ALGEBRAIC MODELS OF DEVIANT MODAL OPERATORS BASED ON DE MORGAN AND KLEENE LATTICES

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Abstract. An algebraic model of a kind of modal extension of de Morgan logic is described under the name MDS5 algebra. The main properties of this algebra can be summarized as follows: (1) it is based on a de Morgan lattice, rather than a Boolean algebra; (2) a modal necessity operator that satisfies the axioms N, K, T, and 5 (and as a consequence also B and 4) of modal logic is introduced; it allows one to introduce a modal possibility by the usual combination of necessity operation and de Morgan negation; (3) the necessity operator satisfies a distributivity principle over joins. The latter property cannot be meaningfully added to the standard Boolean algebraic models of S5 modal logic, since in this Boolean context both modalities collapse in the identity mapping. The consistency of this algebraic model is proved, showing that usual fuzzy set theory on a universe U can be equipped with a MDS5 structure that satisfies all the above points (1)–(3), without the trivialization of the modalities to the identity mapping. Further, the relationship between this new algebra and Heyting-Wajsberg algebras is investigated. Finally, the question of the role of these deviant modalities, as opposed to the usual non-distributive ones, in the scope of knowledge representation and approximation spaces is discussed.

1. Introduction

Modal logic has been extensively used for devising logical accounts of epistemic notions like belief, knowledge, and approximation in the framework of Boolean logic. The standard logic of knowledge is S5 (Halpern et al. Halpern et al. (2003)), and the standard logic of belief is KD45 (Hintikka Hintikka (1962)). In rough set theory, sets are approximated by elements of a partition induced by an equivalence relation, and a natural choice for a rough set logic is S5 again (as proposed by Orlowska Orlowska (1985, 1990)), whereby possibility and necessity modalities express outer and inner approximation operators. In standard modal systems, the basic modalities, called *necessity* (denoted by $\Box$) and *possibility* (denoted by $\Diamond$) satisfy basic properties like their interdefinability via negation ($\Box p = \neg\Diamond\neg p$), and distributivity of $\Box$ over conjunction $\Box(p \wedge q) = \Box p \wedge \Box q$ (hence distributivity of $\Diamond$ over disjunction). Generally, distributivity of $\Box$ over disjunction is not demanded to avoid the collapse to identity of the modalities. In fact, the paper (Cattaneo and Ciucci, 2004) triggered some discussion, especially a terminological debate about what can be considered an acceptable modal system (or a deviant version of a modal system) from the point of view of its algebraic semantics.

The aim of this paper is to start clarifying this issue on the side of modalities. We consider a de Morgan algebra augmented with the unary operators standing for distributive modalities, whose role is to sharpen elements of the algebra, and

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show that this structure is non-trivial, contrary to the Boolean case. This is done by providing two important examples one using fuzzy sets as the basic elements of the algebra, and the other one using pairs of disjoint sets (simply orthopairs). We show that the necessity and possibility correspond to operators extracting inner and outer approximations of elements in the algebra, an example of which is the core and the support of fuzzy sets.

Then we consider the structure of Brouwer-Zadeh (BZ) lattice as introduced by Cattaneo–Nisticò in Cattaneo and Nisticò (1989), in which a Kleene algebra is equipped with an intuitionistic negation. In this BZ context the modalities induced by suitable composition of the two negations generate a modal S5 system based on a Kleene algebra, but without the full distributivity property of modal operators. The latter condition is verified in the more restrictive case of Brouwer Zadeh lattices whose intuitionistic negation satisfies the particular Stone condition. Then we study how the two above mentioned examples of fuzzy sets and orthopairs behave in this stronger setting.

Finally, we consider the case of Heyting-Wajsberg algebras Cattaneo and Ciucci (2002); Cattaneo et al. (2004a,b), where the primitive connectives are two residuated implications (respectively Gödel and Łukasiewicz ones) from which a pair of negations can be retrieved, and whose induced structure is a special case of Stonean Brouwer-Zadeh lattices. It is shown that applying each implication operator to approximation pairs of two elements of the HW algebra yields the approximation pair of a well-defined third element. The significance of this result is discussed, noticing that the latter element is a function not only of the original ones but also of their approximations.

2. De Morgan and Kleene algebras with S5 modal operators

In this section we consider a generalisation of Boolean lattices with operators to lattice structures equipped with an involutive negation, i.e. De Morgan algebras. We use S5-like modal operators that are deviant in the sense that we request that necessity distributes over disjunctions. We introduce an implication connective induced by the canonical ordering on the algebra so as to show that basic modal axioms can be recovered. We show that modalities on this structure called MDS5 become trivial if it is a Boolean algebra, but they are non-trivial in the general case, whenever there are non-Boolean elements, especially on a Kleene algebra. Finally, we study the notion of sharp (or crisp) element on the MDS5 structure, and show two possible, non-equivalent definitions of this notion.

2.1. Modal operators on De Morgan algebras.

In order to lay bare the issues regarding the deviant modal logic appearing in (Cattaneo and Ciucci, 2004), let us recall the corresponding algebraic structure. It consists of a system $\mathbb{A}_{MDS5} = \langle \mathcal{A}, \wedge, \vee, \neg, \nu, \mu, 0, 1 \rangle$ where

(1) The subsystem[1] $\mathbb{A}_K = \langle \mathcal{A}, \wedge, \vee, \neg, 0, 1 \rangle$ is a de Morgan lattice, i.e.,
   (a) $\mathbb{A}_K$ is a (not necessarily distributive) lattice with respect to the meet operation $\wedge$ and the join operation $\vee$. The induced partial order relation $a \leq b$ iff $a = a \wedge b$ (equivalently, $b = a \vee b$) is such that this

[1]By *subsystem* of a given algebra, we mean an algebra based on the same domain but with a subset of operations

lattice turns out to be bounded by the least element 0 and the greatest element 1.

(b) $\neg : \mathcal{A} \to \mathcal{A}$ is a *de Morgan complementation*, i.e., it is a unary operation that satisfies the following conditions for arbitrary elements $a, b \in \mathcal{A}$:
(dM1) $\neg(\neg a) = a$
(dM2) $\neg(a \vee b) = \neg a \wedge \neg b$
Let us remark that, trivially, the greatest element 1, interpreted as *true* (the ever true proposition), is the negation of *false*: $1 = \neg 0$.

(2) $\nu : \mathcal{A} \to \mathcal{A}$ and $\mu : \mathcal{A} \to \mathcal{A}$ are unary operators on the de Morgan lattice (the argument of debate) that satisfy the following conditions for an algebraic version of the (modal like) *necessity* and *possibility* operators (Chellas, 1988):
(N1) $\nu(1) = 1$ (N principle)
(N2) $\nu(a) \leq a$ (T principle)
(N3) $a \leq b$ implies $\nu(a) \leq \nu(b)$ (K principle)
(N4) $\mu(a) = \neg(\nu(\neg a))$ (DF$\Diamond$ principle)
(N5) $\mu(a) = \nu(\mu(a))$ (5 principle).

(3) The operator ν satisfies the following distributivity law with respect to the join operation:
(MD$_\nu$) $\nu(a) \vee \nu(b) = \nu(a \vee b)$

The above is an algebraic semantic of a logical system that resembles S5, but that is deviant from the standard algebraic semantic for the S5 modal system owing to the following two points:

(i) The algebraic semantic of standard modal logics "interprets modal connectives as operators on Boolean algebras" (Goldblatt, 2003), differently from the present case in which modal connectives are interpreted as operators on de Morgan algebras.
(ii) Differently from the standard modal approach to S5, the distributivity law (MD$_\nu$) for the necessity operator with respect to the $\vee$ operation holds.

In some sense, our structure is weaker than the standard one owing to point (i), but it is stronger owing to point (ii). Since condition (MD$_\nu$) plays a key role, in the sequel we call MDS5 *algebra* a structure like the one just introduced, that can be summarized in the following points:

(M1) it is based on a de Morgan lattice, which is NOT necessarily Boolean;
(M2) all S5 principles hold;
(M3) the non standard (with respect to modal logic) (MD$_\nu$) condition holds.

Let us recall (see Cattaneo and Marino (1988)) that in a de Morgan lattice, under condition (dM1), the following properties are mutually equivalent and each of them is equivalent to the *de Morgan* condition (dM2):

(dM2a) $\neg(a \wedge b) = \neg a \vee \neg b$ (dual de Morgan law);
(dM2b) $a \leq b$ implies $\neg b \leq \neg a$ (contraposition law);
(dM2c) $\neg b \leq \neg a$ implies $a \leq b$ (dual contraposition law).

In general, neither the *non contradiction* law $\forall a \in \mathcal{A}$, $a \wedge \neg a = 0$ nor the *excluded middle* law $\forall a \in \mathcal{A}$, $a \vee \neg a = 1$, characterizing Boolean structures, hold (as to the general treatment of de Morgan lattice, see also Monteiro (1960a); Cignoli (1975)).

Let us make some further remarks on the given general lattice structure, where distributivity is not required. This is essentially due to two reasons.

(nD1) All the obtained results, if the distributivity is not explicitly required, hold in the general case of non–distributive lattices.

(nD2) From the very beginning of the algebraic-logical approach to quantum mechanics, expressed in the seminal Birkhoff–von Neumann paper Birkhoff and von Neumann (1936), it has been recognized that the underlying lattice structure is not Boolean, precisely it is an orthomodular lattice. The extension of this quantum logical approach, which is now considered as a crisp version, to the so–called *unsharp* (or, in other terms, fuzzy) quantum mechanics involves in a deep way modal–like operators, of course without any distributivity requirement about the lattice (see for instance Cattaneo and Marino (1984); Cattaneo (1992, 1993); Cattaneo et al. (1993); Cattaneo and Giuntini (1995), with a survey in Cattaneo et al. (2009)).

Let us stress that the following dual modal principles, with respect to a *possibility* operator (Chellas, 1988), can be proved in a straightforward way.

Proposition 2.1. *Let $\mathbb{A}$ be a MDS5 algebra. The operator $\mu : \mathcal{A} \mapsto \mathcal{A}$ satisfies the following properties:*

(P1) $\mu(0) = 0$ *(P principle)*
(P2) $a \leq \mu(a)$ *(T principle for possibility)*
(P3) $a \leq b$ *implies* $\mu(a) \leq \mu(b)$ *(K principle for possibility)*
(P4) $\nu(a) = \neg(\mu(\neg a))$ *(DF$\Box$ principle)*
(P5) $\nu(a) = \mu(\nu(a))$ *(5 principle for possibility).*
(MD_μ) $\mu(a) \wedge \mu(b) = \mu(a \wedge b)$

Note that the modal K principle has been expressed as the non–equational isotonicity (increasing monotonicity) condition (N3). Under conditions (N2), (N4) and (N5), an equational version of this principle can be equivalently formulated as

(K) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$ (multiplicative condition)

This condition summarizes in a unique identity the algebraic realization of two well known modal principles for necessity:

(M) $\nu(a \wedge b) \leq \nu(a) \wedge \nu(b)$ (M principle)
(C) $\nu(a) \wedge \nu(b) \leq \nu(a \wedge b)$ (C principle)

Similarly for possibility, the non–equational version (P3) of the K principle is equivalent to the equational condition:

(K_μ) $\mu(a \vee b) = \mu(a) \vee \mu(b)$ (additive condition)

that describes both the modal principles (M_μ) and (C_μ) for possibility.

Moreover, in any MDS5 algebra, also the following modal principles can be proved.

(D) The D principle $\nu(a) \leq \mu(a)$.
(4) The 4 principle: $\nu\nu(a) = \nu(a)$ and (the 4 principle for necessity) $\mu\mu(a) = \mu(a)$ (the 4_μ principle for possibility).
(B) The B principle: $a \leq \nu\mu(a)$.

Summarizing, the MDS5 structure corresponds to the algebraic semantic of a S5 modal–like system with the further condition (MD_ν), which has the equational form of the *additive condition* for the necessity. Borrowing a comment from Hajek (Hájek, 1998, p. 57), given in the BL_Δ algebraic context (see section 6), "the axioms evidently resemble modal logic with ν as necessity; but in the axiom on $\nu(a \vee b)$, ν behaves as possibility rather than necessity."

Let us recall that the isotonic non-equational version (N3) of the K principle has been formulated in the equational way as the above necessity *multiplicative condition* (K). The two conditions (MD_ν) and (K) are not independent, as shown by the following result.

Proposition 2.2. *In a MDS5 algebra, the (additive) condition (MD_ν) implies the (multiplicative) condition (K).*

Proof. Let $a \leq b$, i.e., $b = a \vee b$, then $\nu(b) = \nu(a \vee b)$ and by (MD_ν), $\nu(b) = \nu(a) \vee \nu(b)$, i.e., $\nu(a) \leq \nu(b)$. So, we have shown that the (MD_ν) condition implies the isotonicity condition (N3), which, as said before, is equivalent to the multiplicative condition (K). □

Vice–versa, the opposite in general does not hold, in the sense that in a system which satisfies all the conditions of the only points 1 and 2 of the definition of MDS5 systems, the condition (K) is equivalent to condition (N3) but it does not imply the distributivity condition (MD_ν). As an example, let us consider the Boolean lattice whose Hasse diagram is depicted in figure 1.

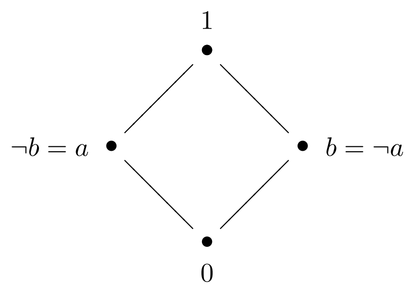


FIGURE 1. Boolean lattice with (K), but not (MD_ν) condition

In this lattice the necessity operator, in the sense that all conditions (N1)–(N5) are satisfied, is defined by the rules: $\nu(1) = 1$ and $\nu(a) = \nu(b) = \nu(0) = 0$. By duality we have the possibility operator defined as $\mu(0) = 0$ and $\mu(a) = \mu(b) = 1$. However, one has that $\nu(a \vee b) = 1$, with $\nu(a) \vee \nu(b) = 0$, i.e., the condition (MD$_\nu$) does not hold, whereas the condition (K) is trivially satisfied. This is a pure (i.e., Boolean based) algebraic model of the S5 modal logic.

Moreover, this result holds in any de Morgan lattice with more than the two elements, if the two modal operators, for x running in the lattice, are defined as:

$$\nu(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases} \quad \text{and} \quad \mu(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \tag{1}$$

Monteiro in Monteiro (1960a) called these operators as *chaotic* or *simple*, whereas the identity operators, as particular modal operators, have been named *discrete*.

2.2. **The Boolean case and genuine MDS5 structures.** All the results discussed in section 2.1, under the weaker condition of de Morgan lattice, should be compared with the axiomatization of system S5 in (Chellas, 1988, p. 14), where of course the underlying basic structure of the algebraic model is the Boolean one: "By a *modal algebra* we mean a structure $\langle \mathcal{B}, \nu \rangle$ in which $\mathcal{B}$ is a Boolean algebra and ν is a unary operation in $\mathcal{B}$, an algebraic counterpart of necessitation" (Chellas, 1988, p. 212).

The critical point of the above MDS5 structure (that has been the source of the discussion about it) arises from the following result.

Proposition 2.3. *If (MD$_\nu$) condition is added to the usual (i.e., based on a Boolean) algebraic model of the system S5,* $\mathbb{A} = \langle \mathcal{A}, \wedge, \vee, \neg, \nu, \mu, 0, 1 \rangle$, *then for all* $a \in \mathcal{A}$, $\nu(a) = a = \mu(a)$.

Proof. Let us suppose that $\mathbb{A}$ is a Boolean algebra. Then it holds $(x \wedge y) \vee (x \wedge \neg y) = x$. Setting $y = \mu(\neg x)$, we have $(x \wedge \mu(\neg x)) \vee (x \wedge \neg\mu(\neg x)) = x$ and applying (P4) and (N3) $(x \wedge \mu(\neg x)) \vee \nu(x) = x$. Now, we show that $(x \wedge \mu(\neg x)) = 0$. Indeed, $x \wedge \mu(\neg x) = (P3) x \wedge \mu(\neg x) \wedge \mu(x) = (\text{MD}_\nu) = x \wedge \mu(x \wedge \neg x) = x \wedge \mu(0) = 0$. Thus, $\nu(x) = x$ and by (P5), $x = \mu(x)$. □

This result means that in any Boolean case the underlying MDS5 algebraic system trivializes because possibility and necessity modalities coincide with the identity operator, and this modal logic reduces to classical logic. This leads, as usual in algebraic semantic of logic or more generally in algebra, to use the term "*genuine*" to denote those MDS5 structures which are not Boolean. A necessary and sufficient condition to ensure the non-triviality of the de Morgan lattice is the existence of an element a such that $a \wedge \neg a \neq 0$ (or equivalently, such that $a \vee \neg a \neq 1$), which prevents the structure from being Boolean.

Of course, all genuine (non–Boolean) structures can be equipped with possibility and necessity modalities both coinciding with the identity operator obtaining a (trivial) MDS5 algebra. Furthermore, the result of proposition 2.3 might induce one to conjecture that also the following extension of this result to all MDS5 algebras is true.

Conjecture.: *Let* $\mathbb{A}_K$ *be a genuine de Morgan lattice and let* $\nu : \mathcal{A} \to \mathcal{A}$ *and* $\mu : \mathcal{A} \to \mathcal{A}$ *be two mappings that satisfy the above properties (N1)–(N5) plus (MD$_\nu$), then* $\nu(a) = \mu(a)$ *for every element* $a \in \mathcal{A}$, *i.e., the modalities* *coincide (and since $\nu(a) \leq a \leq \mu(a)$, then $\nu(a) = \mu(a) = a$ for arbitrary $a \in \mathcal{A}$).*

However, contrary to the above conjecture, we give in section 3 two concrete non-Boolean examples of MDS5 algebra in which this trivialization is avoided.

A sufficient condition to guarantee the non-Booleaneness of a de Morgan algebra is the existence of *complementation-invariant* elements $h \in A$, different from 0 and 1, and such that $\neg h = h$. Indeed, one has that $h \wedge \neg h = h \neq 0$ and $h \vee \neg h = h \neq 1$, and so neither the *noncontradiction* principle nor the *excluded middle* law of the Boolean structure hold. We note that in a de Morgan lattice there might exist more than one complementation–invariant element. For an example, one can consider the non-distributive lattice of 6 elements depicted in figure 2.

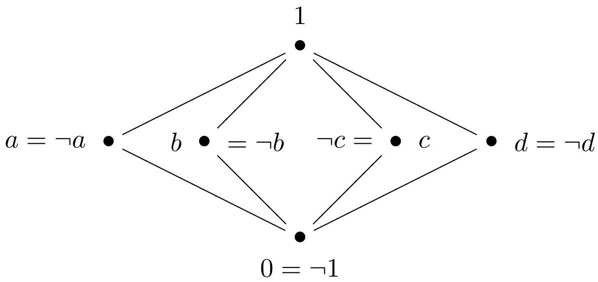


FIGURE 2. Non–distributive de Morgan lattice with two complementation–invariant elements a and d, with $\neg b = c$ and $\neg c = b$

2.2.1. *Kleene algebras.* An interesting strengthening of the de Morgan framework in the MDS5 algebras context is the one in which the negation $\neg$ is Kleene, i.e., besides (dM1) and (dM2) the following Kleene condition holds for arbitrary elements $a, b \in \mathcal{A}$:

(Kl) $\quad a \wedge \neg a \leq b \vee \neg b$

Consequently, the structure $\mathbb{A}_K$ will be called Kleene lattice. When considering a Kleene lattice, excluded middle and non-contradiction laws are substituted by the weaker Kleene condition (Kl), which is trivially satisfied in any Boolean lattice. Note that the lattice described in figure 2 is de Morgan and not Kleene (for instance $a \wedge \neg a$ is incomparable with $d \vee \neg d$). For a systematic treatment of de Morgan and Kleene (distributive) lattices see (Cignoli, 1965, sect. 2) and Cignoli (1975), with the inserted bibliography, or for the non–distributive case also Cattaneo and Manià (1974).

In a Kleene lattice if a *complementation–invariant* element exists, then it is unique. Indeed, if $h = \neg h$ and $k = \neg k$ then from (Kl) one has that $h = h \wedge \neg h \leq k \vee \neg k = k$ and $k = k \wedge \neg k \leq h \vee \neg h = h$, i.e., necessarily $h = k$. Hence, Kleene structures with this kind of privileged element cannot have an even set of elements. The simplest non–linear (i.e., not totally ordered) and non–distributive genuine Kleene lattice is the one in figure 3.

The unique complementation-invariant element in Kleene lattices, usually denoted by 1/2, is interpreted as *half-true*, i.e. halfway between true and false. It

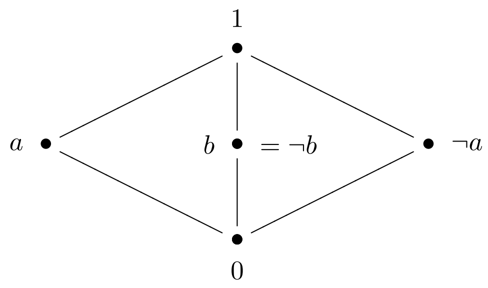


FIGURE 3. Non–distributive Kleene lattice with unique half-true element b

should not be epistemically interpreted as *possible* (as proposed by Lukasiewicz (Borowski, 1970, p. 86)) or *unknown* as in partial logic or Belnap logic, as these are belief or information states. These notions cannot be modelled by genuine elements of a truth set, but rather by subsets thereof (see Dubois Dubois (2008)).

Almost all the examples which are interesting in applications (see the fuzzy set and the ortho-pair cases discussed in section 3 of this paper) are Kleene lattices which possess this half-true element, with the further conditions $\nu(1/2) = 0$ and $\mu(1/2) = 1$. In particular let us quote the unsharp (i.e., fuzzy) approach to quantum mechanics in which this half element is always realized by the one-half identity operator on a (complex separable) Hilbert space, called the semi-transparent effect Cattaneo et al. (2009). In the recent approach to quantum computing this element is concretely realized by a beam–splitter filter widely used in laser quantum optics (see for instance (Nielsen and Chuang, 2000, p. 289)).

2.3. **Some related structures.** Let us note that, in the distributive case, any MDS5 algebra is a particular case of the algebraic semantic of the modal system S5 based on a de Morgan lattice, i.e., a modal algebraic structure formalized according to the only points 1 and 2, and without condition (MD_ν). This kind of S5 algebra has been investigated by Halmos in Halmos (1955, 1956) (collected in Halmos (1962)), but differently from here on the basis of Boolean lattices, disregarding the fact that all the main results do not make use of the noncontradiction and excluded middle laws. In particular, in Halmos (1955) one has (Q'1)=(N1), (Q'2)=(N2), (Q'3)=(K), this latter logically equivalent to (N3), (Q'5)=(N5) making use of (N4) in the equation presented at p. 22 of the book Halmos (1962). Let us only note that the original formulation (Q'2) contains a little misprint. Halmos condition (Q'4) is nothing else than condition (4) which, as remarked in subsection 2.1, can be proved from the other has a derived principle. As affirmed by Goldblatt in Goldblatt (2003) "structure of this kind were later dubbed *monadic algebras* by Halmos in his studies of the algebraic properties of quantifiers. The connection is natural: the modalities have the same formal properties in S5 as do the quantifiers in classical logic." About this argument, Halmos himself recognizes that "the concept of existential quantifier [i.e., possibility in modal terminology] occurs implicitly in a brief announcement of some related work of Tarski and Thompson Tarski and Thompson (1952)."

There are counter-examples of Halmos S5 algebras (i.e., monadic algebras) without condition (MD_ν), as shown in figure 4.

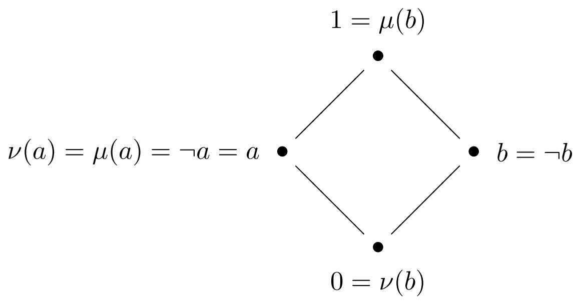


FIGURE 4. An example of S5 algebraic model which is not MDS5

Some years later it has been shown in (Cattaneo and Nisticò, 1989, sect. 3)[2] that the Halmos S5 semantics by monadic algebras is just the one induced by BZ posets, where the condition (MD_ν) is expressed in an equivalent way by point (uq). In sect. 5 of this 1989 paper, in particular in subsect. 5.1 and 5.2, it is described the BZ unsharp (also fuzzy) approach to quantum mechanics with the associated physical interpretation. The corresponding rough approximations by modal necessity–impossibility pairs is described in sect. 7, with the Hilbert space quantum fuzzy–intuitionistic poset treated in sect. 7.2. These results about BZ poset with induced Kleene MDS5 structures have been based on some previous (1984) researches on axiomatic unsharp quantum mechanics Cattaneo and Marino (1984) as starting points of successive researches (for the ones before 1993 see Cattaneo (1992, 1993); Cattaneo et al. (1993)) Let us note that this will be the argument of Section 4.2 in the present paper.

A structure like the S5 monadic algebra introduced by Halmos, or its BZ version, but on the basis of a de Morgan distributive lattice, emerged some year later in Banerjee and Chakraborty (1993) while developing a logic for rough sets, with a formal definition appeared in Banerjee and Chakraborty (1996) with the name of *topological quasi-Boolean algebra* (tqBa). Let us recall that a quasi-Boolean algebra, according to the terminology adopted by A. Bialynicki-Birula and H. Rasiowa in Bialynicki-Birula (1957); Bialynicki-Birula and Rasiowa (1957), is nothing else than a de Morgan algebra according to the term introduced by G. Moisil in Moisil (1935), as recognized by Rasiowa in (Rasiowa, 1974, p. 44, footnote[(4)]) (and for further details see also Monteiro (1960b); Cignoli (1965)).

A more general abstract structure with respect to Halmos S5 monadic Boolean algebraic system, is the one based on a Boolean algebra and satisfying the only conditions (N1), (N2), (4, for necessity, also denoted by (4_ν)), and (K), as algebraic model of the system S4 of Lewis and Langdorf modal logical, called by Monteiro in Monteiro (1960a) *Lewis Algebras.* Note that, from the pointless approach to topology (i.e., topology-without-points treated in an abstract lattice context, and not in the power set of some universe – see Smyth (1992)) these conditions (N1), (N2), (4_ν), and (K) define a Kuratowski interior operator and dually conditions (P1),

[2]in this work, algebras are built on the basis of a Kleene poset, which is not necessarily a lattice

(P2), (4_μ), and (K_μ) a Kuratowski closure operator (or a topological closure according to Birkhoff Birkhoff (1967)). These abstract structures on the Boolean basis have been widely studied by Rasiowa in (Rasiowa, 1974, sect. 5 of Chapt. VI) with the name of *topological Boolean algebras* (tBa), whose relationship with S4 modal logic is treated in sect. 4 of Chapt. XIII (and see also Chapt. III and Chapt. XI of Rasiowa and Sikorski (1970)). In Rasiowa (1974), one can find the remark that "The first results serving to establish a connection between modal logic and topology are due to Tang Tsao–Chen Tsao-Chen (1938). This relationship was pointed out and developed by McKinsey in McKinsey (1941), and jointly with Tarski in McKinsey (1941); McKinsey and Tarski (1944, 1946)". In particular, in McKinsey and Tarski (1944) McKinsey and Tarski proved that any abstract tBa is isomorphic to a subalgebra of a concrete topological space based on the powerset of a universe U equipped with the Kuratowski closure operator induced by a suitable topology for U.

As a terminological remark, if one considers the collection of all elements of S4 tBa (based on a complete lattice) which are fixed points (i.e., such that $\nu(a) = a$) with respect to the necessity (i.e., Kuratowski topological interior) operator ν, then this family is the algebraic version of a real pointless topology of *open elements* since it contains either the least element and the greatest element of the lattice, it is stable with respect to arbitrary join and finite meet. Dually, the collection of all elements which are fixed points with respect to the possibility (i.e., Kuratowski topological closure) operator μ is the family of the *closed elements*. So, the term "topological Boolean algebra" is correct. The term of topological quasi-Boolean algebra, from this point of view, and in our opinion, should be better employed to denote a S4 algebra based on a de Morgan (i.e., quasi–Boolean) distributive lattice. Condition (N5) characterizing S5 algebraic models is equivalent to the property that the families of open and the one of closed elements coincide, i.e., in this case we deal with a family of *clopen* elements. This according to the Goldblatt statement that "Another significant result of the 1948 paper McKinsey and Tarski (1948) is that S5 is characterized by the class of all closure algebras in which each closed element is also open" [i.e., Halmos monadic algebras]. In honor of the Halmos contribution, it will be better to use the name of *monadic Boolean algebra* (mBa) for the Boolean S5 algebra and of *monadic quasi–Boolean algebra* (mqBa) to denote the de Morgan lattice case. In the context of rough set theory these kind of structures are treated for instance in Cattaneo and Ciucci (2008, 2009a).

A further generalization of S4 mqBa is presented in Cattaneo and Ciucci (2006b), on the basis of a previous work (Cattaneo, 1997a, Theoren 2.27). This algebraic structure consists of a de Morgan lattice equipped with a Tarski closure operator, as generalization of S4 possibility connective, characterized by the substitution of the additive condition (K_μ) with a sub–additive condition (see (Cattaneo, 1998, sect. 2.1)). Tarski closure de Morgan lattices are categorically isomorphic to abstract approximation spaces according to the definition presented in Cattaneo (1998), and they turn out to be a very pathological form of pre-BZ lattices.

From another point of view, a strengthening of MDS5 algebras (again, when considering the version based on distributive lattices) are characterized by the further requirement that $\forall x, \neg\nu(x) \vee \nu(x) = 1$. This condition has been introduced by Moisil in his 1940, 1941 papers Moisil (1940, 1941), with the name of *modal principle of excluded middle*, in order to give an algebraic semantic of Łukasiewicz three valued logic, further on investigated in the period 1963–65 by Monteiro in Monteiro (1963, 1965), Cignoli in Cignoli (1965) (and see also the Cignoli and Monteiro contribution Cignoli and Monteiro (1965)) and successively by Becchio in the 1973 paper Becchio (1973) (for a treatment in the BZ poset context see the 1989 paper Cattaneo and Nisticò (1989), in particular condition (ne-1) of sect. 3). These algebraic structures are known in literature with the name of *three valued Łukasiewicz algebras* which, according to Monteiro Monteiro (1967), "play in the study of the three valued Łukasiewicz propositional calculus a role analogous to that of Boolean algebras in the study of the classical propositional calculus." In Banerjee and Chakraborty (1996) this strengthening of distributive MDS5 algebras are called *pre-rough algebras*.

In Cattaneo et al. (1998) the structure of *Łukasiewicz algebra*, as generalization of Łukasiewicz three valued algebra based on possibility as primitive operator defined on a Kleene lattice, has been introduced in definition 4.2 by four conditions (L1)–(L4), showing in Appendix A that they are categorically equivalent to MDS5 algebras.

Let us note that the MDS5 algebra of Figure 5 is not a three valued Łukasiewicz algebra since $\neg\nu(a) \vee \nu(a) = c$.

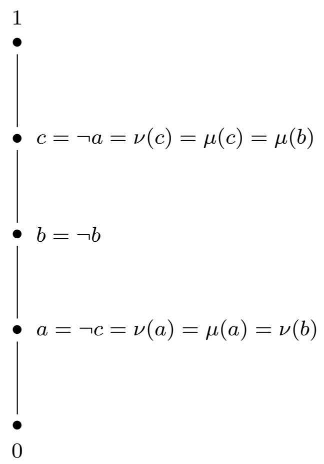


FIGURE 5

Finally, a survey about a hierarchy of topological interior and closure operators, with the corresponding modal interpretations as necessity and possibility connectives, and the relationship with information systems of the rough set theory, can be found in Cattaneo and Ciucci (2009b).

2.4. **Implication and ordering in MDS5 algebras.** From the viewpoint of algebraic semantic of logic, the lattice elements are interpreted as propositions, the meet operation as conjunction, the join operation as disjunction, and the complement as negation. The partial ordering induced from the lattice structure is interpreted as a binary relation of semantical implication on propositions. "The *implication relation*, which is fundamental to logic, must not [...] be confused with an implication or conditional *operation*, which is a logical *connective* that, like conjunction and disjunction, forms propositions out of propositions remaining at the same linguistic level." (Hardegree, 1979) [3]. Thus, the partial order relation $a \leq b$ in $\mathcal{A}$ is rather the algebraic counterpart of the statement "$A \rightsquigarrow B$ is true" with respect to some implication connective $\rightsquigarrow$ involving sentences A and B of the object language, in the sequel denoted by $\mathcal{L}$. In this sense the *implication relation* $a \leq b$ in the lattice $\mathcal{A}$ corresponds to *semantic entailment* among formulas from the language $\mathcal{L}$.

Thus, in order to express the above modal conditions (N1)–(N5) in a formal way nearer to the usual logical formalism, a binary lattice operation $\to$ on the lattice $\mathcal{A}$ is required. This operation should play the role of algebraic counterpart of the logical implication connective $\rightsquigarrow$ in the language $\mathcal{L}$, and should assign to each pair of lattice elements $a, b \in \mathcal{A}$ another lattice element $a \to b \in \mathcal{A}$. "Since not just any binary lattice operation should qualify as a material implication, [...] it seems plausible to require that every implication operation $\to$ be related to the implication relation ($\leq$) in such a way that if a proposition a implies a proposition b, then the conditional proposition $a \to b$ is universally true, and conversely. [...] Translating this into the general lattice context, we obtain

$$a \to b = 1 \quad \text{iff} \quad a \leq b. \tag{2}$$

Here 1 is the lattice unit element, which corresponds to the *universally true proposition*" (Hardegree, 1981). In this just quoted paper, this condition is assumed as one of the *minimal implicative conditions*.

Let us enter in some formal detail supposing that in our de Morgan lattice structure there exists an implication operator $\to$ on $\mathcal{A}$ for which only the minimal condition (2) is required to hold. Moreover, in the relevant examples treated in the present paper none of the involved implication connective has the standard form $a \to_S b := \neg a \vee b$, since in none of these examples does this connective satisfy the required condition (2).
Let us define as a *truth functional tautology* (also *universally valid sentence*) any well formed formula, based on a MDS5 algebraic structure, and equal to the universally true element 1 of the lattice. It is now easy to prove that the ν and μ operators satisfy the algebraic versions of axioms and rules of some celebrated modal principles, once ν is interpreted as a *necessity* operator and μ as a *possibility* operator (L and M in a standard modal language $\mathcal{L}$). Precisely:

(N) The above algebraic version of the N principle can assume also the equivalent form "$a = 1$ implies $\nu(a) = 1$" corresponding to the *necessitation inference rule* of (Hughes and Cresswell, 1984, p. 4) "$\vdash A$ implies $\vdash L(A)$", i.e., if A is true so is $L(A)$. In other words, according to (Chellas, 1988, p. 7), "the necessitation of a valid sentence is itself always valid" also expressed as the *rule of inference* (RN) $\frac{A}{L(A)}$.

(T) According to the above requirement (2) of minimal implicative condition one has that (N2) is the algebraic version of the T modal tautology $L(A) \rightsquigarrow A$.

(K) As a consequence of the *monotonicity* condition "$a \leq b$ implies $\nu(a) \leq \nu(b)$" and under the T principle (N2) one can prove that "If the conditional and

[3]One may question the point about a conditional remaining at the same linguistic level as other connectives. For instance a conditional appearing in conditional probability is not a Boolean connective, but semantically a three-valued entity, not at the same linguistic level as Boolean propositions. See for instance Walker (1994); Dubois and Prade (1994)

its antecedent are both necessarily true, then the consequent is necessarily true" (Chellas, 1988). Formally, that

$$\nu(a \to b) = 1 \text{ and } \nu(a) = 1 \quad \text{imply} \quad \nu(b) = 1. \tag{3}$$

On its hand, property (3) (for proof see Cattaneo and Ciucci (2009a)) is equivalent to

$$\nu(a \to b) \to (\nu(a) \to \nu(b)) = 1. \tag{4}$$

Thus, owing to the equivalence of (4) with (3), the above (N3) is the algebraic version of the characteristic K principle formalized in a modal language $\mathcal{L}$ as the tautology $L(A \rightsquigarrow B) \rightsquigarrow (L(A) \rightsquigarrow L(B))$.

(5) Condition (N5) can be formalized as $\mu(a) \to \nu\mu(a) = 1$, which in a modal language $\mathcal{L}$ corresponds to the 5 tautology $M(A) \rightsquigarrow LM(A)$.

Let us stress the following comment. What we have proved here is that, in order to obtain the algebraic versions of characteristic T, K, 5, and DF$\Diamond$ axioms plus the RN rule of inference of the modal system S5, it is sufficient to consider a de Morgan lattice.

2.5. **Sharpness in MDS5 algebras.** Let us discuss the possibility of distinguishing in the MDS5 algebraic context sharp (or crisp) elements from the generic ones. The elements of a MDS5 algebra can be interpreted as representative of non-classical unsharp situations (that may sometimes be related to gradual predicates, vagueness or uncertainty), and so it is of a certain importance to algebraically select just those elements considered as crisp or sharp counterparts of this interpretation. We have two possible options:

(S1) *μ–sharp elements* are those elements $e \in \mathcal{A}$ such that $\mu(e) = e$ (in general for an element $a \in \mathcal{A}$ it is $a \leq \mu(a)$); note that $\mu(e) = e$ iff $\nu(e) = e$ (whereas in general we have that $\nu(a) \leq a$ for a running on $\mathcal{A}$), i.e., an element is μ–sharp iff it is ν–sharp. For this reasons we also say that in this case we have to do with *modal–sharp* (or M–sharp) elements.

(S2) *Kleene sharp* (or K–sharp) elements are those elements $f \in \mathcal{A}$ such that $f \wedge \neg f = 0$ (in general for an element $a \in \mathcal{A}$ it is $0 \leq a \wedge \neg a$). Also in this case $f \wedge \neg f = 0$ iff $f \vee \neg f = 1$ (whereas for a generic element $a \in \mathcal{A}$ one has that $a \vee \neg a \leq 1$).

Proposition 2.4. *If an element of a* distributive *MDS5 algebra is K–sharp then it is also M–sharp.*

Proof. Let $f \wedge \neg f = 0$. Then $f \vee \neg f = 1$ implies, by (N1), (MD$_\nu$) and (N4), that $1 = \nu(1) = \nu(f) \vee \neg\mu(f)$, but applying to (P2), i.e., to $f \leq \mu(f)$, the contraposition law (dM2a) we have $\neg\mu(f) \leq \neg f$, and so for the just proved result we have a fortiori that $\nu(f) \vee \neg f = 1$. Now, from the hypothesis it follows that $\nu(f) = \nu(f) \vee [f \wedge \neg f] = [\nu(f) \vee f] \wedge [\nu(f) \vee \neg f] = f$ by (N2) and the just obtained result. □

The converse property does not hold as can be seen in the MDS5–algebra of figure 5. Indeed, $\nu(a) = a$, i.e., it is M–sharp, but $a \wedge \neg a = a \wedge c = a \neq 0$.

The set of all M–sharp (resp., K–sharp) elements will be denoted by $\mathcal{A}_{eM}$ (resp., $\mathcal{A}_{eK}$) in the sequel. So, summarizing

$$\mathcal{A}_{eM} = \{e \in \mathcal{A} : \mu(e) = e\} = \{e \in \mathcal{A} : \nu(e) = e\}$$
$$\mathcal{A}_{eK} = \{f \in \mathcal{A} : f \wedge \neg f = 0\} = \{f \in \mathcal{A} : f \vee \neg f = 1\}$$
$$\mathcal{A}_{eK} \subseteq \mathcal{A}_{eM}$$

Trivially $\mathcal{A}_{eK}$ (and so also $\mathcal{A}_{eM}$) is nonempty since $0, 1 \in \mathcal{A}_{eK}$. Moreover, $\mathcal{A}_{eK}$ constitutes a real situation of crispness owing to the fact that the in the distributive case the natural lattice structure describing a sharp context is the Boolean one.

Proposition 2.5. *The K–sharp system* $\mathbb{A}_{eK} = \langle \mathcal{A}_{eK}, \wedge, \vee, \neg, 0, 1\rangle$ *of a* distributive *MDS5 algebra* $\mathbb{A}$ *is a Boolean algebra.*

Proof. First of all, $\mathbb{A}_{eK}$ is closed under complementation: if $f \in \mathbb{A}_{eK}$ then trivially also $\neg f \in \mathbb{A}_{eK}$. Moreover, any element of $\mathbb{A}_{eK}$ is a Boolean one, i.e., for all $f \in \mathcal{A}_{eK}$ there exist $g = \neg f \in \mathcal{A}_{eK}$ such that $f \wedge g = 0$. Hence, the only other thing to prove is that the operators $\wedge, \vee$ are closed in $\mathbb{A}_{eK}$. Trivially, by distributive and de Morgan properties, we have that $(e \wedge f) \wedge \neg(e \wedge f) = (e \wedge f) \wedge (\neg e \vee \neg f) = (e \wedge f \wedge \neg e) \vee (e \wedge f \wedge \neg f) = 0 \vee 0 = 0$. That is $e \wedge f \in \mathbb{A}_{eK}$. Dually, it can be proved that $e \vee f \in \mathbb{A}_{eK}$. □

This result cannot be extended to the case of $\mathbb{A}_{eM}$ since, taking into account the three elements MDS5 algebra $\mathcal{A}_3 = \{0, a, 1\}$, with $0 < a < 1$ under the Kleene negation $\neg a = a$ and equipped with the modalities $\nu(x) = \mu(x) = x$ for every $x \in \mathcal{A}_3$, the set of K sharp elements is the Boolean algebra consisting of the two elements $\{0, 1\}$ whereas the set of M sharp elements is the whole $\mathcal{A}_3$, which is not Boolean.

3. Two models of MDS5 algebras

We give here two models of non-Boolean MDS5 algebras, where necessity modalities that satisfy (MD$_\nu$) are non-trivial: the first one based on the standard collection of fuzzy sets and the second one based on the collection of pairs of orthogonal subsets of a given universe.

After this general treatment of MDS5 in the context of de Morgan lattice basis, in the sequel we consider MDS5 based on Kleene lattices. This is just for coherence with the following canonical examples of fuzzy sets and orthopairs theories that have this basic structure.

3.1. **The Fuzzy Set model of MDS5 algebraic system.** Let us consider a nonempty set of objects U, called the *universe*, in which a *fuzzy set* or *generalized characteristic function* is defined as usual as a $[0, 1]$–valued function on U, $f : U \mapsto [0, 1]$. We shall indicate the collection of all fuzzy sets on U by $\mathcal{F}(U) = [0, 1]^U$. A particular subset of $\mathcal{F}(U)$ is the collection $\{0, 1\}^U$ of all Boolean valued functions $\chi : U \mapsto \{0, 1\}$, whose collection will be denoted also by $\mathcal{F}_e(U)$; this subset coincide with the collection of all the *characteristic functions* of subsets of U. Indeed, given a subset A of U the corresponding characteristic function $\chi_A : U \mapsto \{0, 1\}$ is the mapping that assigns the value 1 (resp., 0) to any point $x \in A$ (resp., $x \notin A$). Moreover, for a generic function $\chi \in \mathcal{F}_e(U)$ let us denote by $A_1(\chi) : \{x \in U : \chi(x) = 1\}$ the corresponding crisp set, then the characteristic function of the latter is such that $\chi_{A_1(\chi)} = \chi$.

The set $[0,1]^U$ contains two special elements: the identically zero fuzzy set $\forall x \in U$, $\mathbf{0}(x) := 0$ (the characteristic function of the empty set: $\mathbf{0} = \chi_\emptyset$) and the identically one fuzzy set $\forall x \in U$, $\mathbf{1}(x) := 1$ (the characteristic function of the whole universe: $\mathbf{1} = \chi_U$). Moreover, there exists the fuzzy set $\mathbf{1/2}$ defined by the law: $\forall x \in U$, $\mathbf{1/2}(x) := (1/2)$, which guarantees that $\mathcal{F}(U)$ is not coincident with $\mathcal{F}_e(U)$. Extending this notation, for any $k \in [0,1]$ we will indicate as $\mathbf{k} \in [0,1]^U$ the constant fuzzy set $\forall x \in U$, $\mathbf{k}(x) = k$.

Then the proof of the following is really straightforward.

Proposition 3.1. *Let us consider the structure $\langle \mathcal{F}(U), \wedge, \vee, \neg, \nu, \mu, \mathbf{0}, \mathbf{1}, \mathbf{1/2} \rangle$ of all fuzzy sets equipped with the operators, defined pointwise ($\forall x \in U$) as follows:*

$$(f_1 \wedge f_2)(x) := \min\{f_1(x), f_2(x)\}$$
$$(f_1 \vee f_2)(x) := \max\{f_1(x), f_2(x)\}$$
$$\neg f(x) := 1 - f(x)$$
$$\nu(f)(x) := \begin{cases} 1 & \textit{if} \quad f(x) = 1 \\ 0 & \textit{otherwise} \end{cases}$$
$$\mu(f)(x) := \begin{cases} 1 & \textit{if} \quad f(x) \neq 0 \\ 0 & \textit{otherwise} \end{cases}$$

The partial order relation induced on $\mathcal{F}(U)$ by the above lattice operators is the usual pointwise ordering on fuzzy sets:

$$\forall f_1, f_2 \in \mathcal{F}(U), \quad f_1 \leq f_2 \;\; \textit{iff} \;\; \forall x \in U : f_1(x) \leq f_2(x)$$

Then all properties (1), (2), and (3) of Section 2 characterizing a MDS5 algebra are true. This algebra is genuine (non-Boolean) of distributive Kleene type since it has the half-true *element $\neg(\mathbf{1/2}) = \mathbf{1/2}$.*

The corresponding set of K–sharp and M–sharp elements coincide with the collection of all characteristic functions: $\big(\mathcal{F}(U)\big)_{eM} = \big(\mathcal{F}(U)\big)_{eK} = \mathcal{F}_e(U)$, which is a Boolean algebra isomorphic to the power set $\mathcal{P}(U)$ of the universe U by the one-to-one correspondence $A \longleftrightarrow \chi_A$ between a subset A of U and its characteristic function χ_A that allows one to identify any subset $A \subseteq U$ with its characteristic function $\chi_A \in \mathcal{F}_e(U)$, written $A \equiv \chi_A$.

Then, under this formal result it is not true that in standard fuzzy set theory, as consequence of axiom (MD_ν), for any genuine fuzzy set f the following identity should hold: $\nu(f) = \mu(f)$. For instance, if this claim should be true, then in any nonempty universe U it necessarily must be $\emptyset \equiv \nu(\mathbf{1/2}) = \mu(\mathbf{1/2}) \equiv U$, i.e., any subset of U is simultaneously empty and nonempty, a devastating result. In fact, $\nu(f)$ is the core of the fuzzy set f and $\mu(f)$ is its support. These sets always differ when the fuzzy set is non–crisp.

Let us recall that in this fuzzy set example there exist several implication connectives that share the above discussed Hardegree minimal condition (2). Two important ones are the following:

$$(f_1 \to_L f_2)(x) : = \min\{1, 1 - f_1(x) + f_2(x)\} \tag{5a}$$

$$(f_1 \to_G f_2)(x) : = \begin{cases} 1 & \text{if } \; f_1(x) \leq f_2(x) \\ f_2(x) & \text{otherwise} \end{cases} \tag{5b}$$

Trivially, none of them has the form $(\neg f \vee g)(x) = \max\{1 - f(x),\, g(x)\}$. In particular, for this connective $\neg f \vee g$ the two fuzzy sets $(\mathbf{1/3})(x) = 1/3$ and $(\mathbf{1/2})(x) = 1/2$ are such that $(\mathbf{1/3}) \leq \mathbf{1/2}$, but $\neg(\mathbf{1/3}) \vee \mathbf{1/2} \neq \mathbf{1}$, i.e., it does not satisfy the minimal implication condition under discussion.

3.2. **The orthopair model of MDS5 algebraic system.** Let us now give another model of the MDS5 algebraic system always based on a nonempty *universe* U and consisting of the collection $L_3(U)$ of all pairs (A_1, A_0) of disjoint $(A_1 \cap A_0 = \emptyset)$ subsets A_1 and A_0 of U, also called *orthopairs* of subsets (for the relationship, and the corresponding terminological debate, between these orthopairs and the so–called intuitionistic fuzzy sets of Atanassov, see Cattaneo and Ciucci (2003a,b, 2006a)). To the best of our knowledge, the notion of pair of subsets has been introduced for the first time by M. Yves Gentilhomme in Gentilhomme (1968) (see also Moisil (1972c)) in an equivalent way with respect to the one described here. Indeed, in these papers Gentilhomme considers pairs of ordinary subsets of the universe U of the kind (A_1, A_p), under the condition $A_1 \subseteq A_p$. Of course, the mapping $(A_1, A_0) \to (A_1, (A_0)^c)$ institutes a one-to-one and onto correspondence that allows one to identify the two approaches.

In particular we denote by $\mathbf{0} := (\emptyset, U)$, $\mathbf{1} := (U, \emptyset)$. Also in this case the following is straightforward.

Proposition 3.2. *Let us consider the structure $\langle L_3(U), \sqcap, \sqcup, \neg, \nu, \mu, \mathbf{0}, \mathbf{1}\rangle$ of all orthopairs from U equipped with the following operations:*

$$(A_1, A_0) \sqcap (B_1, B_0) := (A_1 \cap B_1, A_0 \cup B_0)$$
$$(A_1, A_0) \sqcup (B_1, B_0) := (A_1 \cup B_1, A_0 \cap B_0)$$
$$\neg(A_1, A_0) := (A_0, A_1)$$
$$\nu(A_1, A_0) := (A_1, U \setminus A_1)$$
$$\mu(A_1, A_0) := (U \setminus A_0, A_0)$$

Then all properties (1), (2), and (3) of Section 2 characterizing a MDS5 algebra are satisfied, with the further property of being a genuine distributive Kleene lattice, where the orthopair $(\emptyset, \emptyset) = \neg(\emptyset, \emptyset)$ stands for the unique half-true *element* $\mathbf{1/2}$.

The collection of K–sharp elements and M–sharp elements coincide with the collection of all orthopairs of the form (A, A^c), with $A \in \mathcal{P}(X)$ the generic subset of the universe X. These "sharp" orthopairs constitute a Boolean algebra isomorphic to the Boolean algebra $\mathcal{P}(X)$ by the one-to-one and onto correspondence $(A, A^c) \longleftrightarrow A$.

The partial order relation induced on $L_3(U)$ by the now considered lattice operations is:

$$(A_1, A_0) \sqsubseteq (B_1, B_0) \quad \text{iff} \quad A_1 \subseteq B_1 \text{ and } B_0 \subseteq A_0\,. \tag{6}$$

In the present case, the condition $\nu(A_1, A_0) = \mu(A_1, A_0)$ holds iff the orthopair is of the very particular form $A_0 = (A_1)^c$ in which there is no uncertainty in the *boundary region*: $A_b := U \setminus (A_1 \cup A_0) = \emptyset$. Moreover, if one argues that as a consequence of the (MD$_\nu$) condition this must be a universal property of any orthopair of the structure, then for the particular case of the *half-true* element one has $(\emptyset, U) = \nu(\mathbf{1/2}) = \mu(\mathbf{1/2}) = (U, \emptyset)$, which should lead inexorably to the conclusion that also in this case any universe U (and so any of its subsets) is simultaneously empty and nonempty.

As to implication connectives that satisfy the minimal Hardegree condition for the material implication connective (2), we can quote at least the following two cases (whose definition is functional):

$$(A_1, A_0) \Rightarrow_L (B_1, B_0) := ((A_1^c \cap B_0^c) \cup A_0 \cup B_1,\ A_1 \cap B_0) \tag{7a}$$

$$(A_1, A_0) \Rightarrow_G (B_1, B_0) := ((A_1^c \cap B_0^c) \cup A_0 \cup B_1,\ A_0^c \cap B_0) \tag{7b}$$

As a particular remark, the implications given by equations (7) are a generalization of Pagliani's work Pagliani (1998).

Proposition 3.3. *The operators* $\Rightarrow_L$ *and* $\Rightarrow_G$ *as defined in* (7) *are truth-functional since, once set* $\alpha = (A_1, A_0)$ *and* $\beta = (B_1, B_0)$*, they can be equivalently expressed as follows:*

$$\alpha \Rightarrow_L \beta = (\neg\nu(\alpha) \sqcap \mu(\beta)) \sqcup \neg\alpha \sqcup \beta$$
$$\alpha \Rightarrow_G \beta = (\neg\nu(\alpha) \sqcap \mu(\beta)) \sqcup \neg\mu(\alpha) \sqcup \beta$$

Proof. We prove only the first equivalence, the second proof being similar.

$$\begin{aligned}(\neg\nu(A_1, A_0) \sqcap \mu(B_1, B_0)) &\sqcup \neg(A_1, A_0) \sqcup (B_1, B_0)\\ &= ((A_1^c, A_1) \sqcap (B_0^c, B_0)) \sqcup (A_0, A_1) \sqcup (B_1, B_0)\\ &= ((A_1^c \cap B_0^c) \cup A_0 \cup B_1, A_1 \cap B_0)\end{aligned}$$

□

Let us stress that the implication $\neg(A_1, A_0) \sqcup (B_1, B_0) = (B_1 \cup A_0, A_1 \cap B_0)$ does not satisfy the minimal condition (2) since for any $A_1 \neq \emptyset, U$ the following holds $(\emptyset, \emptyset) \sqsubseteq (A_1, \emptyset)$, but $\neg(\emptyset, \emptyset) \sqcup (A_1, \emptyset) = (A_1, \emptyset) \neq (U, \emptyset) = \mathbf{1}$.

Note that under the (bijective) identification of any orthopair (A_1, A_0) with the three valued fuzzy set

$$f_{A_1,A_0} := \chi_{A_1} + \tfrac{1}{2}\chi_{U\setminus(A_1\cup A_0)} \tag{8}$$

the restriction of the implications (5) to fuzzy sets of this kind leads to the above formulae (7). As an immediate consequence of this fact we have that the operators $\Rightarrow_L$ and $\Rightarrow_G$ as defined in (7) satisfy the minimal Hardegree condition for the material implication connective (2) since this property holds in the more general fuzzy set context.

Moreover, in Banerjee and Chakraborty (1996) another implication on orthopairs is introduced as follows:

$$(A_1, A_0) \Rightarrow_{GR} (B_1, B_0) := ((A_1^c \cap B_0^c) \cup A_0 \cup B_1,\ (A_0^c \cap B_0) \cup (A_1 \cap B_1^c))$$

This implication is different from both $\Rightarrow_L$ and $\Rightarrow_G$ and putting $\alpha = (A_1, A_0)$ and $\beta = (B_1, B_0)$ as generic orthopairs, the following chain of inclusions, according to the above partial order relation (6), holds:

$$(\alpha \Rightarrow_{GR} \beta) \sqsubseteq (\alpha \Rightarrow_G \beta) \sqsubseteq (\alpha \Rightarrow_L \beta)\ .$$

Also $\Rightarrow_{GR}$ satisfies the minimal condition (2) and, as we will discuss in a forthcoming work, it corresponds to the Gaines-Rescher implication on three values.

Let us remark that the collection of all fuzzy sets $\mathcal{F}(U)$ equipped with the implication connectives of Łukasiewicz $\to_L$ and Gödel $\to_G$ type from one side, and from the other side the collection of all orthopairs $L_3(U)$ with the implication connectives of Łukasiewicz $\Rightarrow_L$ and Gödel $\Rightarrow_G$ type are both examples of the more abstract lattice structure of Heyting Wajsberg (HW) algebra, widely studied in Cattaneo and Ciucci (2002); Cattaneo et al. (2004a,b) and discussed in the forthcoming subsection 4.4.

3.2.1. *Some further remarks and generalizations.* As said before, the now described orthopair structure $L_3(U)$ has been introduced in an equivalent way by Gentilhomme, with the de Morgan distributive lattice structure widely described by Moisil in Moisil (1972c) and the modal operators introduced in (Moisil, 1972b, sect. 2). In this latter paper one can find the comment that "every three valued Łukasiewicz algebra is isomorphic to some algebra of orthopairs," and in a previous paper about three valued Łukasiewicz algebras the condition (MD$_\nu$) is explicitly required to hold. For a successive treatment of this argument see also Cattaneo and Nisticò (1986), where the following important results are stressed:

(EM1) the mapping $ext : \mathcal{F}(U) \to L_3(U)$ assigning to any fuzzy set $f \in \mathcal{F}(U)$ its *semantical extension* $ext(f) := (A_1(f), A_0(f)) \in L_3(U)$, where $A_1(f) = \{x \in U : f(x) = 1\}$ and $A_0(f) = \{x \in U : f(x) = 0\}$, is an *epimorphism* of structures. The two subsets $A_1(f)$ and $A_0(f)$ are called the *core* (or *necessity*) and the *impossibility* of f, respectively. The subset $A_p(f) := U \setminus A_0(f) = \{x \in U : f(x) \neq 0\}$ is the *support* (or *possibility*) of f;

(EM2) the surjective property of this mapping φ is consequence of the fact that for any orthopair $(A_1, A_0) \in L_3(U)$, the three-valued fuzzy set f_{A_1,A_0} previously defined by equation (8) is such that $ext(f_{A_1,A_0}) = (A_1, A_0)$;

(EM3) the restriction of the extensional mapping to the collection of all these three-valued fuzzy sets $\mathcal{F}_t(U) := \{f_{A_1,A_0} \in \mathcal{F}(U) : (A_1, A_0) \in L_3(U)\}$ is an isomorphism, which allows one to identify $L_3(U)$ with the subalgebra $\mathcal{F}_t(U)$ of the algebra $\mathcal{F}(U)$ of all fuzzy sets;

(EM4) the restriction of the extensional mapping to the collection $\mathcal{F}_e(U)$ of all sharp fuzzy sets, assigning to any characteristic functional χ_A the orthopair $ext(\chi_A) = (A, A^c)$, is an isomorphism of Boolean lattices, which allows one to identify $\mathcal{F}_e(U)$ and $\mathcal{P}(U)$.

A generalization of the Gentilhomme-Moisil approach to orthopairs can be found in Cattaneo (1997a), in the context of generalized rough set theory. This generalized approach is based on a so–called *preclusive*, i.e., irreflexive and symmetric, binary relation $\#$ on a universe U. Note that the logical negation of such a kind of relation is a *similarity* (or tolerance), i.e., reflexive and symmetric, but in general not transitive, relation. On the basis of a preclusive relation it is possible to extend to the power set $\mathcal{P}(U)$ the notion of preclusive pair (or $\#$–pair), written $A\#B$, by the law: $\forall a \in A$, $\forall b \in B$, $a\#b$. Then one can consider the collection $L_3(U, \#)$ of all such preclusive pairs with the involved algebraic structure, very close to the one of proposition 3.2, but with some pathological behavior which is inessential to discuss here (see Cattaneo (1998) for details). The binary relation on U of being different $\neq$ is just a particular case of preclusive relation and the structure $L_3(U, \neq)$ is the one discussed in proposition 3.2, since in this case trivially $A \neq B$ iff $A \cap B = \emptyset$.

Note that also this generalization is based on the power set of a concrete universe. Another abstraction can be developed on the basis of a de Morgan lattice $\mathbb{A}_K = \langle \mathcal{A}, \wedge, \vee, \neg, 0, 1\rangle$, considered as a primitive structure, without any other further requirement. According to (Cattaneo and Nisticò, 1989, sect. 6), the binary relation on the lattice $\mathcal{A}$ defined as $a \perp b$ iff $a \leq \neg b$ (or equivalently iff $b \leq \neg a$) is an orthogonality relation on a lattice, according to Cattaneo and Manià (1974) (and see also (Cattaneo and Nisticò, 1989, conditions (og-1)–(og-5))). In this abstract de Morgan lattice context, an *orthopair* is any pair $(a_1, a_0) \in \mathcal{A}^2$ such that $a_1 \perp a_0$, whose collection is denoted by $L_3(\mathcal{A})$. The unary operation on orthopairs $\neg(a_1, a_0) := (a_0, a_1)$ is a Kleene complementation (also if the starting structure is de Morgan), with the unique half–true element $1/2 = (0, 0)$. Moreover, the further unary operations $\nu(a_1, a_0) := (a_1, \neg a_1)$ and $\mu(a_1, a_0) := (\neg a_0, a_0)$ are the necessity and possibility operators of a full MDS5 structure, respectively.

4. Brouwer-Zadeh lattices and MDS5 modal structures

In the first part of the present paper we have considered a basic structure called MDS5, which is a de Morgan (or in interesting applications Kleene) lattice equipped with modal unary operations of necessity and possibility satisfying an unusual condition named (MD_ν). As discussed before, in MDS5 algebra, the existence of a half element $\frac{1}{2}$ invariant by negation, is a sufficient condition to have a non-Boolean structure. In this second part we investigate another structure on the basis of a Kleene lattice, but equipped with a unary operation of complementation as algebraic model of a Brouwer (or intuitionistic) negation.

4.1. **BZ lattices.** This structure as been treated in many papers by the first two authors, with the name of Brouwer Zadeh (BZ) lattice Cattaneo and Nisticò (1989); Cattaneo et al. (1998); Cattaneo and Ciucci (2004, 2006a), and formalized in the following way.

Definition 4.1. A system $\mathbb{A}_{BZ} = \langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$ is a *Brouwer Zadeh (BZ) lattice* iff the following properties hold:

(1) The subsystem $\mathbb{A}_K = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ is a Kleene lattice with respect to the join and the meet operations $\vee$ and $\wedge$, bounded by the least element 0 and the greatest element 1: $\forall a \in A,\ 0 \leq a \leq 1$. The unary operation $\neg : A \mapsto A$ is a *Kleene* (or *fuzzy*) complementation.
(2) The unary operation $\sim : A \mapsto A$ is a *Brouwer* (or *intuitionistic*) complementation. In other words for arbitrary $a, b \in A$:
   (B1) $a \wedge {\sim\sim} a = a$ (i.e., $a \leq {\sim\sim} a$)
   (B2) $\sim (a \vee b) = {\sim} a \wedge {\sim} b$
   (B3) $a \wedge {\sim} a = 0$.
(3) The two complementations are linked by the interconnection rule that must hold for arbitrary $a \in A$:
   (in) ${\sim\sim} a = \neg \sim a$

Given a BZ lattice, as we shall widely discuss in the forthcoming section 4.2, the modalities are defined as $\nu(a) := \sim \neg a$ and $\mu(a) := \neg \sim a$. Differently from the result of proposition 2.3, even if the BZ structure is based on a Boolean algebra, these operators in general do not collapse to the identity operator, as shown in the following example.

Example 4.2. Let us consider the Boolean BZ lattice of figure 6.

We can easily see that for every non trivial element $x \neq 0, 1$ of this lattice, $\nu(x) = \sim \neg x = 0 \neq x \neq 1 = \neg \sim x = \mu(x)$. Moreover, from $\sim a = \sim b = 0$ and $\sim 0 = 1$ it follows that ${\sim\sim}\, a = {\sim\sim}\, b = 1$. Notice that these two modal operators

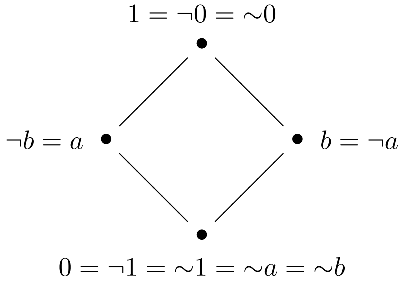


FIGURE 6. Boolean BZ lattice

are just the trivial ones described in equation (1) and the corresponding lattice is the one depicted by the Hasse diagram of figure 1, as pure algebraic model of the S5 modal logic. ■

This example also shows that we can consider the special case of BZ structures based on Boolean lattices, since Boolean lattices are in particular distributive Kleene lattices. This is different from the MDS5 algebra where in the Boolean case the modalities will necessarily collapse to identity.

Another remark about BZ structures is that they can be applied to quantum logics Cattaneo and Nisticò (1989); Cattaneo (1993); Cattaneo et al. (1993); Cattaneo and Giuntini (1995); Cattaneo (1997b), which basically are non distributive (precisely orthomodular) lattices, as discussed in point (nD2) of section 2.1.

Example 4.3. The BZ lattice depicted in the figure 7 is not Boolean with respect to the negation $\neg$.

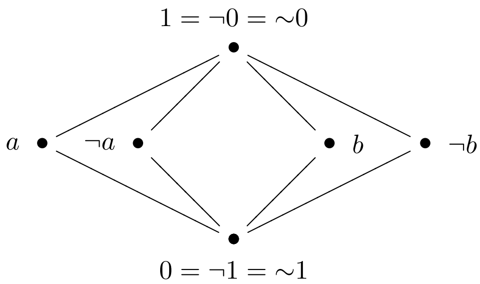


FIGURE 7. Non–distributive BZ lattice in which $\sim a = \sim \neg a = \sim b = \sim \neg b = 0$

This non-distributive, indeed modular (see (Birkhoff, 1967, p. 13)), lattice can be "covered" by the two "local" Boolean BZ structures $\mathcal{B}_a = \{0, a, \neg a, 1\}$ and $\mathcal{B}_b = \{0, b, \neg b, 1\}$, each of which coincides with the lattice shown in figure 6.
In quantum logic this is the very important non-Boolean lattice describing a spin 1/2 particle. The Boolean sublattice $\mathcal{B}_a$ is the range of the observable measuring the spin of the particle along the z-axis, with the element a (resp., $\neg a$) describing the event "the spin of the particle along the z -axis is up (resp., down)." Similarly, the Boolean sublattice $\mathcal{B}_b$, with its two elements b and $\neg b$, is the range of the

observable measuring the spin up and down along the x-axis, respectively.
The fact that there is no global Boolean lattice that covers both $\mathcal{B}_a$ and $\mathcal{B}_b$ means that there is no observable which can simultaneously measure them, and this has to do with the Heisenberg uncertainty principle (see (Varadarajan, 1968, p. 118, 123)). ■

As seen in the previous examples, there exist Boolean BZ structures without the collapse of both modalities to the identity. Anyway, one can consider the case of BZ structures that correspond to a subsystem of genuine Kleene lattice, in particular the sufficient case in which a (unique) element $1/2 \in A$ exists such that $\neg(1/2) = 1/2$.

In any BZ lattice with *half-true* element, condition $\sim(1/2) = 0$ is equivalent to (B3), as we are going to prove. First, let us show a preliminary result.

Lemma 4.4. *Let us assume that $\mathbb{A}_{BZ}$ is a BZ lattice. Then for all $a \in A$*

$$\sim a \leq \neg a.$$

Proof. Applying de Morgan condition (dM2b) to (B1) we get $\neg \sim\sim a \leq \neg a$. Using the interconnection rule (in) and (dM1) we have the thesis. □

Proposition 4.5. *Let $\mathbb{A}_{BZ}$ be a BZ lattice with half-true element. Then $\sim(1/2) = 0$. Conversely, let us assume that $\mathbb{A}_{BZ}$ is a BZ lattice with half-true element where axiom (B3) is replaced by condition $\sim(1/2) = 0$. Then for all $a \in A$*

$$a \wedge \sim a = 0.$$

Proof. For any element a condition $\neg a \wedge \sim a = \sim a$ holds. Applying this result to $1/2$ we get $\neg 1/2 \wedge \sim 1/2 = \sim 1/2$. The thesis easily follows from property $1/2 = \neg 1/2$ and (B3).
Let us prove the converse. In any lattice the property $(a \wedge b) \vee a = a$ holds. Thus, $\sim [(a \wedge b) \vee a] = \sim a$. Now, setting $a := z \vee \neg z$ and $y := 1/2$, we have $\sim [((z \vee \neg z) \wedge 1/2) \vee (z \vee \neg z] = \sim (z \vee \neg z)$. But, by (Kl) $1/2 = 1/2 \wedge \neg 1/2 \leq a \vee \neg a$. That is, $1/2 \wedge (a \vee \neg a) = 1/2$. Thus, by the last property, $\sim [1/2 \vee (z \vee \neg z)] = \sim (z \vee \neg z)$ and by (B2) $\sim 1/2 \wedge \sim (z \vee \neg z) = \sim z \wedge \sim\neg z$. By hypothesis ($\sim 1/2 = 0$), $\sim z \wedge \sim\neg z = 0$. By this applied to $z := \sim a$ and the interconnection rule, it follows $\sim a \wedge \neg \sim a = 0$. Finally, $a \wedge \sim a = a \wedge (\neg \sim a \wedge \sim a) = 0$. □

Let us remark that under the *weak double negation law* (B1), the de Morgan property (B2) of the Brouwer negation can be equivalently expressed as the following *contraposition law* (see Cattaneo and Marino (1988)):

(B2b) $a \leq b$ implies $\sim b \leq \sim a$

In general, in BZ lattices the following dual de Morgan law is not equivalent to (B2):

(B4) $\sim (a \wedge b) = \sim a \vee \sim b$

Example 4.6. Let us consider the (non–distributive) BZ lattice whose Hasse diagram is depicted in figure 8.

As to the (B4) condition we have that $\sim (a \wedge \neg a) = \sim 0 = 1$, but $\sim a \vee \sim\neg a = 0 \vee 0 = 0$, i.e., $\sim(a \wedge \neg a) \neq \sim a \vee \sim\neg a$. For the non-distributivity of this lattice, let us stress that $(\neg b \wedge b) \vee a = a$ with $(\neg b \vee a) \wedge (b \vee a) = \neg b$. ■

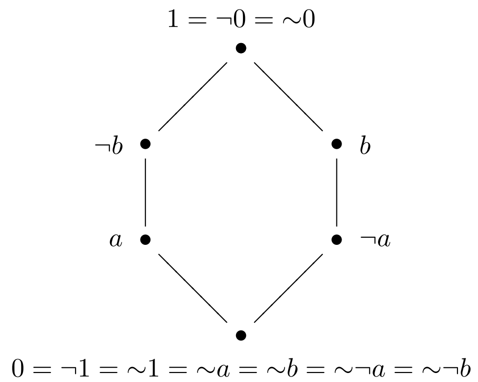


FIGURE 8. BZ lattice without property (B4)

Proposition 4.7. *In any BZ lattice, the following two conditions involving Brouwer negation are equivalent*

1) $\sim(a \wedge b) = \sim a \vee \sim b$ *(dual de Morgan law)*
2) $\sim a \vee \sim\sim a = 1$ *(Stone condition)*

Proof. From (B3) we have $\sim(a \wedge \sim a) = \sim 0$, and applying (B4) we get $\sim a \vee \sim\sim a = 1$. That from the Stone condition we obtain the de Morgan law (B4) is proved in (Monteiro and Monteiro, 1968) in a structure even weaker than BZ lattices. □

Definition 4.8. A structure $\langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$ is a *Stone BZ lattice*, denoted by $\mathrm{BZ}^{(S)}$, if it is a BZ lattice satisfying also the Stone condition.

Let us note that in other papers (for instance (Cattaneo et al., 1999)) this structure has been called *de Morgan* (with respect to the Brouwer negation) BZ lattices and denoted by $\mathrm{BZ}^{(dM)}$.

4.2. **Modal algebras induced from BZ structures.** Now, in the case of a BZ lattice $\mathbb{A}_{BZ} = \langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$, if two unary operations are defined for arbitrary $a \in A$ as

$$\nu(a) := \sim\neg(a) \quad \text{and} \quad \mu(a) := \neg\sim(a) \tag{9}$$

the following results can be proved (Cattaneo et al., 2004a):

(i) The induced structure $\mathbb{A}_{NP} = \langle A, \wedge, \vee, \neg, \nu, \mu, 0, 1\rangle$ is a Kleene lattice with necessity (N) and possibility (P) operators, ν and μ respectively, that satisfy conditions (N1)–(N5) of point 2 section 2.1. That is, the structure of BZ lattice induces an algebraic semantic of a S5 modal system based on a Kleene lattice since principles N, T, K (or the equivalent M and C), and 5 are all verified.
(ii) In general it does not induce an algebraic model of a MDS5 logical system (i.e., a system whose algebraic model besides points 1 and 2, satisfies *also* point 3 of section 2.1).

Example 4.9. The BZ lattice of figure 8 gives a concrete situation in which condition (MD_ν) does not hold. Indeed, one has that $\nu(a \vee b) = 1$, with $\nu(a) = \nu(b) = 0$

and so $\nu(a \vee b) \neq \nu(a) \vee \nu(b)$. In this example the two modalities ν and μ are described by the equation (1). ■

In this way BZ lattices $\mathbb{A}_{BZ}$, or better the induced structures $\mathbb{A}_{NP}$, are algebraic versions of a S5–like modal system that is *less deviant* from the standard one, at least with respect to the discussed condition (MD$_\nu$). Indeed, this system can be summarized in the following points (compare with the analogous points (M1)–(M3) of section 2.1).

(BZ-M1) it is based on a Kleene lattice, which in general is NOT Boolean;
(BZ-M2) all S5 principles hold;
(BZ-M3) the (MD$_\nu$) condition in general does NOT hold.

The following can be proved.

Lemma 4.10. *Cattaneo and Nisticò (1989); Cattaneo and Ciucci (2004) Let $\mathbb{A}_{BZ}$ be a BZ lattice. Then, for arbitrary $a, b \in A$ one has*

(1) $\nu(a) = \nu(\nu(a))$ *(4-principle)*
(2) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$ *(MC-principle)*
(3) $\nu(a) \vee \nu(b) \leq \nu(a \vee b)$ *(sub-additive condition)*
(4) $\sim a = \sim\sim\sim a$ *(triple negation law Frink (1938))*

Conditions (1)–(3) can be expressed in a dual way for the possibility μ.

In this BZ lattice context, making use of the unary modal operations ν and μ defined by (9), one has the following important result relatively to the (MD$_\nu$) condition.

Proposition 4.11. *In any BZ lattice $\mathbb{A}_{BZ}$, condition (MD$_\nu$) is equivalent to the dual de Morgan condition for the Brouwer negation (B4) (and so to the Stone condition).*
Thus, in the case of Stone BZ lattices the induced modal system $\mathbb{A}_{NP}$ is a MDS5 algebra, with respect to which the Stone condition 2) of proposition 4.7 assumes the form:

$$\forall a, \;\; \mu(a) \vee \neg\mu(a) = 1 \tag{10}$$

Proof. From the (MD$_\nu$) condition in the equivalent form (MD$_\mu$), $\mu(a \wedge b) = \mu(a) \wedge \mu(b)$, we obtain that $\sim(a \wedge b) = \neg\mu(a \wedge b) = \neg(\mu(a) \wedge \mu(b)) = \neg\mu(a) \vee \neg\mu(b) = \sim a \vee \sim b$. Conversely, if the (B4) condition holds, from $\sim(a \wedge b) = \sim a \vee \sim b$ it follows that $\neg\sim(a \wedge b) = \neg(\sim a \vee \sim b) = \neg\sim a \wedge \neg\sim b$, i.e., $\mu(a \wedge b) = \mu(a) \wedge \mu(b)$. □

Also when dealing with BZ lattices, and making use of the induced modal operators defined according to (9), we can define two kinds of crispness, respectively:

$$A_{eK} := \{e \in A : e \wedge \neg e = 0\} = \{e \in A : e \vee \neg e = 1\}$$
$$A_{eM} := \{f \in A : \nu(f) = f\} = \{f \in A : \mu(f) = f\}$$

As usual, elements from A_{eK} are said to be K–crisp and elements from A_{eM} M-crisp.

Proposition 4.12. *Cattaneo et al. (1999) Let $\langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$ be a BZ lattice. Then, the set A_{eM} of all M–crisp elements is a standard complemented distributive lattice $\langle A_{eM}, \wedge_e, \vee_e, ', 0, 1\rangle$ with respect to the operations:*
(1) The lattice join $\forall e, f \in A_{eM}, \;\; e \vee_e f = e \vee f$.

(2) The lattice meet $\forall e, f \in A_{eM}$, $e \wedge_e f = e \wedge f$.
(3) The two complementations coincide on elements from A_{eM}*,*

$$\forall e \in A_{eM}, \quad \neg e = \sim e \in A_{eM}$$

and the mapping $\neg : A_{eM} \mapsto A_{eM}$, $e \to \neg e$ *turns out to be a standard complementation in the sense that the following are satisfied:*
(SC1) $\neg\neg e = e$,
(SC2) $\neg(e \vee f) = \neg e \wedge \neg f$,
(SC3) $e \wedge \neg e = 0$ *(equivalently,* $e \vee \neg e = 1$*).*

As a consequence of this result, and contrary to the behavior of the MDS5 case where in proposition 2.4 we have seen that any K-crisp element is M-crisp (i.e., the inclusion $\mathcal{A}_{eK} \subseteq \mathcal{A}_{eM}$ holds), we have the following.

Proposition 4.13. *In any BZ structure one has the set theoretical inclusion:*

$$A_{eM} \subseteq A_{eK}$$

i.e., any M-crisp element is also K-crisp.

Proof. Let $f \in A_{eM}$, i.e., let $f = \nu(f)$. Then by Proposition 4.12 we have that also $\neg f \in A_{eM}$, i.e., one has that $\neg f = \nu(\neg f)$. Hence, $f \wedge \neg f = \nu(f) \wedge \nu(\neg f) =$ (2) of Lemma 4.10 $= \nu(f \wedge \neg f) =$ (SC3) of Proposition 4.12 $= \nu(0) = 0$. □

Note that from condition 1 of Lemma 4.10 relative to necessity, and its dual relative to possibility, we have that the modalities of each element are M–crisp. So from this result we have that they are also K–crisp:

$$\forall a \in A, \quad \nu(a), \mu(a) \in A_{eM} \subseteq A_{eK} \tag{11}$$

In general the converse does not hold, as can be seen in the example of figure 8, where $a \wedge \neg a = 0$ but $\mu(a) = 1$.

As a consequence of proposition 4.13, in the case of Stone BZ lattices $\mathbb{A}_{BZ}$, whose induced modal structure $\mathbb{A}_{NP}$ is according to proposition 4.11 a MDS5 algebra for which the inclusion $\mathcal{A}_{eK} \subseteq \mathcal{A}_{eM}$ holds, we have the equality of the two crisp subsets, simply denoted by $A_e := A_{eM} = A_{eK}$, whose elements are said tout court *crisp* (sharp, exact).

In the BZ context, the notion of M-crispness has some relationship with approximation in rough set theory. Indeed, rough approximations generate pairs of mutually orthogonal M–crisp elements (*orthopairs*) of the kind (x_1, x_0), with $x_1, x_0 \in A_{eM}$ and $x_1 \leq \neg x_0$ (equivalently, $x_0 \leq \neg x_1$), this latter condition also written as $x_1 \perp x_0$. With respect to the orthopair (x_1, x_0), the M–crisp element x_1 is called the *interior* of the pair, the M–crisp element x_0 its *exterior*, and $x_b := \neg x_1 \wedge \neg x_0$ the *boundary* (see for instance (Cattaneo, 1998; Cattaneo and Ciucci, 2004)).

The *approximation* of a generic (approximable) element a of the lattice A is defined as the M–crisp orthopair $r_e(a) = (\nu(a), \sim a) = (\nu(a), \neg\mu(a))$, whose boundary is $a_b := \neg\nu(a) \wedge \mu(a) = \mu(\neg a) \wedge \mu(a)$. In general, not all orthopairs are generated by a rough set approximation (see section 5). The rough approximation $r_e(a)$ of an approximable element a is such that $a_b = 0$ (i.e., there is no uncertainty on the boundary) iff $\nu(a) = a = \mu(a)$, and this is true iff $r_e(a)$ is of the form $(a, \neg a)$, i.e., a is M–crisp. Sometimes, elements that are not crisp are also called *rough.*

Let us prove that in a BZ structure whose lattice is *distributive*, condition (MD_ν) is verified by a particular pair of elements a, b if at least one of them, either a or b, is M-*crisp*, i.e., are such that $\nu(x) = x$ (or equivalently $\mu(x) = x$).

Proposition 4.14. *Let $\mathbb{A}_{BZ}$ be a BZ* distributive *lattice. If either a or b is M–crisp, then*

$$\nu(a \vee b) = \nu(a) \vee \nu(b) \tag{12a}$$

$$\mu(a \wedge b) = \mu(a) \wedge \mu(b) \tag{12b}$$

Proof. Without loss in generality, let us assume that $e \in A_{eM}$ and $b \in A$, and let us set $c := \mu(e \wedge b)$, with $c \in A_{eM}$ since according to (11) it is the possibility of element $e \wedge b$. By $e \wedge b \leq e$ and monotonicity of μ we have $c = \mu(e \wedge b) \leq \mu(e) = e$. Hence, taking into account the (SC3) of Proposition 4.12 and the *distributivity* of the lattice,

$$c = e \wedge c = (e \wedge \neg e) \vee (e \wedge c) = e \wedge (\neg e \vee c) \tag{1}$$

From Proposition 4.12 we have that $\neg e$ is M–crisp, moreover we have seen that c is exact; hence, by point (1) of Proposition 4.12, also $\neg e \vee c$ is exact. So, recalling the point (SC3) of Proposition 4.12, for the exact element e we have that $e \vee \neg e = 1$,

$$\begin{aligned} b \leq \mu(b) &= \mu(b \wedge (e \vee \neg e)) = \mu[(b \wedge e) \vee (b \wedge \neg e)] = (MC) \\ &= \mu(b \wedge e) \vee \mu(b \wedge \neg e) = c \vee \mu(b \wedge \neg e) \\ &\leq c \vee \mu(\neg e) = c \vee \neg e \end{aligned}$$

Hence, $\mu(b) \leq c \vee \neg e$, from which it follows that $e \wedge \mu(b) \leq e \wedge (c \vee \neg e) = (1) = c = \mu(e \wedge b)$. Since e is exact the latter can be written as $\mu(e) \wedge \mu(b) \leq \mu(e \wedge b)$, and from the dual of point (3) of Lemma 4.10 we obtain (12b).
From (12b), applied to the pair $\neg e$ (exact) and $\neg b$, it follows that $\neg \sim(\neg e \wedge \neg b) = \neg \sim\neg e \wedge \neg \sim\neg b$, that is (applying to both members the Kleene complementation) $\sim(\neg e \wedge \neg b) = (\neg \sim\neg e \wedge \neg\neg \sim\neg b) = \sim\neg e \vee \sim\neg b$. Finally, $\nu(e \vee b) = \sim\neg(e \vee b) = \sim(\neg e \wedge \neg b) = \sim\neg e \vee \sim\neg b = \nu(e) \vee \nu(b)$, that is (12a). □

The fact that in general a BZ lattice does not satisfy condition (MD_ν) allows one to partition the class of all BZ lattices into two subclasses: the first one consisting of all BZ lattices for which B4 (equivalently, the Stone condition) does not hold, and the second one consisting of all BZ lattices for which this condition holds. The two examples discussed in sections 3.1 and 3.2 are models of Stone BZ distributive lattices, as shown below.

4.2.1. *The Stone BZ lattice of fuzzy sets.* In the framework of all fuzzy sets on a universe U the structure $\langle \mathcal{F}(U), \wedge, \vee, \neg, \sim, \mathbf{0}, \mathbf{1} \rangle$ where the operations $\wedge, \vee$, and $\neg$ of the Kleene distributive lattice subsystem are defined as in subsection 3.1 and the Brouwer negation as

$$\sim f(x) := \begin{cases} 1, & \text{if } f(x) = 0 \\ 0, & \text{otherwise} \end{cases}$$

is a Stone BZ lattice. Let us note that with respect to the modal operators defined in generic BZ structures $\nu(f) = \sim\neg(f)$ and $\mu(f) = \neg \sim(f)$, one just obtains the two modalities introduced in subsection 3.1. As we have seen, the crisp elements, i.e., the fuzzy sets f such that $\nu(f) = \mu(f) = f$, are the $\{0,1\}$–valued characteristic functions, that is classical (Boolean) sets. Hence, with the now defined Brouwer negation, from the Stone BZ lattice of fuzzy sets $\mathcal{F}(U)$ one induces the standard MDS5 structure of fuzzy sets.

4.2.2. *The Stone BZ lattice of orthopairs.* In the framework $L_3(U)$ of all orthopairs from a nonempty universe U the structure $\langle L_3(U), \sqcap, \sqcup, \neg, \sim, \mathbf{0}, \mathbf{1}\rangle$ where the operations $\sqcap$, $\sqcup$, and $\neg$ of the Kleene distributive lattice subsystem are defined as in proposition 3.2 of subsection 3.2 and the Brouwer negation as

$$\sim(A_1, A_0) := (A_0, (A_0)^c)$$

is a Stone BZ lattice. Also in this case we have that the modal operators induced from the BZ structure, $\nu(A) = \sim\neg(A_1, A_0) = (A_1, (A_1)^c)$ and $\mu(A) = \neg\sim(A_1, A_0) = ((A_0)^c, A_0)$, coincide with the necessity and possibility operators introduced in subsection 3.2. Therefore, with the now defined Brouwer negation, from the Stone BZ lattice of orthopairs $L_3(U)$ one induces the standard MDS5 structure of orthopairs.

As a summary of the present subsection we have that

- Any Stone BZ distributive lattice $\mathbb{A}_{BZ} = \langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$, in which condition (B4) plays a fundamental role, it is possible to induce a MDS5 algebra $\mathbb{A}_{NP} = \langle A, \wedge, \vee, \neg, \nu, \mu, 0, 1\rangle$ where the modal operators are defined according to (9).

4.3. **From MDS5 modal algebras to induced Brouwer–Zadeh structures.** In the previous proposition 4.11 of subsection 4.2 a standard procedure to induce MDS5 algebras from Stone BZ lattices has been investigated. In this section we discuss the converse problem. So, if one starts from a MDS5 algebra $\mathbb{A} = \langle A, \wedge, \vee, \neg, \nu, \mu, 0, 1\rangle$ and naturally defines the unary operator

$$\sim a := \neg\mu(a)$$

then one obtains a structure $\mathbb{A}_{BZ} = \langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$, which resembles a BZ lattice. The point is that all the requirements of Definition 4.1 are satisfied but the contradiction law (B3). Note that the form of this condition with respect to the primitive MDS5 operations is the following:

(B3μ) $\quad \neg a \vee \mu(a) = 1.$

As an example, let us consider the lattice of figure 5. Clearly, this structure is a MDS5 lattice, which does not satisfy property (B3), indeed, $a \wedge \sim a = a \wedge c = a \neq 0$.

As a conclusion, figure 9 lays bare the relationship among MDS5, BZ and Stone BZ lattices. Of course, the category of tqBa algebras contains all these structures, and it is wider, as shown by the previously discussed example of figure 4.

Recall that both the collection of all orthopairs and of all fuzzy sets are particular models of BZ$^{(S)}$.

4.4. **BZ $^{(S)}$ algebras induced from HW algebras.** In both examples of fuzzy sets and, according to the identification of point (EM3) of subsection 3.2.1, its formal special case of orthopairs, we have seen that it is possible to construct two implication connectives $\to_L$ and $\to_G$ satisfying the Hardgree minimal condition of implication (2). This remark naturally suggests to investigate abstract algebraic structures based on two primitive implication connectives, some possible concrete

FIGURE 9. relationship among BZ lattices, MDS5 algebras and tqBa algebras

models of which are just $\mathcal{F}(U)$ and $L_3(U)$. The following structure addresses this question (see (Cattaneo et al., 2004a)).

Definition 4.15. A system $\mathbb{A}_{HW} = \langle A, \rightarrow_L, \rightarrow_G, 0\rangle$ is a *Heyting Wajsberg* (HW) *algebra* iff A is a nonempty set, $0 \in A$, and $\rightarrow_L, \rightarrow_G$ are binary operators, such that, given the definitions

1) $a \vee b := (a \rightarrow_L b) \rightarrow_L b$
2) $a \wedge b := \neg((\neg a \rightarrow_L \neg b) \rightarrow_L \neg b)$
3) $\neg a := a \rightarrow_L 0$
4) $\sim a := a \rightarrow_G 0$
5) $1 := \neg 0$

the following are satisfied:

(HW1) $a \rightarrow_G a = 1$
(HW2) $a \rightarrow_G (b \wedge c) = (a \rightarrow_G c) \wedge (a \rightarrow_G b)$
(HW3) $a \wedge (a \rightarrow_G b) = a \wedge b$
(HW4) $(a \vee b) \rightarrow_G c = (a \rightarrow_G c) \wedge (b \rightarrow_G c)$
(HW5) $1 \rightarrow_L a = a$
(HW6) $a \rightarrow_L (b \rightarrow_L c) = \neg(a \rightarrow_L c) \rightarrow_L \neg b$
(HW7) $\neg \sim a \rightarrow_L \sim\sim a = 1$
(HW8) $(a \rightarrow_G b) \rightarrow_L (a \rightarrow_L b) = 1$

It turns out that the operations $\wedge$ and $\vee$ are just the meet and join operators of a *distributive* lattice structure whose partial order is as usual defined as $a \leq b$ iff $a \wedge b = a$ (equivalently, iff $a \vee b = b$). The two primitive binary operations $\rightarrow_L$ and $\rightarrow_G$ are interpreted as *implication* connectives since it is easy to prove that under the above axioms the minimal implication condition (2) holds for individually each of them:

$$a \leq b \quad \text{iff} \quad a \rightarrow_L b = 1 \tag{13a}$$

$$\text{iff} \quad a \rightarrow_G b = 1 \tag{13b}$$

Let us remark that from any HW algebra it is possible to induce a Wajsberg algebra (Wajsberg, 1931, 1935) (equivalently, Chang MV algebra (Chang, 1958)) with respect to the unique implicative connective $\rightarrow_L$ and a symmetric Heyting algebra (Monteiro, 1980) with respect to the unique implicative connective $\rightarrow_G$. We refer the interested reader to (Cattaneo et al., 2004a).

As to the scope of the present work, we remark that it is possible to prove that all the axioms of Stone BZ lattice are satisfied in any HW algebra relatively to the subsystem involving the only two negations $\neg$ and $\sim$ induced according to the above points 3) and 4). Hence, also all the axioms of MDS5 algebras are satisfied in any HW algebra.

Proposition 4.16. *Let $\mathbb{A}_{HW}$ be a HW algebra. Then,*

(a) a Stone BZ distributive lattice structure $\mathbb{A}_{BZ} = \langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$ can be induced from it according to the following points:

1) By defining $\wedge$ and $\vee$ as in points 1) and 2) of Definition 4.15, the structure $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive lattice (where as usual (13) *defines the partial order relation);*

2) the unary operation $\neg : A \mapsto A$, $\neg a := a \to_L 0$ is a Kleene negation;

3) the unary operation $\sim : A \mapsto A$, $\sim a := a \to_G 0$ is a Brouwer negation which satisfies the Stone condition in the equivalent form (B4);

4) the interconnection law (in) is satisfied.

(b) In this Stone BZ context the modal operators of necessity and possibility defined according to (9) *are the following ones:*

$$\nu(a) = (a \to_L 0) \to_G 0 \tag{14a}$$

$$\mu(a) = (a \to_G 0) \to_L 0 \tag{14b}$$

which, according to proposition 4.11, give rise to a MDS5 algebra of Stone type (i.e., such that the modal form (10) *of the Stone condition holds).*

5. The algebraic structure of approximation spaces

In this section, the pair $\langle \nu(a), \mu(a)\rangle$ is understood as an approximate description of the approximable element a lying in a BZ lattice.

5.1. Approximation spaces induced from BZ lattices.

Definition 5.1. Let $\mathbb{A}_{BZ} = \langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$ be a BZ lattice. For any element of the BZ support $a \in A$, the *rough approximation* of a induced by $\mathbb{A}_{BZ}$ is the orthopair $r(a) := \langle \nu(a), \mu(a)\rangle$.

Remark 5.2. We use the term *approximations* for the two mappings ν, μ since they satisfy the conditions defining an approximation space according to Cattaneo (1998); Cattaneo and Ciucci (2004, 2008). That is, let $\mathbb{A}_{BZ}$ be a BZ lattice and consider the collection of M-crisp elements A_{eM}. Then, the following hold:

(AS1) $\nu(a) \le a \le \mu(a)$;
(AS2) $\nu(a), \mu(a) \in A_{eM}$;
(AS3) $\forall \alpha \in A_{eM}$, "$\alpha \le a$ implies $\alpha \le \nu(a)$" and "$a \le \alpha$ implies $\mu(a) \le \alpha$".

The above three conditions (AS1)–(AS3) can be summarized by the statement: "$\nu(a)$ (resp., $\mu(a)$) is the best approximation of the element a from the bottom (resp., top) by M–crisp elements". Adopting the rough set terminology, $\nu(a)$ is also called the *lower approximation* (also *necessity*) of a and $\mu(a)$ its *upper approximation* (also *possibility*).

In the sequel we denote the collection of all approximations on the BZ lattice $\mathbb{A}$, equivalently written as the *necessity–impossibility* orthopairs $r_e(a) = \langle \nu(a), \neg\mu(a)\rangle$

for a running on $\mathcal{A}$, as $\mathbb{D}(A)$. Thus,

$$\mathbb{D}(A) := \{r_e(a) = \langle \nu(a), \neg\mu(a)\rangle : a \in A\}$$

In this context, $\nu(a)$ is also called the interior and $\neg\mu(a)$ the exterior of a.

5.1.1. *HW algebra induced by modalities from distributive BZ lattices.* Now let us show that the collection of all such approximation pairs definable in a *distributive* BZ lattice has a HW algebraic structure. As remarked below, the distributivity of the lattice is an essential condition in order to obtain the desired results.

Theorem 5.3. *Let* $\mathbb{A} = \langle A, \wedge, \vee, \neg, \sim, 0, 1\rangle$ *be a* distributive *BZ lattice. Let* $r_e(a) = \langle \nu(a), \neg\mu(a)\rangle$ *and* $r_e(b) = \langle \nu(b), \neg\mu(b)\rangle$ *be two elements from* $\mathbb{D}(A)$ *and let us define*

$$\begin{aligned} r_e(a) \Rightarrow_L r_e(b) :&= \langle \nu(a), \neg\mu(a)\rangle \Rightarrow_L \langle \nu(b), \neg\mu(b)\rangle \\ &= \langle (\neg\nu(a) \wedge \mu(b)) \vee \neg\mu(a) \vee \nu(b),\ \nu(a) \wedge \neg\nu(b)\rangle \end{aligned}$$

$$\begin{aligned} r_e(a) \Rightarrow_G r_e(b) :&= \langle \nu(a), \neg\mu(a)\rangle \Rightarrow_G \langle \nu(b), \neg\mu(b)\rangle \\ &= \langle (\neg\nu(a) \wedge \mu(b)) \vee \neg\mu(a) \vee \nu(b),\ \mu(a) \wedge \neg\nu(b)\rangle \end{aligned}$$

Then, setting for the sake of simplicity $a_i := \nu(a)$ *(the necessary or* interior*) and* $a_e = \neg\mu(a)$ *(the impossible or* exterior*), one obtains the following:*

i) the operators $\Rightarrow_L$ *and* $\Rightarrow_G$ *are closed on* $\mathbb{D}(A)$ *since the following hold:*

$$\begin{aligned} r_e(a) \Rightarrow_L r_e(b) &= r_e\big((\neg\nu(a) \wedge \mu(b)) \vee \neg a \vee b\big) \\ &= \langle (\neg a_i \wedge \neg b_e) \vee a_e \vee b_i,\ a_i \wedge b_e\rangle \end{aligned} \tag{15a}$$

$$\begin{aligned} r_e(a) \Rightarrow_G r_e(b) &= r_e\big((\neg\nu(a) \wedge \mu(b)) \vee {\sim}a \vee b\big) \\ &= \langle (\neg a_i \wedge \neg b_e) \vee a_e \vee b_i,\ \neg a_e \wedge b_e\rangle \end{aligned} \tag{15b}$$

ii) the structure $\langle \mathbb{D}(A), \Rightarrow_L, \Rightarrow_G, \langle 0, 1\rangle\rangle$ *is a HW algebra, whose least element is* $\mathbf{0} = r_e(0) = \langle 0, 1\rangle$ *and corresponding greatest element* $\mathbf{1} = \neg\mathbf{0} = r_e(1) = \langle 1, 0\rangle$.

Proof. i) If $\langle a_i, a_e\rangle, \langle b_i, b_e\rangle \in \mathbb{D}(A)$ then,

$$\begin{aligned} &r_e((\neg\nu(a) \wedge \neg{\sim}b) \vee \neg a \vee b) = \\ &\quad = \langle ({\sim}\neg\neg{\sim}\neg a \wedge {\sim}\neg\neg{\sim}b) \vee {\sim}\neg\neg a \vee \nu(b), ({\sim}\neg{\sim}\neg a \vee {\sim}\neg{\sim}b) \wedge {\sim}\neg a \wedge {\sim}b\rangle \\ &\quad = \langle (\neg{\sim}{\sim}{\sim}\neg a \wedge \neg{\sim}{\sim}{\sim}b) \vee {\sim}a \vee \nu(b), ({\sim}\neg a \vee {\sim}{\sim}{\sim}b) \wedge {\sim}\neg a \wedge {\sim}b\rangle \\ &\quad = \langle (\neg a_i \wedge \neg b_e) \vee a_e \vee b_i, a_i \wedge b_e\rangle = \langle a_i, a_e\rangle \Rightarrow_L \langle b_i, b_e\rangle \end{aligned}$$

and dually, for the Gödel implication.

ii) By point i) the operators $\Rightarrow_L$ and $\Rightarrow_G$ define two binary operators on $\mathbb{D}(A)$. It is straightforward to prove that under the essential condition of distributivity of the lattice all the axioms (HW1–8) of definition 4.15 are satisfied.

□

Let us remark that the partial order induced on $\mathbb{D}(A)$ by the lattice operations is

$$\langle \nu(a), \neg\mu(a)\rangle \sqsubseteq \langle \nu(b), \neg\mu(b)\rangle \quad \text{iff} \quad \nu(a) \leq \nu(b) \text{ and } \mu(a) \leq \mu(b)$$

Clearly, this theorem a fortiori holds also in the case that $\mathbb{A}_{BZ}$ is a Stone BZ lattice, i.e., it is equivalently a MDS5 algebra satisfying the modal version (10) of Stone condition. On the other hand, if $\mathbb{A}$ is a MDS5 algebra we do not even know if $\mathbb{D}(A)$ has a lattice structure.

5.2. **The HW algebra of orthopairs and induced BZ and MDS5 structures.** As previously remarked, the collection $L_3(U)$ of all orthopairs of subsets (A_1, A_0) from a universe U equipped with the two binary implication operations defined by equations (7) is a typical example of HW algebra $\langle L_3(U), \Rightarrow_L, \Rightarrow_G, (\emptyset, U)\rangle$.

According to points 2), 3)–(a) of proposition 4.16, from this HW algebra one can induce a structure of Stone BZ distributive lattice, whose two negations are just the Kleene one of proposition 3.2 and the Brouwer one considered in subsection 4.2.2:

$$\neg(A_1, A_0) = (A_1, A_0) \Rightarrow_L (\emptyset, \emptyset) = (A_0, A_1)$$
$$\sim (A_1, A_0) = (A_1, A_0) \Rightarrow_G (\emptyset, \emptyset) = (A_0, A_0^c)$$

On the other hand, it is easy to prove that the induced MDS5 algebra of Stone type obtained according to the modalities defined by equations (14) of point (b) of proposition 4.16 is just the one described in proposition 3.2.

Finally, on the basis of this BZ distributive lattice based on $L_3(U)$ one can consider the induced collection $\mathbb{D}(L_3(U))$ of all rough approximations

$$r_e(A_1, A_0) = \langle \nu(A_1, A_0),\ \neg\mu(A_1, A_0)\rangle = \langle (A_1, A_1^c),\ (A_0, A_0^c)\rangle \tag{16}$$

which, according to point ii) of Theorem 5.3 is a MDS5 algebra. Trivially, the orthopair $r_e(A_1, A_0) \in \mathbb{D}(L_3(U))$, by the identifications $(A_1, A_1^c) \equiv A_1$ and $(A_0, A_0^c) \equiv A_0$, can be identified with the original orthopair (A_1, A_0) in $L_3(U)$. So we can further on identify the two structures $L_3(U) \equiv \mathbb{D}(L_3(U))$, which according to subsection 4.2.2 are a Stone BZ distributive lattice (and so also a MBS5 algebra of Stone type).

5.3. **The HW algebra of fuzzy sets and induced BZ and MDS5 structures.** As in the case of orthopairs on a universe U, also the collection $\mathcal{F}(U)$ of all fuzzy sets on U equipped with the two implication connectives defined according to equations (5) turns out to be a HW algebra $\langle \mathcal{F}(U), \rightarrow_L, \rightarrow_G, 0\rangle$.

The Stone distributive BZ lattice induced from this HW algebra according to the procedure described in point (a) of proposition 4.16, is just based on the Kleene negation introduced in proposition 3.1 and the Brouwer one considered in subsection 4.2.1.

Also in this case the MDS5 algebra induced from this fuzzy-set HW algebra on the basis of the modal operators defined according to equations (14) of point (b) of proposition 4.16 is just the one described in proposition 3.1.

On this MDS5 structure based on the collection of fuzzy sets $\mathcal{F}(U)$ it is possible to introduce, according to definition 5.1, the rough approximation of a generic fuzzy set f by the orthopair of crisp sets as

$$r_e(f) := \langle \nu(f), {\sim} f\rangle = \left\langle \chi_{A_1(f)}, \chi_{A_0(f)} \right\rangle$$

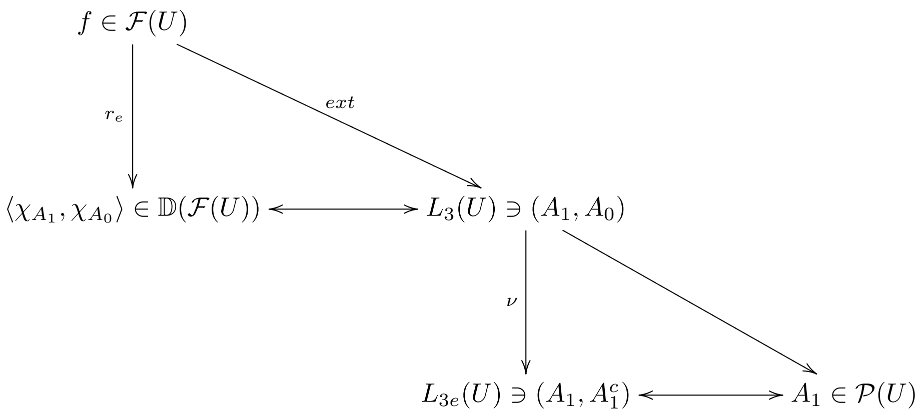
obtaining the MDS5 algebra $\mathbb{D}(\mathcal{F}(U))$, according to point ii) of Theorem 5.3. Note that one can set the identification $r_e(f) \equiv (A_1(f), A_0(f))$, this latter being (according to point (EM1) of subsection 3.2.1) the orthopair from $L_3(U)$ consisting of the necessity (interior) $A_1(f)$ and the impossibility (exterior) $A_0(f)$ of the fuzzy set f. Therefore, we have obtained the algebraic structure identification $\mathbb{D}(\mathcal{F}(U)) \equiv L_3(U)$, in such a way that the Stone condition of $L_3(U)$ is inherited also by $\mathbb{D}(\mathcal{F}(U))$. Let us recall the point (EM3) of the same subsection which institute the further identification of $L_3(U)$ with the subset of fuzzy sets $\mathcal{F}_t(U) \subseteq \mathcal{F}(U)$.

In the sequel, for the sake of simplicity we set $f_i := \nu(f) = \chi_{A_1(f)}$ and $f_e := \sim f = \chi_{A_0(f)}$, and for any point $x \in U$ by $\langle f_i, f_e \rangle (x)$ we will mean $\langle f_i(x), f_e(x) \rangle$. Now, in the present $\mathbb{D}(\mathcal{F}(U))$ case the implication connectives defined according to Theorem 5.3, for $x \in U$ are expressed in the following way:

$$(\langle f_i, f_e \rangle \Rightarrow_L \langle g_i, g_e \rangle)(x) = \begin{cases} r_e(g)(x) & \text{if } f(x) = 1 \\ \langle f_e, f_i \rangle (x) & \text{if } g(x) = 0 \\ \langle 1, 0 \rangle & \text{otherwise} \end{cases}$$

$$(\langle f_i, f_e \rangle \Rightarrow_G \langle g_i, g_e \rangle)(x) = \begin{cases} r_e(g)(x) & \text{if } f(x) = 1 \\ \langle 0, 1 \rangle & \text{if } g(x) = 0 \quad \text{and} \quad f(x) \neq 0 \\ \langle 1, 0 \rangle & \text{otherwise} \end{cases}$$

5.3.1. *A summary of the most relevant results about the Stone MDS5 algebras of rough sets and of orthopairs.* All the results discussed in the previous two subsections, with the corresponding identifications of Stone BZ distributive lattices on the same universe, can be summarized in the following diagram, where the rough approximation mapping on fuzzy sets $r_e : \mathcal{F}(U) \to \mathbb{D}(\mathcal{F}(U))$ is expressed by the correspondence associating with any fuzzy set $f \in \mathcal{F}(U)$ the simplified pair $\langle \chi_{A_1}, \chi_{A_0} \rangle$:



5.4. **Approximations of fuzzy sets vs. rough approximations of sets.** As an important remark it is of interest to comment on some of the results obtained here in terms of approximations of the composition of elements expressible or not in terms of approximation of these elements. For instance, in proposition 3.3, implications between orthopairs are directly expressed in terms of MDS5 primitive operations $\neg, \sqcap, \sqcup, \nu, \mu$ acting on orthopairs. Also, in equation (15) of Theorem 5.3, the same situation occurs with elements in the HW algebra of modal orthopairs generated by a distributive BZ lattice.

The deviant modalities ν and μ are defined from scratch in the algebraic structures and their role is to extract specific features of the elements of the algebra (for instance, one side of an orthopair, or the core or the support of a fuzzy set).

So it is no surprise that the debatable property (MD_ν) may hold, as the obtained approximation of an element a is intrinsic to a itself and does not involve any external information (it is an invariant). So, for instance, the expression of the elements $\alpha \Rightarrow_L \beta$ and $\alpha \Rightarrow_G \beta$ in proposition 3.3 in terms of α and β is intrinsic, in the sense that it is devised in terms of elements of the lattice only.

The situation is very different from the case of rough sets according to Pawlak Pawlak (1982); Pawlak and Skowron (2007), which also lead to a BZ algebra, but where MD does not hold. The rough approximation of a Boolean subset A of a universe S, say the nested pair $(R_*(A), R^*(A))$ where $R_*(A)$ is the inner approximation of A and $R^*(A)$ its outer approximation, is no longer intrinsic nor invariant. It depends on an external ingredient, namely the equivalence relation R that reflects the imperfect perception of an observer trying to grasp some unreachable object with a language that is not expressive enough. $R_*(A)$ and $R^*(A)$ are made of unions of equivalence classes, hence depend on the partition chosen, hence on the observer. Bonikowski in Bonikowski (1992) has proved that, given two subsets A and B of S there exists another subset C of S such that $R_*(C) = R_*(A) \cup R_*(B)$, $R^*(C) = R^*(A) \cup R^*(B)$, and this set C can be expressed in terms of A, B and their approximations (contrary to the case studied in this paper where $C = A \cup B$ due to MD, but where these entities are NOT Boolean)[4] . While the existence of a partition-dependent set C may have interesting mathematical consequences (like bridging the gap between rough sets and a truth-functional many-valued logics), the fact that the set C depends not only on A and B but on R as well is rather troublesome, as this object is not intrinsic to the space S but would change if another partition of the space S (another observer) is chosen. This would be problematic in a dynamic environment (where the equivalence relation would change, adding an attribute to the information system, for instance), or in a multiagent setting (where each agent would refer to a different set C) since C would not be an invariant of the space. This point would deserve a further study in settings like rough sets, where property MD cannot hold, and modalities are defined via an external entity like a relation representing perception by an outside entity. For a deeper discussion about this issue see also Ciucci and Dubois (2010).

6. Comparison with Delta operator and truth stressers

Let us note that the operator ν shares some property with the Baaz Delta operator, that is a projection modality Δ in Gödel logics defined as $\Delta(1) = 1$ and $\Delta(x) = 0$ for $x \neq 1$. Indeed, in (Baaz, 1996) a logic is introduced as "an axiomatization consisting of the axiom schemas of intuitionistic propositional logic and of modal logic S4 for Δ plus [...] the following schemas:

$$(A \rightsquigarrow B) \vee (B \rightsquigarrow A)$$
$$\Delta A \vee -\Delta A$$
$$\Delta(A \vee B) \rightsquigarrow \Delta A \vee \Delta B\text{''}$$

where $-$ is the intuitionistic negation defined as $-0 = 1$ and $-a = 0$ for $a \neq 1$.

Subsequently, this operator has been introduced from an algebraic standpoint in the context of BL algebras (see Hájek (1998)), giving rise to the stronger structure of BL_Δ algebras. Let us recall these definitions.

Definition 6.1. A BL algebra is a system $\langle A, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ such that

[4]There are also constructive definitions of these sets that are approximated by unions (or intersections)of upper and lower approximations Gehrke and Walker (1992); Banerjee and Chakraborty (1996). These definitions do not require the axiom of choice.

(BL1) $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice;
(BL2) $\langle A, *, 1\rangle$ is a commutative monoid;
(BL3) $(*, \Rightarrow)$ form an adjoint pair, i.e., $c \leq (a \Rightarrow b)$ iff $a * c \leq b$.
(BL4) $(a \Rightarrow b) \vee (b \Rightarrow a) = 1$
(BL5) $a \wedge b = a * (a \Rightarrow b)$

A BL_Δ algebra is a BL algebra plus a unary operation $\Delta : A \mapsto A$ satisfying:

$$\Delta x \vee -\Delta x = 1$$
$$\Delta(x \vee y) \leq \Delta x \vee \Delta y$$
$$\Delta x \leq x$$
$$\Delta x \leq \Delta\Delta x$$
$$(\Delta x) * (\Delta(x \Rightarrow y)) \leq \Delta y$$
$$\Delta 1 = 1$$

where $-x := x \Rightarrow 0$.

First of all, we note that in all BL algebras (see Cattaneo et al. (2004a) for a proof) $-x := x \Rightarrow 0$ is a *minimal negation* according to (Dunn, 1986), i.e., it satisfies the weak double negation law "$x \leq --x$" and the contraposition law "$x \leq y$ implies $-y \leq -x$". Moreover, also the Kleene property (Kl) is satisfied by $-$. In general, the double negation law "$x = --x$", the non contradiction principle "$x \wedge -x = 0$" and the excluded middle law "$x \vee -x = 1$"are not satisfied. Hence $-$ is a very weak form of negation.

Two further important results about BL_Δ (Hájek, 1998, pages 57–60) are:

(1) if the underlying lattice A is linearly ordered then the operation Δ is defined as

$$\Delta(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases} \tag{17}$$

and the operator $\sim x := \Delta(-x)$ (where $-x = x \Rightarrow 0$) has always the following form

$$\sim x = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Further, it behaves as an intuitionistic Stone negation, in the sense that it satisfies properties (B1)–(B3) and the Stone condition. Consequently, it is possible to introduce the possibility as:

$$\nabla(x) := \sim\sim x = \Delta(-\Delta(-x)) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \end{cases}$$

(2) A formula ψ is a tautology for each linearly ordered BL_Δ algebra iff it is a tautology for each BL_Δ algebra.

As a consequence of the above two points we have that in any BL_Δ algebra, the operators (Δ, ∇) satisfy all the properties (N1), (N2), (N3), and (N5) plus (MD_ν) introduced in section 2. The property (N4) in general does not hold with respect to $-$, but can be substituted by the new identity $\nabla(x) = \sim\Delta\sim(x)$ involving the intuitionistic (Stone) negation $\sim$.

Now, on the basis of a BL_Δ algebra, the structure involving the negation $\sim$ and the pair Δ, ∇ is not a MDS5 algebra since in general condition (dM1) of Kleene lattices does not hold (also if it shares all the other conditions). On the other hand, if one chooses $-$ as negation in general the condition (dM1) of Kleene lattice is not verified and (N4) is not true. As a consequence, BL_Δ algebras in general do not induce MDS5 algebras.

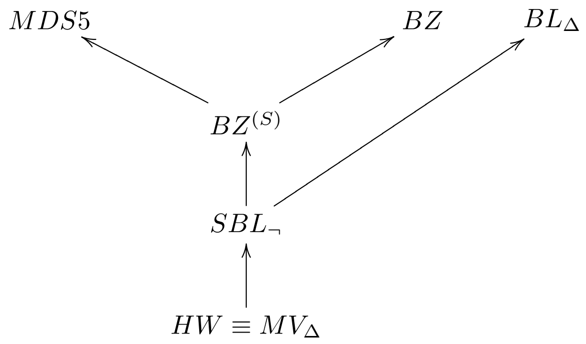
Example 6.2. Let us consider the standard $[0,1]$–models of BL_Δ algebras in which $*_j$ is a generic continuous t–norm and $\Rightarrow_j$ is the implication induced by $*_j$. Since $[0,1]$ is linearly ordered with respect to the standard ordering on real numbers, the Δ operation is uniquely given by equation (17).
Now, if we choose the implication $x \Rightarrow_L y = \min\{1, 1-x+y\}$ induced by the Łukasiewicz t–norm, then the negation $-_L x = 1-x$ is Kleene and the structure $\langle [0,1], \wedge, \vee, \nu = \Delta, -_L\rangle$ is a MDS5 algebra.
On the other hand, if we choose the implication induced from the Gödel t–norm $x \Rightarrow_G y = 1$ if $x \leq y$, and $= y$ otherwise, then the negation $-_G x = 1$ iff $x = 0$ (and so $= 0$ otherwise) is not Kleene and the structure $\langle [0,1], \wedge, \vee, \nu = \Delta, -_G\rangle$ is not a MDS5 algebra. ■

Conversely, an adjoint pair $(*, \Rightarrow)$ is not present in MDS5 algebras as primitive structure. We cannot exclude that, under the constraint $\Delta = \nu$, an adjoint pair is definable in any MDS5 algebra. However, the resulting structure will be far from the purposes of MDS5 algebras. What we can prove is that the new structure will have three different negations. Indeed, the negation $-$ induced from $\Rightarrow$ should be different from the original Kleene negation $\neg$ and the induced negation $\sim$. Recall that the Δ operator must satisfy the property $\Delta x \vee -\Delta x = 1$. However, the MDS5 example of figure 5 shows that for the particular element a it is $\nu(a) \vee \sim \nu(a) = c \neq 1$ and $\nu(a) \vee \neg\nu(a) = c \neq 1$.

In conclusion,

- when considering only the properties satisfied by ν and Δ, as modal–like connectives of necessity, we have that ν is weaker, i.e., it satisfies less properties, than Δ. Indeed, Δ satisfies (N1)–(N5), (MD_ν), and also the excluded middle-like property $\Delta x \vee \sim\Delta x = 1$ with respect to the induced negation $\sim a := \Delta(-a)$ (but it must be stressed that this negation is not Kleene), whereas ν (as shown by the above discussed example of figure 5) in general satisfies the latter property for none of the negations $\neg$ and $\sim$;
- when considering also the underlying algebraic structure, we have that MDS5 algebras cannot be induced in BL_Δ algebras. Of course there are structures that are, at the same time, a model of BL_Δ and MDS5. To be precise, the following diagram summarizes the relationship among the

algebras we have met until now:



where

- SBL$_\neg$ algebras (Esteva et al., 2000) are BL algebras satisfying also the axiom $-(x * y) = -x \vee -y$ plus an involutive (de Morgan in our terminology) negation $\neg$, such that $\Delta x := -\neg x$ is a Delta operator;
- MV$_\Delta$(Hájek, 1998) are BL$_\Delta$ algebras plus the axiom $x = --x$ or equivalently MV algebras plus the Delta operator;
- the relation among SBL$_\neg$, MV$_\Delta$, HW and BZ$^{(S)}$ is discussed in (Cattaneo et al., 2004a,b).

Finally, we note that the Δ operator as well as some of its generalizations has been considered also in other structures (for instance (Esteva et al., 2000; Hájek, 2001; Ciabattoni et al., 2005)), but always based on a residuated lattice.

Also the work of Hájek (2001) is of particular interest since it relates our ν operator to fuzzy *truth stressers* expressing linguistic hedges such as "very", "little", and so on as discussed by Zadeh (1972). Indeed, the two basic properties required for the "very true" operator are the following (quoting from Hájek (2001)): "$vt(x) \leq x$ (if ψ is very true then it is true) and $vt(1)$ must be 1 (1 is absolute true)", which are both satisfied by the ν operator. More in detail, the algebra of "very true" is the BL$_{vt}$ algebra, i.e., a BL algebra plus a unary operator vt satisfying:

$$vt(1) = 1$$
$$vt(x \vee y) \leq vt(x) \vee vt(y)$$
$$vt(x) \leq x$$
$$vt(x \Rightarrow y) \leq (vt(x) \Rightarrow vt(y))$$

Now, as discussed in section 2, the first three axioms are also required for ν and the fourth corresponds to the K principle of our lattice setting (see axiom N3 and equation (4)). That is, ν satisfies all the properties required for vt plus some more. So, a further interpretation that we can give to the ν operator is a particular axiomatization of a linguistic hedge of the truth-stresser kind, based on a de Morgan (Kleene) lattice. It is also possible to construct a dual truth-weakener using a counterpart of axiom N4, but N5 is unlikely to hold in the general case, and MD is a strengthening of the second above axiom. The ν operation applied to a fuzzy set as in section 3.1, that yields its core, can be viewed as an extreme case of truth-stresser in the above sense.

7. Conclusions

The MDS5 algebra has been introduced as a de Morgan, not Boolean, lattice with a further unary operator that satisfies all S_5 modal properties and the MD law. Two non trivial models of this structure (one being a particular case of the other) are given. Further, the relationship between BZ lattices and MDS5 algebras is studied and it is shown that from any HW algebra it is possible to induce a MDS5 algebra. Clearly, since HW algebras are equivalent to MV_Δ algebras (Hájek, 1998; Cattaneo et al., 2004b), MDS5 algebras can also be induced by this well known structure. Moreover, if one takes into account other structures related to MV_Δ algebras with modal operators it can be seen that

- MDS5 algebras can be induced in $SBL_\neg$ algebras (Esteva et al., 2000), since as proved in (Cattaneo et al., 2004a) also Stone BZ algebras can be induced in $SBL_\neg$.
- MDS5 algebras cannot be defined in all the BL_Δ (Hájek, 1998) algebras. Indeed, BL_Δ algebras are not based on a Kleene lattice. On the other side, MDS5 algebra has not an adjoint pair as primitive operators. If it would be possible to residuate it and define BL_Δ algebras by any MDS5 algebra, then the new structure will have three different negations.

The MD axiom enables combinations of approximations of elements to be expressed in terms of approximations of *intrinsic* combinations of these elements (i.e. involving intrinsically defined connectives only), which as discussed above is atypical, since this is not true for basic approximation settings like rough sets. So, this result suggests further investigations on the significance of casting rough sets in the framework of truth-functional many-valued logics.

Another interesting issue is that HW algebras can be viewed as too powerful enrichment of BZ-algebras, as they involve Łukasiewicz disjunction (i.e. an addition operation) while BZ structures are much more qualitative. One idea to define qualitative counterparts of HW algebras that would still be strengthening of BZ-algebras is to use the implication of the NM logic (see Godo Esteva and Godo (2001)) that is the residuation of the nilpotent minimum (see Fodor Fodor (1195)). This operation can be defined on a Kleene lattice as follows: $a \wedge_0 b = a \wedge b$, if $a > \neg b$, and 0 otherwise. The corresponding implication is $a \rightarrow_0 b = 1$ if $a \leq b$ and $\neg a \vee b$ otherwise. It holds that $\neg a = a \rightarrow_0 0$. Moreover, this implication satisfies all properties (HW5–8) of Łukasiewicz implication. Thus the obtained structure is similar to a HW algebra (except that $\vee$ and $\wedge$ cannot be defined in terms of $\rightarrow_0$, and $\wedge$ must be assumed on top of $\rightarrow_0$ and $\rightarrow_G$). Moreover, if the underlying scale is just the chain $\{0, 1/2, 1\}$, the nilpotent implication $\rightarrow_0$ coincides with Lukasiewicz's (so the set of orthopairs seem to be an example of HW algebras and the new one). A full-fledged study of the role of the nilpotent implication in approximation spaces is thus in order.

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