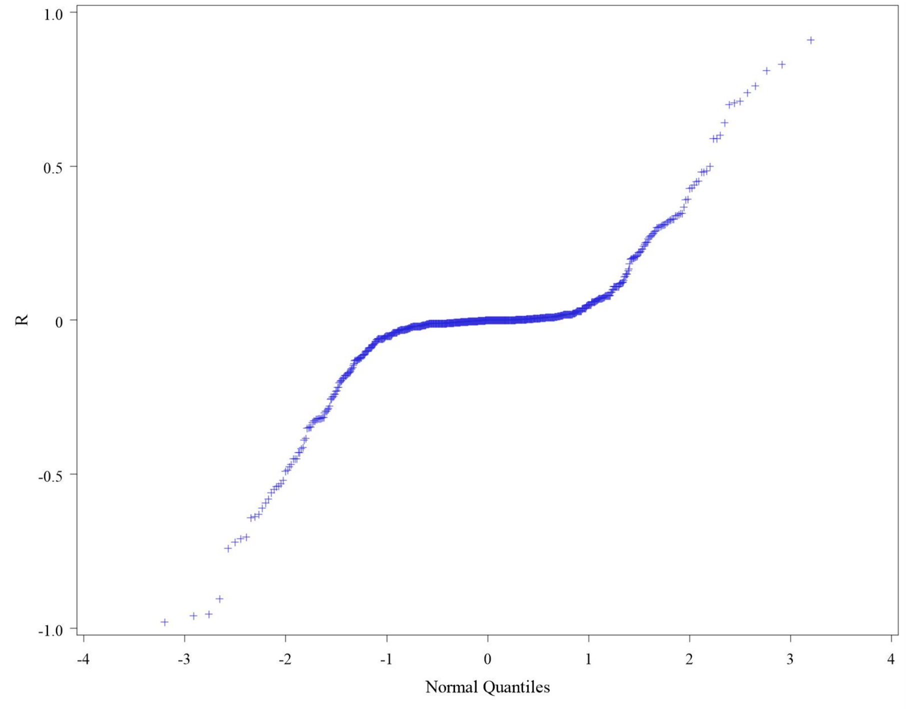

Figure A.12: qq-plot empirical distribution vs Normal distribution

Oat 1 year

Mean	-0.0039
Standard deviation	0.0736
Skewness	-2.6352
Excess Kurtosis	22.5323
alpha of Pareto	1.4581
autocorrelation (lag 1)	0.1387**
autocorrélation (lag 2)	0.0546
autocorrelation (lag 3)	0.1811**
autocorrelation (lag 4)	0.0895*
autocorrelation (lag 5)	0.0277
autocorrelation squared (lag 1)	0.3939**
autocorrelation squared (lag 2)	0.0571
autocorrelation squared (lag 3)	0.1134**
autocorrelation squared (lag 4)	0.053
autocorrelation squared (lag 5)	0.0971**

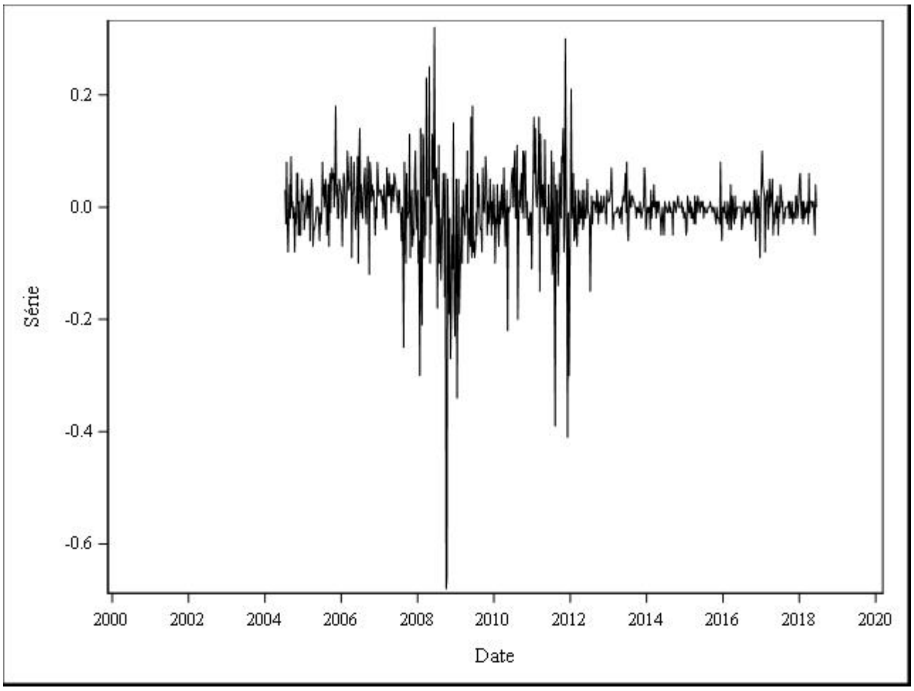

Figure A.13: Historical series of the Oat 1y return

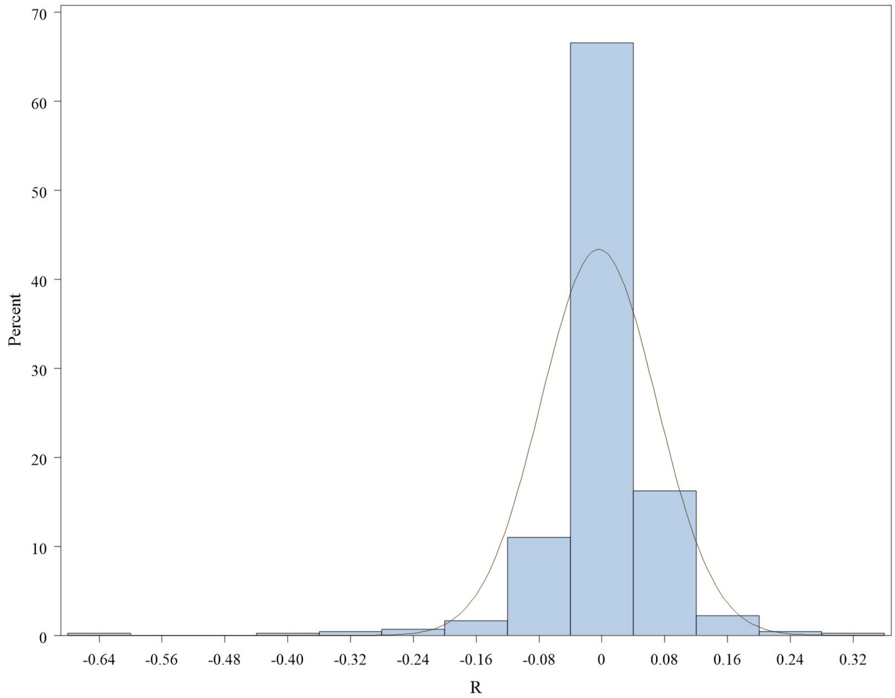

Figure A.14: Empirical distribution of the Oat 1y return vs Normal distribution

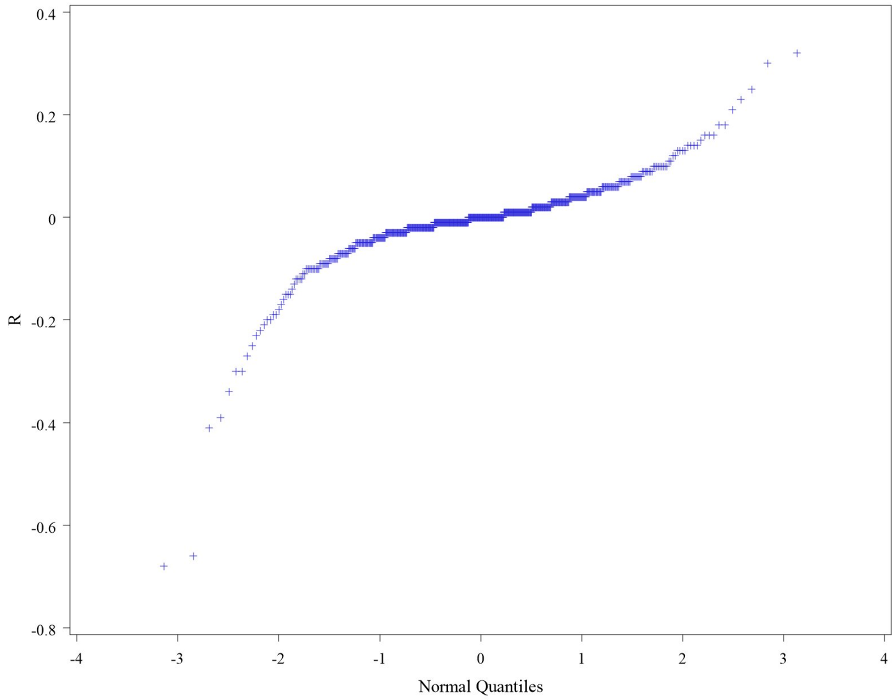

Figure A.15: qq-plot empirical distribution vs Normal distribution

Oat 10 years

Mean	-0.0046
Standard deviation	0.0944
Skewness	0.2497
Excess Kurtosis	1.4866
alpha of Pareto	2.6457
autocorrelation (lag 1)	0.0048
autocorrelation (lag 2)	-0.0461
autocorrelation (lag 3)	0.0762*
autocorrelation (lag 4)	0.0229
autocorrelation (lag 5)	-0.0277
autocorrelation squared (lag 1)	0.1001**
autocorrelation squared (lag 2)	0.1663**
autocorrelation squared (lag 3)	0.261**
autocorrelation squared (lag 4)	0.1604**
autocorrelation squared (lag 5)	0.2034**

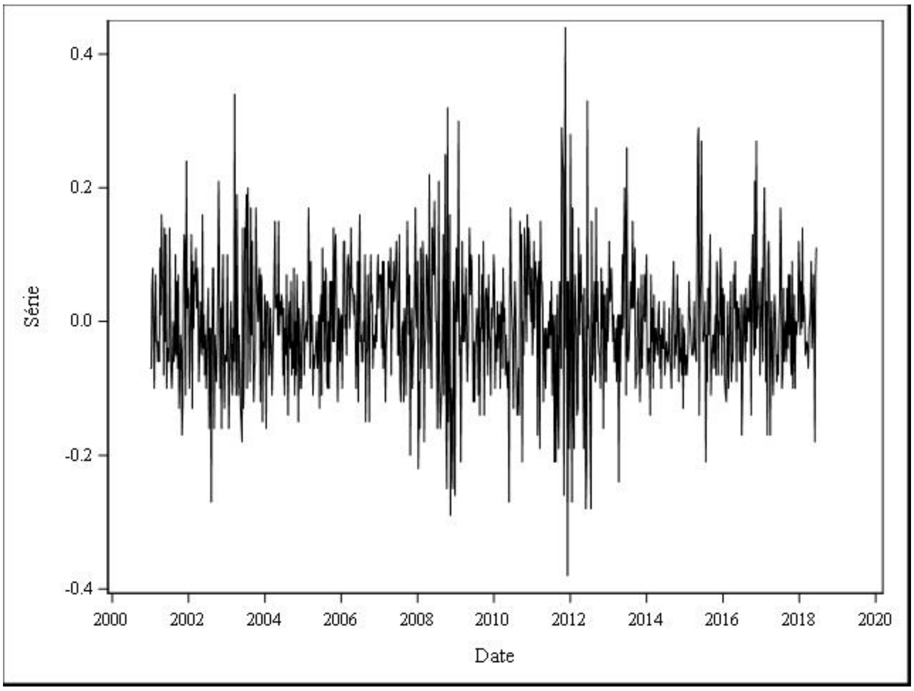

Figure A.16: Historical series of the Oat 10y weekly return

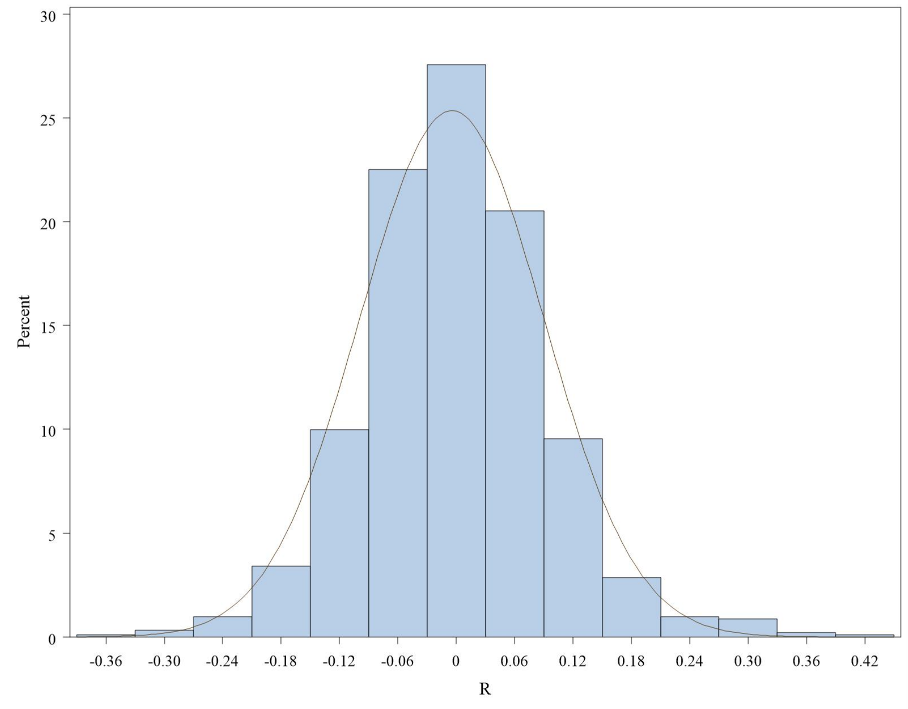

Figure A.17: Empirical distribution of the Oat 10y return vs Normal distribution

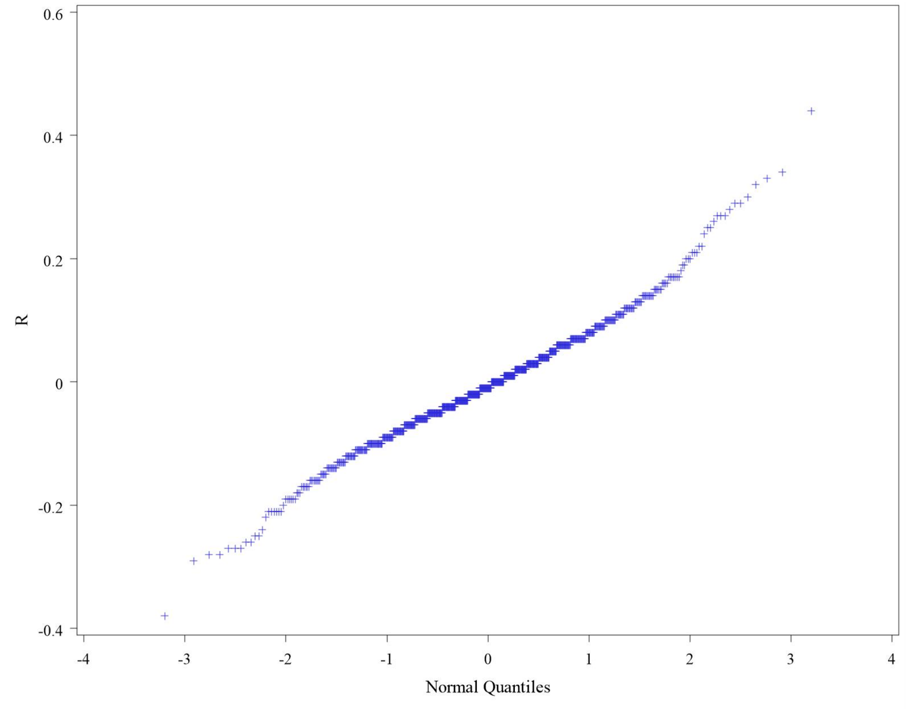

Figure A.18: qq-plot empirical distribution vs Normal distribution

For what concerns the interest rates, we can observe the following facts:

i) absence of autocorrelation, for all indices

ii) an high excess kurtosis for the Eonia and the Oat 1y and a low excess kurtosis for the Oat 10 y, this is due to the fact that we are considering a risk free rate of 10 years.

iii) volatility clustering

iv) a slight negative/positive skewness, respectively for the Eonia and the Oat 10 y, and an high negative skewness for the Oat 1 y.

Oil

Mean	0.0007
Standard deviation	0.0504
Skewness	-0.4459
Excess Kurtosis	3.1277
alpha of Pareto	2.1785
autocorrelation (lag 1)	-0.0757*
autocorrelation (lag 2)	-0.0496
autocorrelation (lag 3)	0.0728*
autocorrelation (lag 4)	-0.0084
autocorrelation (lag 5)	0.0355
autocorrelation squared (lag 1)	0.2422**
autocorrelation squared (lag 2)	0.2616**
autocorrelation squared (lag 3)	0.1795**
autocorrelation squared (lag 4)	0.1018**
autocorrelation squared (lag 5)	0.1228**

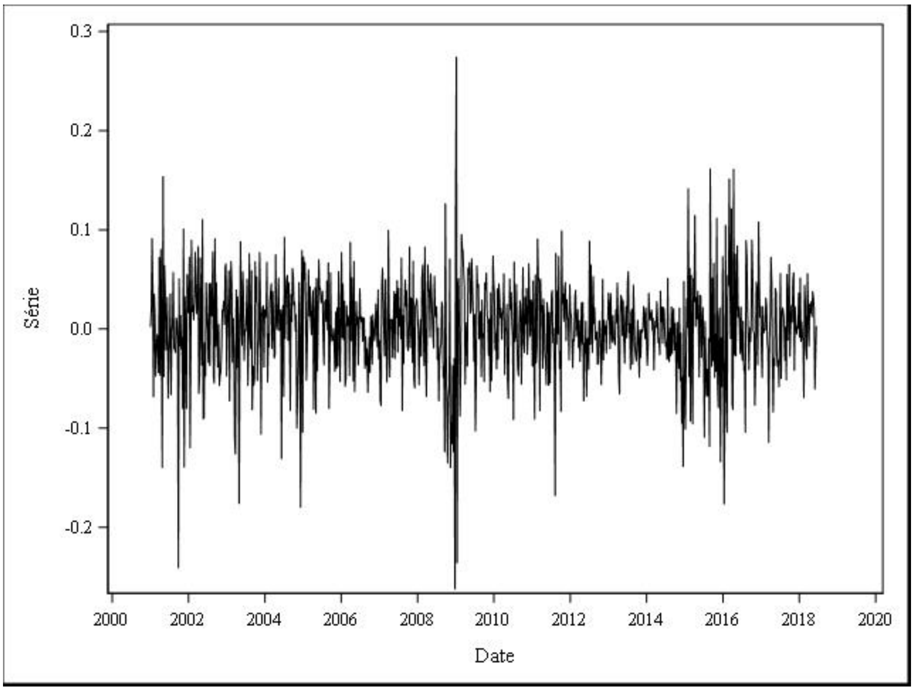

Figure A.19: Historical series of the Oil weekly return

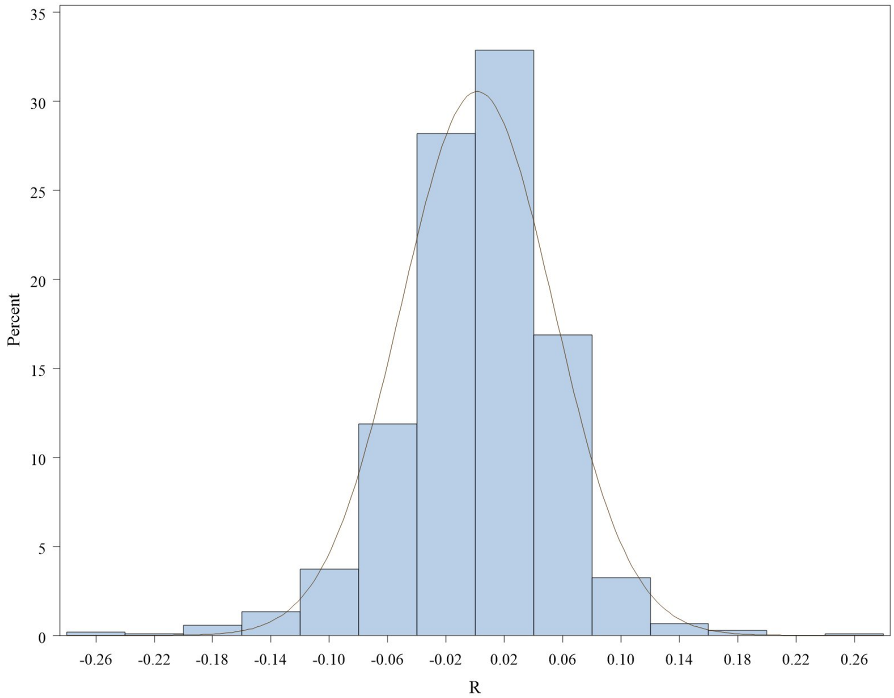

Figure A.20: Empirical distribution of the Oil return vs Normal distribution

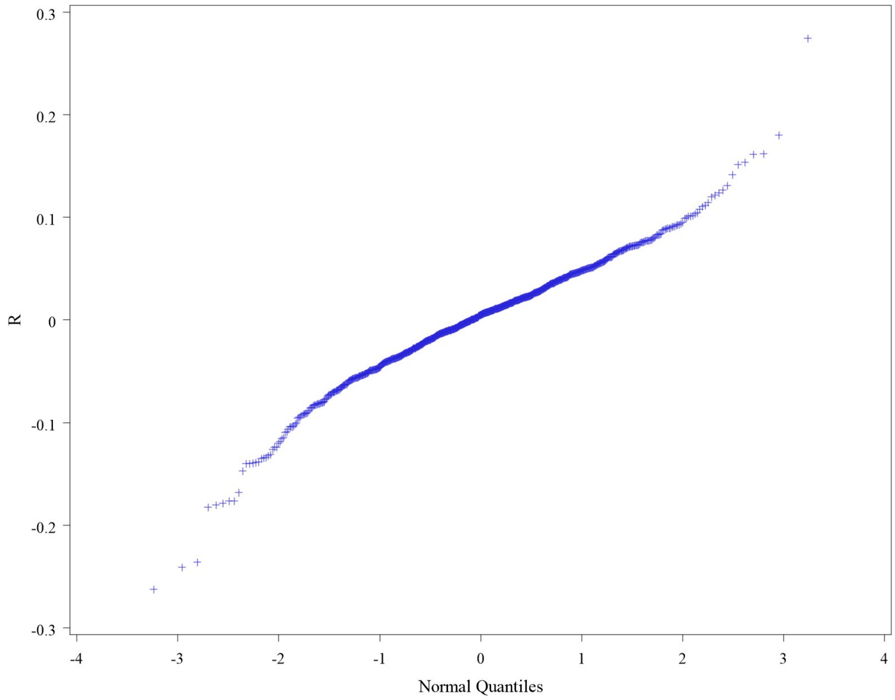

Figure A.21: qq-plot empirical distribution vs Normal distribution

We can notice

i) absence of autocorrelation

ii) high excess kurtosis

iii) volatility clustering

iv) slight negative skewness

Change rate Dollar-Euro

Mean	0.0003
Standard deviation	0.017
Skewness	-0.1909
Excess Kurtosis	1.6655
alpha of Pareto	2.8615
autocorrelation (lag 1)	-0.0064
autocorrelation (lag 2)	0.0186
autocorrelation (lag 3)	0.0157
autocorrelation (lag 4)	0.0006
autocorrelation (lag 5)	-0.0129
autocorrelation squared (lag 1)	0.183**
autocorrelation squared (lag 2)	0.1372**
autocorrelation squared (lag 3)	0.2797**
autocorrelation squared (lag 4)	0.1852**
autocorrelation squared (lag 5)	0.2134**

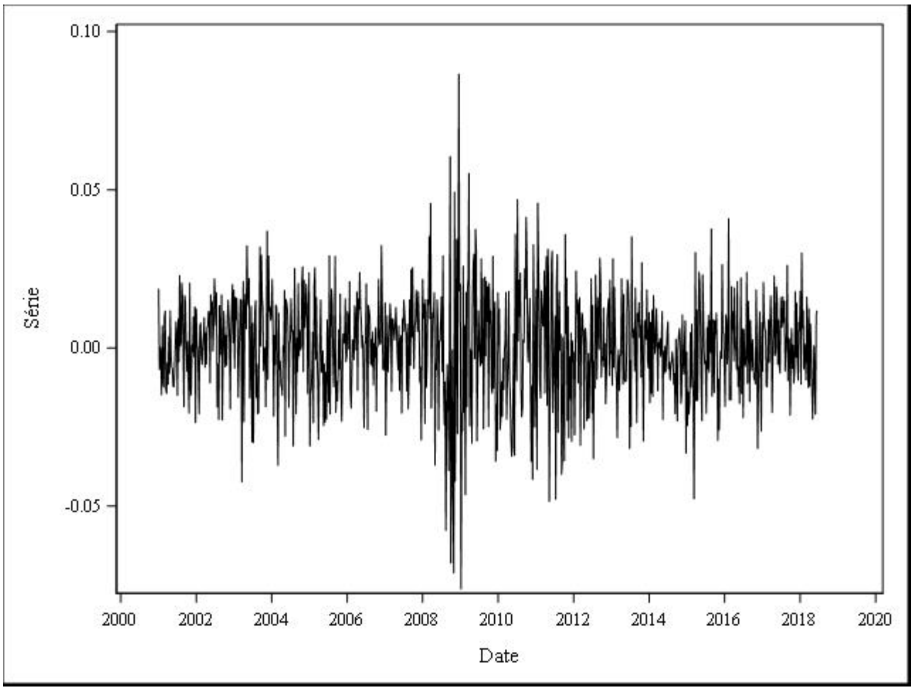

Figure A.22: Historical series of the US/EUR change rate weekly return

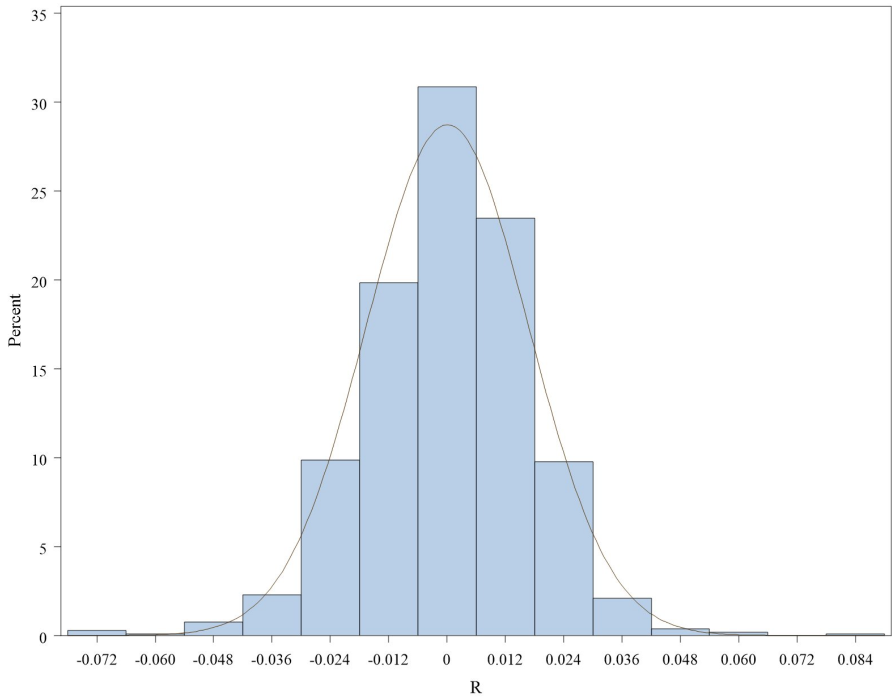

Figure A.23: Empirical distribution of the US/EUR change rate return vs Normal distribution

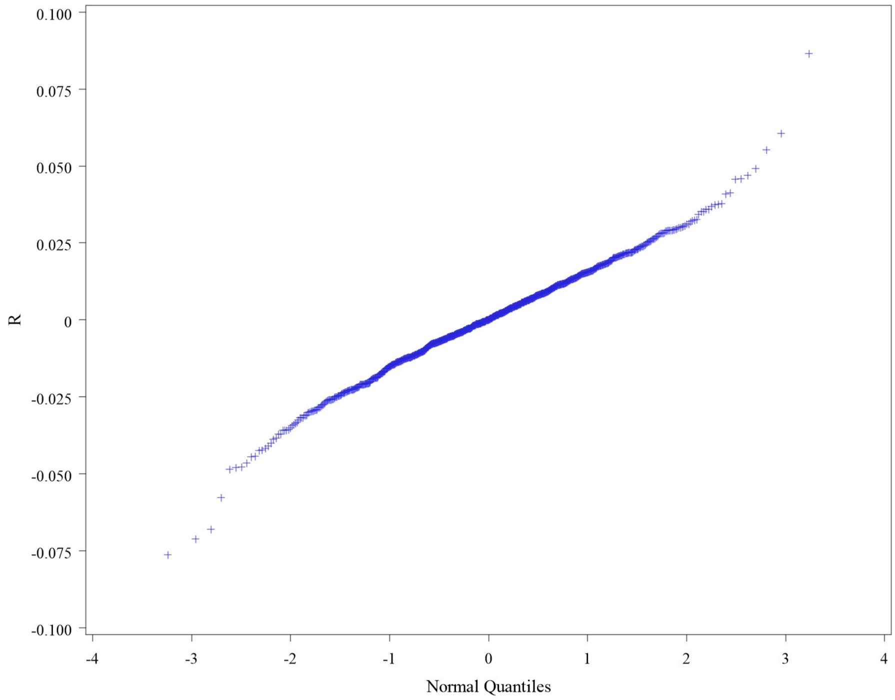

Figure A.24: qq-plot empirical distribution vs Normal distribution

As for the Oil returns we can observe the same stylized facts as before, except for a value of excess kurtosis which is lower but always positive.

Inflation Europe

Mean	0.0014
Standard deviation	0.0044
Skewness	-0.6288
Excess Kurtosis	2.027
alpha of Pareto	2.2173
autocorrelation (lag 1)	0.0336
autocorrelation (lag 2)	-0.2485**
autocorrelation (lag 3)	-0.0988
autocorrelation (lag 4)	-0.2563**
autocorrelation (lag 5)	-0.02513
autocorrelation squared (lag 1)	-0.03871
autocorrelation squared (lag 2)	0.2612**
autocorrelation squared (lag 3)	-0.1477*
autocorrelation squared (lag 4)	-0.0541
autocorrelation squared (lag 5)	-0.1752*

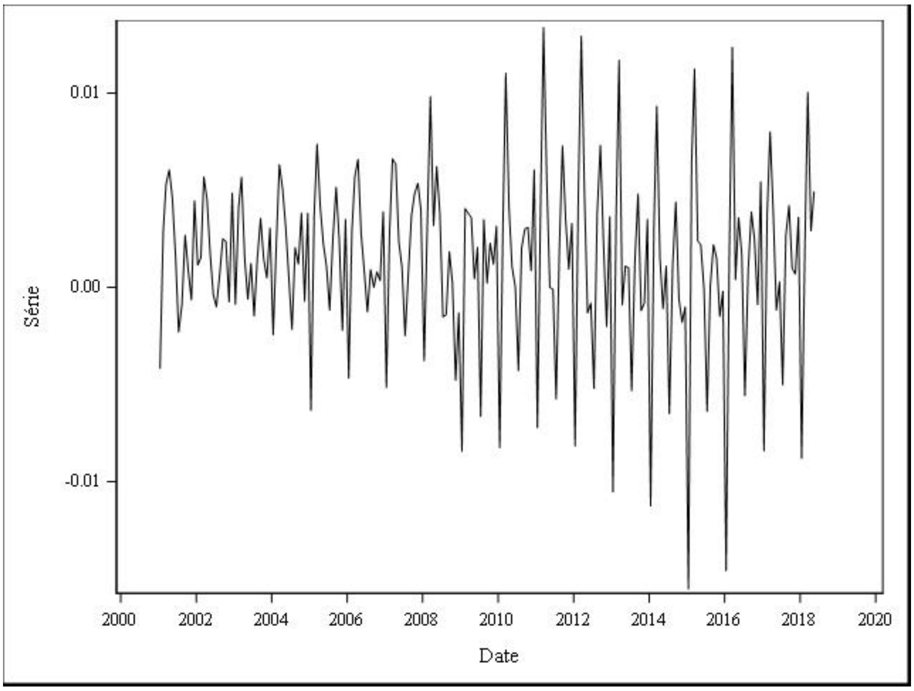

Figure A.25: Historical series of the European Inflation monthly variation

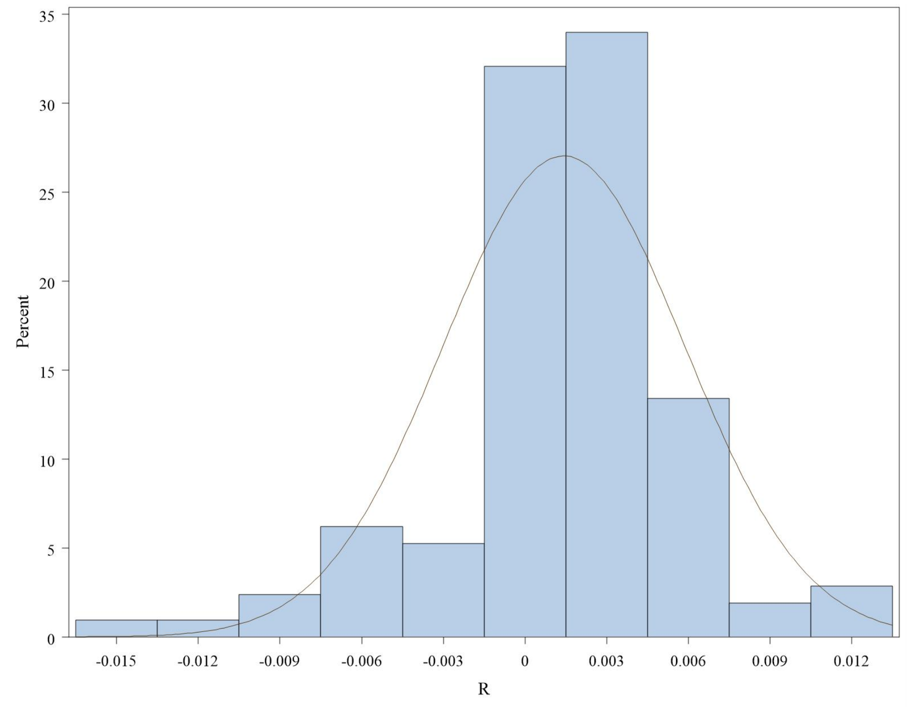

Figure A.26: Empirical distribution of the European Inflation variation vs Normal distribution

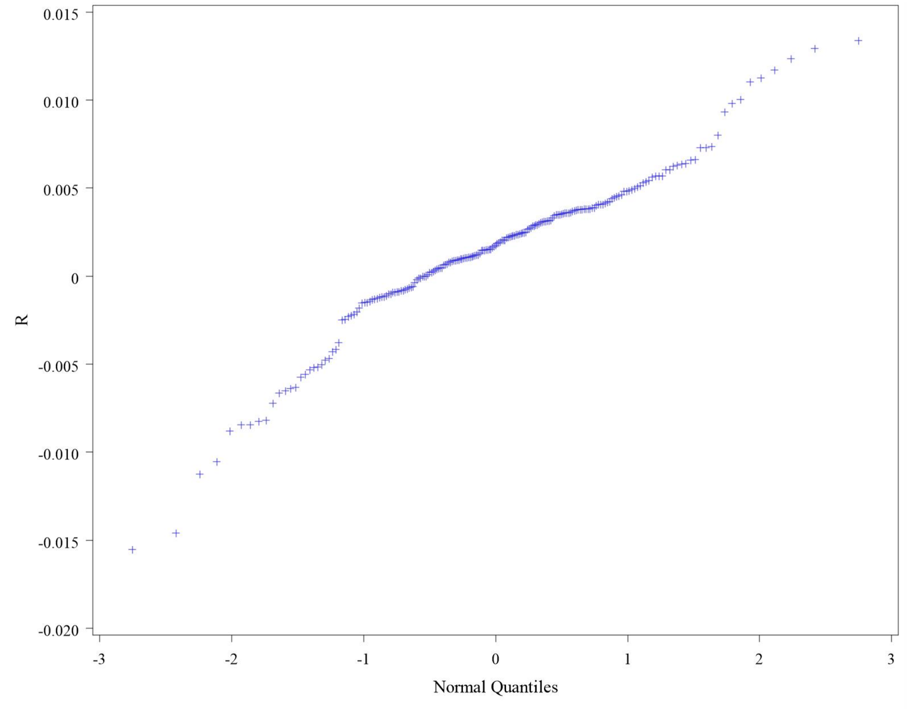

Figure A.27: qq-plot empirical distribution vs Normal distribution

Inflation France

Mean	0.0012
Standard deviation	0.0018
Skewness	-0.1779
Excess Kurtosis	1.1443
alpha of Pareto	1.2297
autocorrelation (lag 1)	0.1276
autocorrelation (lag 2)	0.0725
autocorrelation (lag 3)	0.0882
autocorrelation (lag 4)	0.0645
autocorrelation (lag 5)	0.0395
autocorrelation squared (lag 1)	0.1482*
autocorrelation squared (lag 2)	0.0794
autocorrelation squared (lag 3)	-0.0261
autocorrelation squared (lag 4)	0.0261
autocorrelation squared (lag 5)	0.0014

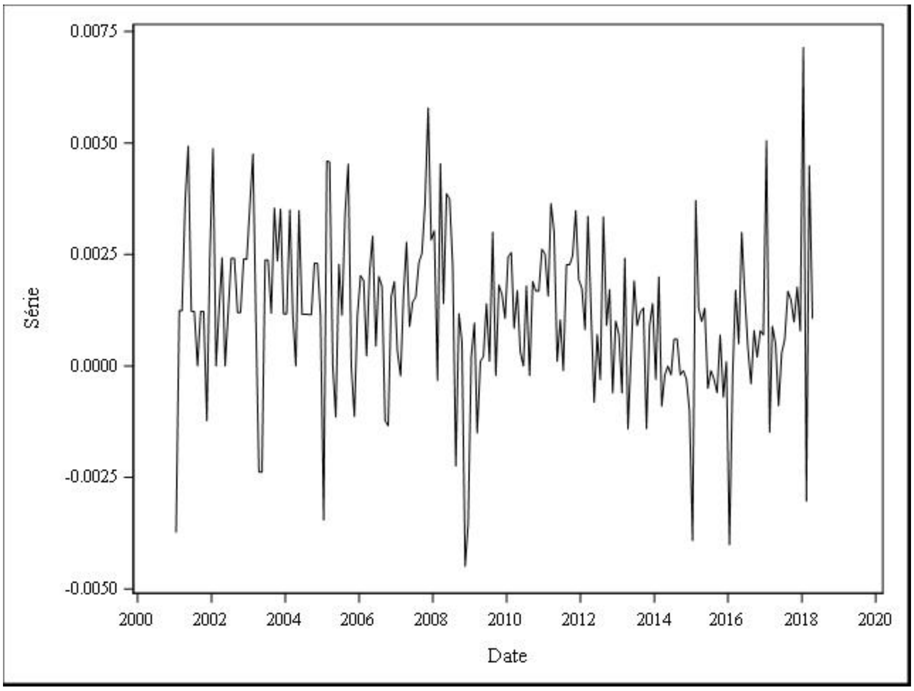

Figure A.28: Historical series of the French Inflation monthly variation

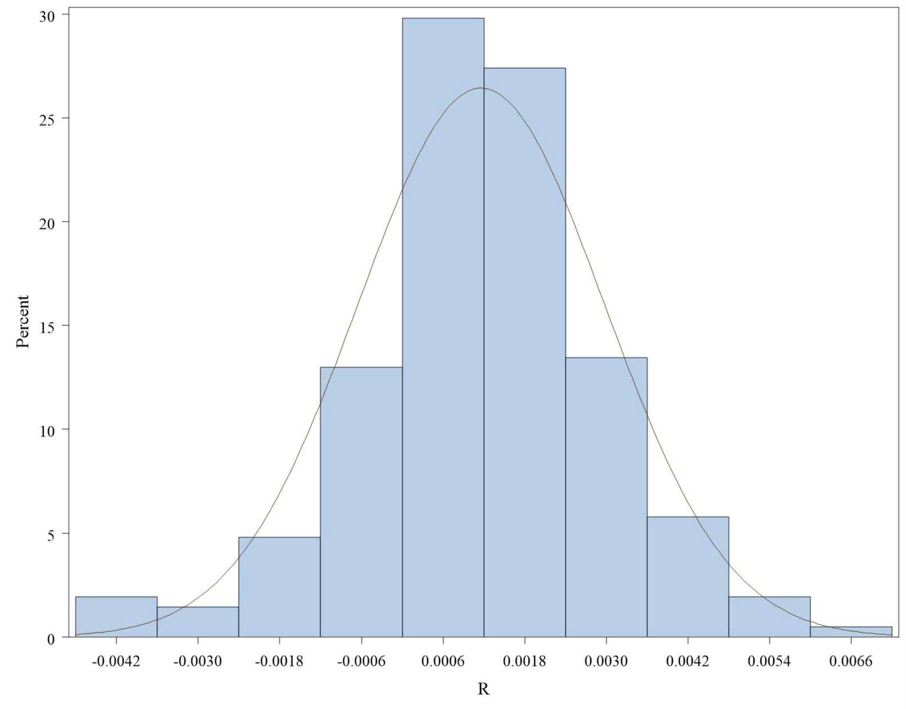

Figure A.29: Empirical distribution of the French Inflation variation vs Normal distribution

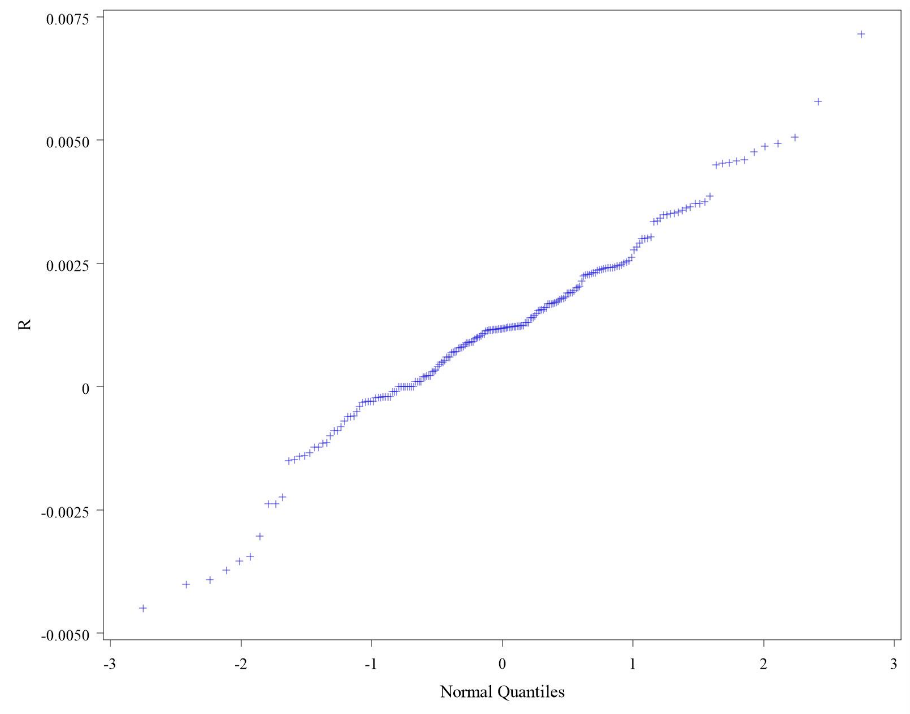

Figure A.30: qq-plot empirical distribution vs Normal distribution

Concerning the Inflation we can state

i) absence of autocorrelation

ii) positive excess kurtosis

iii) volatility clustering, especially in the last part of the historical series (European Inflation)

iv) slight negative skewness for both variables

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