Rapporto n. 211

Testing Serial Independence via Density-Based Measures of Divergence

Autori

Luca Bagnato, Lucio De Capitani, Antonio Punzo

Giugno 2011

Dipartimento di Metodi Quantitativi per le Scienze Economiche ed Aziendali
Università degli Studi di Milano Bicocca
Via Bicocca degli Arcimboldi 8 - 20126 Milano - Italia
Tel +39/02/64483102/3 - Fax +39/2/64483105
Segreteria di redazione: Andrea Bertolini

Testing Serial Independence via Density-Based Measures of Divergence

Luca Bagnato · Lucio De Capitani · Antonio Punzo

Abstract This article reviews the nonparametric serial independence tests based on measures of divergence between densities. Among others, the well-known Kullback-Leibler, Hellinger and Tsallis divergences are analyzed. Moreover, the copula-based version of the considered divergence functionals is defined and taken into account. In order to implement serial independence tests based on these divergence functionals, it is necessary to choose a density estimation technique, a way to compute p-values and other settings. Via a wide simulation study, the performance of the serial independence tests arising from the adoption of the divergence functionals with different implementation is compared. Both single-lag and multiple-lag test procedures are investigated in order to find the best solutions in terms of size and power.

Keywords Serial independence · Divergence measures · Nonparametric density estimation · Copula

Luca Bagnato
Dipartimento di Metodi Quantitativi per le Scienze Economiche ed Aziendali,
Università di Milano-Bicocca (Italy)
Tel.: +39-02-64483186
Fax: +39-02-64483105
E-mail: luca.bagnato@unimib.it

Lucio De Capitani
Dipartimento di Metodi Quantitativi per le Scienze Economiche ed Aziendali,
Università di Milano-Bicocca (Italy)
Tel.: +39-02-64483186
Fax: +39-02-64483105
E-mail: lucio.decapitani1@unimib.it

Antonio Punzo
Dipartimento di Impresa, Culture e Società,
Università di Catania (Italy)
Tel.: +39-095-7537640
Fax: +39-095-7537610
E-mail: antonio.punzo@unict.it

1 Introduction

In the analysis of a strictly stationary time series, a primary concern is whether the observations are serially independent or not. Detection of the presence of serial dependence is usually a preliminary step carried out before proceeding with further analysis (Kendall and Stuart, 1966, p. 350) like modeling and prediction (Lo, 2000). Since the theoretical models incorporate independent and identically distributed (*i.i.d.*) innovation (noise), the analysis of the dependence is also useful in model-checking and in testing important economic and financial postulates, such as the random walk hypothesis (Delgado, 1996; Darrat and Zhong, 2000). For example, independence of log-returns is an essential feature of the Black-Scholes option pricing equation (Hull, 1999).

The considerations stated above represent only some of the examples justifying both the importance of testing serial independence and the considerable attention devoted to this problem in the literature (see, e.g., the recent work of Diks, 2009, for a comprehensive review, and an extensive bibliography, on the subject). Nevertheless, as underlined by Genest et al (2002) and Genest and Verret (2005), test procedures merely based on the serial correlation coefficient (see, e.g., the proposals of Wald and Wolfowitz 1943, Moran 1948, Durbin and Watson 1950, 1951, Ljung and Box 1978, and Dufour and Roy 1985) continue to be the most commonly used in practice (see King, 1987, for a survey). Although they perform well when the dependence structure is linear and the innovations are normal, they are not consistent against alternatives with zero serial correlation (see, e.g., Hall and Wolff, 1995) and can behave rather poorly, both in terms of level and power, when applied to non-Gaussian and nonlinear time series (see, e.g., the simulations reported by Hallin and Mélard, 1988). Nowadays, as highlighted by Hong (1999), it is widely recognized that many time series arising in practice, display non-Gaussian and nonlinear features (see, e.g., Brock et al, 1991; Granger and Andersen, 1978; Granger and Teräsvirta, 1993; Priestley, 1988; Subba Rao and Gabr, 1980, 1984; Tong, 1990).

This has motivated the development of tests for serial independence which are powerful against general types of dependence (*omnibus tests*; see Diks, 2009, pp. 6256–6257 for details). Since the concept of independence can be characterized in terms of distributions, a line of research in these terms consists of basing tests of serial independence on the fact that the null hypothesis holds if, and only if, the joint density equals the product of the marginals. In these procedures, the test statistic is based on a distance or, more in general, on a divergence measure between the estimated joint density and the product of the estimated marginal densities. Naturally, both the *divergence measure*, and the *density estimation technique* adopted, make the procedures different.

The principal literature on this topic can be summarized as follows. Chan and Tran (1992) propose an estimation of the joint and marginal densities by the histogram and introduce a statistic based on the L_1-norm. Bagnato and Punzo (2010) and Dionísio et al (2006) consider the χ^2-statistic and the entropy of Shannon respectively, using a histogram-based estimator (see also Bagnato and Punzo, 2011, for a contextualization in the model validation phase). Robinson (1991) adopts the Kullback-Leibler information criterion using smoothing kernel density estimators (see also Granger and Lin, 1994; Hong and White, 2005); Skaug and Tjøstheim (1993b) extend Robinson's framework to other measures of divergence between densities, including the Hellinger distance. Maasoumi and Racine (2002), Granger et al (2004), and Racine and Maasoumi (2007) contribute to this line of research considering a normalization of the Hellinger distance, while Fernandes and Néri (2010) extend all these proposals taking into consideration the generalized entropic measure suggested by Tsallis (1998). Rosenblatt (1975) and Ahmad and Li (1997), among others, use smoothing kernel density estimators with the von-Mises L_2-norm; Robinson (1991) and Skaug and Tjøstheim (1996) also use kernel density estimation but they consider, respectively, the Kullback-Leibler information criterion and the Hellinger measure (beyond several other measures).

It is worthwhile to note that in this context a uniformly most powerful test does not exist. Indeed, different tests will be more powerful against different alternatives. Motivated by this consideration, this paper reviews the nonparametric serial independence tests based on measures of divergence between densities. This comparison is not as simple as it might seem since, in addition to the divergence measure and the density estimation technique, there are other quantities to specify (the way p-values are computed, use of trimming functions, and some computational aspects). Moreover, the copula-based versions of all the considered divergence functionals are defined. Thus, the performance of the discussed tests are investigated by varying the considered quantities through a wide simulation study. The general aim is to provide a guideline for the use of these testing procedures.

This paper is structured into two main parts: review of existing methodologies (Section 3 and Section 4) and a comparative simulation study (Sections 5-8). Section 3 introduces some general and preliminary aspects such as the distinction between single-lag testing procedures, developed to be powerful against dependence in a particular lag, and multiple-lag testing procedures controlling dependence on a finite set of pre-specified lags. All the divergence measures and their corresponding copula versions, are introduced here. Focusing on the single-lag procedures, Section 4 summarizes the techniques used to estimate densities (histogram-based in Section 3.1, Gaussian kernel in Section 3.2, and jackknife kernel in Section 3.3) and to estimate the copula density. For the latter problem, the Kallenberg (2009) method is considered (see Section 3.4). Section 4 describes some computational aspects related to the divergence functionals estimated through both raw densities and copula densities (Section 4.1). Details on resampling-based approaches (bootstrap and permutation) to compute p-values, and extensions to multiple-lag testing procedures of the discussed methodologies, are given in Section 4.2. The design of the simulation study is illustrated in Section 5 while results for single-lag, and multiple-lag, procedures are described in Section 6 and Section 7, respectively. Results on simulations are further summarized in Section 8.

2 Density-Based Measures of Dependence

Let $\{X_t\}_{t\in\mathbb{N}_+}$ be a real-valued and strictly stationary stochastic process and assume that X_1 is continuous with density g and support $\mathcal{S}$. Moreover, let l be a positive integer and f_l the joint distribution function of (X_1, X_{1+l}). Under the assumption of strict stationarity, the components of $\{X_t\}_{t\in\mathbb{N}_+}$ are independent if, and only if, X_1 and X_{1+l} are independent for all the integers $l \geq 1$. As a consequence, the hypothesis of serial independence can be expressed as follows:

$$H_0: \; f_l(x,y) = g(x)g(y) \qquad \forall\,(x,y) \in \mathcal{S}^2 \quad \text{and} \quad \forall\, l \geq 1. \tag{1}$$

From a practical point of view, testing hypothesis (1) is impossible. Usually, a maximum value for l, say p, is fixed and the presence of serial independence is checked by testing the following simpler hypothesis:

$$H_0: \; f_l(x,y) = g(x)g(y) \qquad \forall\,(x,y) \in \mathcal{S}^2 \quad \text{and} \quad \forall\, l \in \{1,\ldots,p\}\,. \tag{2}$$

Naturally, if (2) is false for a particular $p \geq 1$, then also (1) is false. On the contrary, (2) may be true even if (1) is false; this happens if X_1 and X_{1+r} are dependent only for some lag(s) $r > p$. However, the aforementioned problem seems to have only a little practical relevance since it can be substantially avoided by choosing a sufficiently large value of p. In the following (2) will be referred to as *multiple-lag testing problem*. On the contrary

$$H_0: \; f_l(x,y) = g(x)g(y) \qquad \forall\,(x,y) \in \mathcal{S}^2\,, \tag{3}$$

for a particular value of l, will be called *single-lag testing problem*; the case $l = 1$ is particularly interesting since, in almost all the cases of practical interest, it is reasonable to retain that the process exhibits the higher dependence between consecutive components. Accordingly, special attention in the literature (see, e.g., Robinson, 1991; Hong and White, 2005; Fernandes and Néri, 2010) is devoted to it. Section 4.2 will show how to face the multiple testing problem (2) by using the information arising from the first p single tests in (3). If there is no ambiguity, f_l will be written as f from here onwards.

Several density-based measures of dependence can be used for testing (3). All the functionals considered here evaluate the discrepancy between $f(x,y)$ and $g \cdot g(x,y) = g(x)g(y)$ as follows:

$$\Delta = \int_{\mathcal{S}^2} B\{f(x,y), g(x), g(y)\} f(x,y)\,dxdy, \tag{4}$$

where B is a real-valued function. An example of these functionals is the Tsallis (1988) generalized entropy, which in this context coincides with:

$$\Delta_\gamma = \begin{cases} \dfrac{1}{1-\gamma} \displaystyle\int_{\mathcal{S}^2} \left[1 - \left(\dfrac{g(x)g(y)}{f(x,y)}\right)^{1-\gamma}\right] f(x,y)dxdy & \gamma \neq 1 \\[2ex] \displaystyle\int_{\mathcal{S}^2} \log\left(\dfrac{f(x,y)}{g(x)g(y)}\right) f(x,y)dxdy & \gamma = 1. \end{cases} \tag{5}$$

It is easy to note that $\Delta_{1/2}$ coincides with the Hellinger metric while Δ_1 with the Kullback-Leibler divergence. Moreover, when $\gamma = 2$ it turns out that

$$\Delta_2 = \int_{\mathcal{S}^2} \frac{(f(x,y) - g(x)g(y))^2}{g(x)g(y)} dxdy,$$

which can be interpreted as the "continuous counterpart" of the Pearson Chi-square of independence.

A further intuitive dependence measure is the L_1 distance:

$$\Delta_{L_1} = \int_{\mathcal{S}^2} |f(x,y) - g(x)g(y)|\,dxdy. \tag{6}$$

Note that (6) can be interpreted as the "continuous counterpart" of the well-known Mortara dependence index (Mortara, 1922). In a similar fashion, the following functionals can be introduced:

$$\Delta_{SD} = \int_{\mathcal{S}^2} \left(f(x,y) - g(x)g(y)\right)^2 dxdy \tag{7}$$

$$\Delta_{ST} = \int_{\mathcal{S}^2} \left(f(x,y) - g(x)g(y)\right) f(x,y)dxdy \tag{8}$$

$$\Delta_{2^*} = \int_{\mathcal{S}^2} \frac{\left(f(x,y) - g(x)g(y)\right)^2}{g(x)g(y)} f(x,y)dxdy \tag{9}$$

$$\Delta_{L_1^*} = \int_{\mathcal{S}^2} \left|f(x,y) - g(x)g(y)\right| f(x,y)dxdy. \tag{10}$$

Functionals (9) and (10) are a slight modification of Δ_2 and Δ_{L_1}, respectively.

While in the literature (5)-(8) have been largely used, as far as we know, (9) and (10) have not been used for testing serial independence. A serial independence test based on Δ_1 was first proposed by Robinson (1991) and later refined by Hong and White (2005). A test based on $\Delta_{1/2}$ was studied by Granger et al (2004). More recently, Fernandes and Néri (2010) proposed a test based on Δ_γ for arbitrary γ, and they studied its finite sample properties both for $\gamma = 2$ and $\gamma = 4$. The functional (6) was used to test serial independence by Chan and Tran (1992) while (7) was employed by Rosenblatt (1975) and Rosenblatt and Wahlen (1992). Skaug and Tjøstheim (1993b) proposed the use of (8).

Naturally, all the aforementioned functionals are sensitive to departures from independence. Moreover, with the exception of (8), they satisfy the following condition (see, e.g. Tjøstheim, 1996, p. 261):

$$\Delta\left(f, g \cdot g\right) \geq 0 \quad \text{and} \quad \Delta(f, g \cdot g) = 0 \quad \text{iff} \quad f = g \cdot g. \tag{11}$$

Functional (8) satisfies condition (11) only in the case of Gaussian processes (Skaug and Tjøstheim, 1993b).

As pointed out by Tjøstheim (1996), invariance under monotonic transformations is another important requirement for Δ. Let h be a strictly monotonic transformation from $\mathcal{S}$ to $\mathcal{S}' \subset \mathbb{R}$ and let f^* and g^* be the densities of $(h(X_1), h(X_2))$ and $h(X_1)$, respectively. The functional Δ is invariant under strictly monotonic transformations if

$$\Delta(f^*, g^* \cdot g^*) = \Delta(f, g \cdot g) \qquad \text{for all } h \text{ strictly monotonic.} \tag{12}$$

As noted by Diks (2009), if the density g is unknown, it plays the role of an infinite dimensional nuisance parameter. However, if the test is based on an invariant functional, it is possible to "remove" the impact of the particular shape of g. Let G be the cumulative distribution function associated to g and let $h = G$. In this case, the random variable $G(X_1)$ has a uniform distribution on $\mathcal{I} = [0, 1]$ and the distribution of the random vector $(G(X_1), G(X_2))$ is a copula (Nelsen, 2006), that is

$$(G(X_1), G(X_2)) \sim c(u, v) \qquad (u, v) \in \mathcal{I}^2,$$

where c denotes the copula density. Then, if (12) holds, Δ depends only on the copula of (X_1, X_2) and not on the marginal distribution of X_1. Such a result is particularly meaningful since it underlines that an invariant functional relies only on the dependence structure between X_1 and X_2 (that is, it depends only on the copula implicit in f). It is worthwhile to note that the dependence functional (5) is invariant (Tsallis, 1998). It follows that it is possible to express it in terms of the copula density:

$$\Delta_\gamma^c = \Delta_\gamma = \begin{cases} \dfrac{1}{1-\gamma} \displaystyle\int_{\mathcal{I}^2} \left[1 - \left(\dfrac{1}{c(u,u)}\right)^{1-\gamma}\right] c(u,v)dudv & \gamma \neq 1 \\ \displaystyle\int_{\mathcal{I}^2} \log\left[c(u,v)\right] c(u,v)dudv & \gamma = 1. \end{cases} \tag{13}$$

In order to avoid confusion, the superscript "c" will be used to denote the copula-based version of Δ. Although (6)-(10) do not satisfy (12), it is interesting to propose their copula-based version by substituting f with c and $g \cdot g$ with the independence copula density (the uniform distribution on $\mathcal{I}^2$). In particular, for (7)-(9), we obtain:

$$\Delta_{SD}^c = \int_{\mathcal{I}^2} \left(c(u,v) - 1\right)^2 dudv \tag{14}$$

$$\Delta_{ST}^c = \int_{\mathcal{I}^2} \left(c(u,v) - 1\right) c(u,v)dudv \tag{15}$$

$$\Delta_{2^*}^c = \int_{\mathcal{I}^2} \left(c(u,v) - 1\right)^2 c(u,v)dudv \tag{16}$$

Simple algebra leads to the equalities $\Delta_{SD}^c = \Delta_{ST}^c = \Delta_2^c = \Delta_2$. Furthermore, note that $\Delta_2^c = 3\Delta_4^c - 4\Delta_3^c$. Finally, as regards (6) and (10) we obtain:

$$\Delta_{L_1}^c = \int_{\mathcal{I}^2} \left|c(u,v) - 1\right| dudv \tag{17}$$

$$\Delta_{L_1^*}^c = \int_{\mathcal{I}^2} \left|c(u,v) - 1\right| c(u,v)dudv. \tag{18}$$

Note that all the copula-based dependence functionals (13)-(18) have the following general form

$$\Delta^c = \int_{\mathcal{I}^2} B\left\{c(u,v), 1, 1\right\} c(u,v)dudv = \int_{\mathcal{I}^2} B^c\left\{c(u,v)\right\} c(u,v)dudv. \tag{19}$$

3 Density Estimation Techniques

In order to estimate δ the estimators of the densities f, g and c must be defined. In the following paragraphs three (nonparametric) estimators for f and g commonly used in the context of testing serial independence are reviewed: the histogram-based, the Gaussian kernel, and the jackknife kernel. Regarding c, in the fourth paragraph the copula density estimator recently proposed by Kallenberg (2009) is outlined. As far as we know, this copula density estimator has never been used in the context of testing serial independence.

The vector $(X_1, \ldots, X_n)$ will denote the n-dimensional random vector describing the segment of the process $\{X_t\}_{t \in \mathbb{N}_+}$ from time 1 to time n. A particular realization of $(X_1,\dots,X_n)$ will be denoted by $(x_1,\dots,x_n)$. For a chosen value of the lag l, the bivariate density f_l will be estimated using the $n-l$ couples $(X_1,X_{l+1}),\dots,(X_{n-l},X_n)$. Finally, it is interesting to note that the number of sample observations for the estimation of g depends on the used methodology.

3.1 Histogram-based estimators

Once l is fixed, the most straightforward approach is to estimate f_l by histogram-based estimators. Let $\{C_i\}_{i=1}^k$ and $\{D_j\}_{j=1}^k$ be two generic sets of adjacent intervals so that $X_t \in \bigcup_{i=1}^k C_i$ for all $t=1,\dots,n-l$ and $X_t \in \bigcup_{j=1}^k D_j$ for all $t=l+1,\dots,n$. Starting from the sets of intervals defined above, all the couples $(X_1,X_{l+1}),\dots,(X_{n-l},X_n)$ can be classified obtaining the contingency table reported in Table 1 where

$$\widehat{n}_{ij} = \#\left\{(X_t,X_{t+l}) : (X_t,X_{t+l}) \in C_i \times D_j, t=1,\dots,n-l\right\}, \quad i,j=1,\dots,k.$$

Now, let L_{C_i} and L_{D_j} denote the length of the interval C_i and D_j, respectively.

Table 1: *Contingency table related to the l-th lag*

	D_1	$\cdots$	D_j	$\cdots$	D_k	
C_1	$\widehat{n}_{11}$	$\cdots$	$\widehat{n}_{1j}$	$\cdots$	$\widehat{n}_{1k}$	$\widehat{n}_{1\cdot}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
C_i	$\widehat{n}_{i1}$	$\cdots$	$\widehat{n}_{ij}$	$\cdots$	$\widehat{n}_{ik}$	$\widehat{n}_{i\cdot}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
C_k	$\widehat{n}_{k1}$	$\cdots$	$\widehat{n}_{kj}$	$\cdots$	$\widehat{n}_{kk}$	$\widehat{n}_{k\cdot}$
	$\widehat{n}_{\cdot 1}$	$\cdots$	$\widehat{n}_{\cdot j}$	$\cdots$	$\widehat{n}_{\cdot k}$	$n-l$

Starting from Table 1 it is possible to define the densities estimators

$$\begin{aligned} \widehat{f}_l(x,y) &= \frac{\widehat{p}_{ij}}{L_{C_i}L_{D_j}} && \text{if } (x,y)\in C_i\times D_j \\ \widehat{g}_1(x) &= \frac{\widehat{p}_{i\cdot}}{L_{C_i}} && \text{if } x\in C_i \\ \widehat{g}_2(y) &= \frac{\widehat{p}_{\cdot j}}{L_{D_j}} && \text{if } y\in D_j \end{aligned} \tag{20}$$

where

$$\widehat{p}_{ij} = \frac{\widehat{n}_{ij}}{n-l}, \quad \widehat{p}_{\cdot j} = \frac{\widehat{n}_{\cdot j}}{n-l}, \quad \widehat{p}_{i\cdot} = \frac{\widehat{n}_{i\cdot}}{n-l}, \quad i,j=1,\dots,k. \tag{21}$$

Note that the estimate of g may vary between the two marginals in the same table, and between the same marginal referred to different lags l, $l=1,\dots,p$.

Usually, once k is fixed, the sets of intervals $\{C_i\}_{i=1}^k$ and $\{D_j\}_{j=1}^k$ are defined by setting

$$\begin{aligned} \inf\{C_1\} &= \min\{X_1,\dots,X_{n-l}\}, \quad \sup\{C_k\} = \max\{X_1,\dots,X_{n-l}\}, \\ \inf\{D_1\} &= \min\{X_{l+1},\dots,X_n\}, \quad \sup\{D_k\} = \max\{X_{l+1},\dots,X_n\}, \end{aligned} \tag{22}$$

and according to the following relations

$$L_{C_i} = L_{C_j} \quad \text{and} \quad L_{D_i} = L_{D_j} \quad \forall\, i, j = 1, \dots, k. \tag{23}$$

Note that in this way C_i and D_j (and then also their lengths) are random and, for a fixed realization $(x_1, \dots, x_n)$, they depend on the lag taken into account. The estimators (20), obtained via (22) and (23), are known as "Histogram-based with Equi-Distant cells" estimators, hereafter simply denoted as HED. An example of HED estimator, used for checking serial independence, can be found in Chan and Tran (1992) who consider the L_1-distance.

A different technique to build Table 1 is known as marginal equiquantization (Dionísio et al, 2006). In this case the classes $\{C_i\}_{i=1}^{k}$ and $\{D_j\}_{j=1}^{k}$ are defined so that they satisfy the following conditions:

$$\widehat{n}_{i\cdot} = \widehat{n}_{j\cdot} \quad \text{and} \quad \widehat{n}_{\cdot i} = \widehat{n}_{\cdot j} \quad \forall\, i, j = 1, \dots, k. \tag{24}$$

From now on these estimators will be referred to simply as HEF, "Histogram-based with Equi-Frequent cells". This technique is also adopted in Bagnato and Punzo (2010, 2011). Note that HEF can be interpreted as a histogram-based copula density estimator.

3.2 Gaussian kernel density estimator

The Gaussian Kernel (GK) density estimator is used in the context of serial dependence by several authors such as Robinson (1991), Tjøstheim (1996), Granger et al (2004), and Fernandes and Néri (2010). The GK estimator for the univariate density g is:

$$\widehat{g}(x) = \frac{1}{n} \sum_{i=1}^{n} K_h\left(x; X_i\right), \tag{25}$$

where $K_h\left(x; X_i\right) = \left(2\pi h^2\right)^{-1/2} \exp\left\{-\frac{1}{2}\left[h^{-1}\left(x - X_i\right)\right]^2\right\}$ and $h > 0$ is the bandwidth. Similarly, the GK estimator for the bivariate density f_l is:

$$\widehat{f}_l(x, y) = \frac{1}{n - l} \sum_{i=1}^{n-l} K_h\left(x; X_i\right) K_h\left(y; X_{i+l}\right), \tag{26}$$

where K_h is defined as in (25). Note that for simplicity the bivariate kernel function in (26) is the product of two one-dimensional Gaussian kernels with equal bandwidths. In order to find h, the likelihood cross-validation (see, e.g., Silverman, 1986, p. 52) which produces "optimal" density estimators according to the Kullback-Leibler criterion will be adopted (Granger et al, 2004, p. 654). After finding the bandwidth in (25), we use this value also in (26). This choice allows coherence between the two estimators $\widehat{g}$ and $\widehat{f}_l$. In fact, integrating (26) with respect to x or y, a unidimensional density almost equal to $\widehat{g}$ is obtained. The term "almost" is due to the fact that in (26) l observations are used less than in (25).

3.3 Jackknife Kernel density estimator

The boundedness of the support of f_l and g is a key assumption for the asymptotic theory concerning several serial independence tests such as those in Robinson (1991), Hong and White (2005) and Fernandes and Néri (2010). In order to estimate a density on a compact support, a particular kernel estimator is needed since common estimators, such as the GK estimator, suffer from the so called *boundary effect* (see Härdle 1992 for details on this topic). This effect can be avoided, for example, by using the jackknife kernel density estimator introduced by Rice (1984) in the context of kernel regression. We refer to Hong and White (2005) which employs this estimator in the context of testing serial independence. A brief and non-technical illustration of this methodology is reported below.

Assume $\mathcal{S} \equiv \mathcal{I}$. Thus, the bivariate density f_l is defined on $\mathcal{S}^2 \equiv \mathcal{I}^2$. Let $k(x)$ denote the well-known *quartic kernel*:

$$k(x) = \frac{15}{16}(1-x^2)^2 \mathbb{1}(|x|<1)$$

where $\mathbb{1}(\cdot)$ is the indicator function. The quartic kernel is the building block in order to define the following jackknife kernel:

$$k_b(x) = (1+r)\frac{k(x)}{\omega_k(0,b)} - \frac{r}{(2-b)}\frac{k(x/(2-b))}{\omega_k(0,b/(2-b))}, \tag{27}$$

where $\omega_k(l,b) = \int_{-b}^{1} x^l k(x)dx$ for $l = 0, 1$, and

$$r = \frac{\omega_k(1,b)/\omega_k(0,b)}{(2-b)\omega_k(1,b/(2-b))/\omega_k(0,b/(2-b)) - \omega_k(1,b)/\omega_k(0,b)}.$$

The global kernel function is defined as follows:

$$K_h(x,y) = \begin{cases} h^{-1}k_{(x/h)}\left(\dfrac{x-y}{h}\right) & \text{if } x \in [0,h] \\ h^{-1}k\left(\dfrac{x-y}{h}\right) & \text{if } x \in (h, 1-h) \\ h^{-1}k_{[(1-x)/h]}\left(\dfrac{x-y}{h}\right) & \text{if } x \in [1-h, 1]. \end{cases} \tag{28}$$

Note that in the interior region $(h, 1-h)$, K_h uses the simple quartic kernel, while in the boundary region $[0,h] \cup [1-h,h]$ it adopts the modified quartic kernel (27). In this way the boundary effect is avoided.

In order to estimate the densities g and f_l it is possible to use (25) and (26) with the kernel defined as in (28). As regards the estimation only over the observed values, the following kernel-based "leave-one-out" density estimators can be used:

$$\widehat{g}(X_j) = \frac{1}{n-1}\sum_{i=1}^{n} K_h(X_j, X_i)\,\mathbb{1}(i \neq j), \tag{29}$$

$$\widehat{f}_l(X_j, X_{j+l}) = \frac{1}{n-l-1}\sum_{i=1}^{n-l} K_h(X_j, X_i) K_h(X_{j+l}, X_{i+l}) \mathbb{1}(i \neq j). \tag{30}$$

As for the GK estimator, the problem of choosing h is limited only to $\widehat{g}$; the obtained value is then used also for $\widehat{f}_l$. In the following, the asymptotically bandwidth that yields to the optimal convergence rate for the Kullback-Leibler information criterion derived in Hong and White (2005) is estimated through:

$$h = 2.0236\, n^{-1/5} \left\{ \frac{1}{n} \sum_{i \in \tilde{T}} \left[\frac{\tilde{g}^{(2)}(X_i)}{\tilde{g}(X_i)} \right]^2 \right\}^{-1/5}$$

where $\tilde{g}$ and $\tilde{g}^{(2)}$ are preliminary estimators for g and the second order derivative of g, respectively. In detail, $\tilde{T} = \{1 \leq i \leq n : X_i \in [h_0, 1-h_0]\}$ and $\tilde{g}$ is obtained from (29) using $h_0 = \widehat{\sigma} n^{-1/6}$, where $\widehat{\sigma}$ is the sample standard deviation. In order to obtain $\tilde{g}^{(2)}$, the following naive estimator is adopted:

$$\tilde{g}^{(2)}(X_i) = \frac{2\tilde{g}(X_i) - \tilde{g}(X_i - \epsilon) - \tilde{g}(X_i + \epsilon)}{\epsilon^2} \quad \text{with} \quad \epsilon = \frac{h_0}{100}.$$

It is important to note that the assumption related to the boundedness of the support is not too restrictive since it can be ensured by transforming the original time series $(Y_1, \ldots, Y_n)$ with a strictly monotonic function like the logistic one

$$X_t = \frac{1}{1 + \exp(-Y_t)}, \tag{31}$$

or the relative rank function

$$X_t = \frac{\text{rank}\,(Y_t)}{n}. \tag{32}$$

The resulting density estimators will be simply referred to as JKL and JKR: "Jakknife Kernel with Logistic transformation" and "Jakknife Kernel with Rank transformation", respectively. While the JKL estimator was adopted by Hong and White (2005), here the use of the jackknife kernel estimator will be extended to rank transformation. As for HEF, JKR can be interpreted as copula density estimator.

3.4 Kallenberg's copula density estimator

Recently, Kallenberg (2009) proposed an estimator for c based on a Legendre's polynomials approximation. Let c_0 be a *starting copula* and let b_r be the r-th normalized Legendre polynomial on $(0,1)$ (see Szegö, 1959). Assume both c and c_0 belonging to L_2, that is: $\|c\|_2 < \infty$and $\|c_0\|_2 < \infty$ where

$$\|c\|_2 = \left\{ \int_{\mathcal{I}^2} c(u,v)^2 du dv \right\}^{1/2}$$

denotes the L_2-norm of c. Denote the L_2-inner product of c and c' by

$$\langle c, c' \rangle = \int_{\mathcal{I}^2} c(u,v) c'(u,v) du dv.$$

Since the normalized Legendre polynomials on $(0,1)$ are orthogonal with respect to the L_2-inner product and they have unitary L_2-norm, the following orthogonal series representation can be introduced:

$$c(u,v) - c_0(u,v) = \sum_{r,s} \gamma_{rs} b_r(u) b_s(v), \tag{33}$$

where the coefficients of the series are given by

$$\gamma_{rs} = \langle c - c_0, b_r b_s \rangle = \rho_c(b_r(U), b_s(V)) - \rho_{c_0}(b_r(U), b_s(V)).$$

In the previous formula $\rho_c(b_r(U), b_s(V))$ denotes the correlation coefficient of $b_r(U)$ and $b_s(V)$ under c. Since the orthogonal representation (33) depends on infinite unknown parameters, it is not directly useful in order to build a copula density estimator. A possible solution is to approximate the series in (33) by a finite sum, obtaining

$$c(u,v) \approx c_0(u,v) + \sum_{j=1}^{\kappa} \gamma_{r_j s_j} b_{r_j}(u) b_{s_j}(v), \tag{34}$$

where κ is an integer and $(r_1, \ldots, r_\kappa)$, $(s_1, \ldots, s_\kappa)$ denote two sets of indexes identifying the order of the maintained Legendre polynomials.

In order to use expression (34), the coefficients $\gamma_{r_j s_j}$ need to be estimated. Furthermore, it is necessary to define an “optimal” rule to choose κ and the couples of indexes (r_j, s_j), $j = 1, \ldots, \kappa$, of the maintained Legendre's polynomials. Let $(U_1, V_1), \ldots, (U_n, V_n)$ be a random sample drawn from c. The coefficients γ_{rs} can be easily estimated by their sample counterparts

$$\widehat{\gamma}_{rs} = \frac{1}{n} \sum_{i=1}^{n} b_r(U_i) b_s(V_i) - E_{c_0}\left[b_r(U) b_s(V) \right] \ .$$

Regarding the rule proposed by Kallenberg (2009) to select κ and (r_j, s_j), let $m_n : \mathbb{N}_+ \to \mathbb{N}_+$ be a function of the sample size n defining the largest dimension for r and s. The value assumed by m_n restricts the possible values of κ and the possible choices of (r_j, s_j): $0 \leq \kappa \leq m_n^2$ and $(r_j, s_j) \in \mathcal{A}$ for all $j = 1, \ldots, \kappa$ where $\mathcal{A} = \{(w,z) : w, z = 1, \ldots, m_n\}$. Now, let $\{\widehat{\gamma}_{wz}, (w,z) \in \mathcal{A}\}$ be the set of the estimated coefficients of order belonging to $\mathcal{A}$ and place them in descending order

$$|\widehat{\gamma}_{R_1 S_1}| \geq \ldots \geq \left|\widehat{\gamma}_{R_{m_n} S_{m_n}}\right|. \tag{35}$$

In (35) the indexes R_j and S_j are capitalized since they are random variables. Now, the selection rule can be expressed as follows:

$$\widehat{\kappa} = \begin{cases} 0 & \text{if } \widehat{\gamma}^2_{R_1 S_1} < \Delta_n \\ \max\left\{1 \leq \kappa \leq m_n^2 : \widehat{\gamma}^2_{R_k S_k} \geq \Delta_n \right\} & \text{otherwise} \end{cases} \tag{36}$$

and the Kallenberg Copula density (KC) estimator is

$$\widehat{c}(u,v) = c_0(u,v) + \sum_{j=1}^{\widehat{\kappa}} \widehat{\gamma}_{R_j S_j} b_{R_j}(u) b_{S_j}(v). \tag{37}$$

Following Kallenberg (2009) the independence copula is chosen as the starting one $(c_0(u,v) = 1, (u,v) \in \mathcal{I}^2)$ and

$$m_n = \lfloor \log(n) \rfloor \qquad \text{and} \qquad \Delta_n = \frac{\log(n)\log(m_n)}{n},$$

with $\lfloor \cdot \rfloor$ denoting the integer part. Note that in this case $\rho_{c_0}(b_r(U), b_s(V)) = 0$.

4 Estimation of the dependence functionals and computation of their critical values

The density estimators described in the previous section can be employed in order to estimate the dependence functionals which allow to built serial independence tests. In section 4.1 the estimators of the dependence functionals will be introduced while in Section 4.2 instructions will be given on how to use them in order to test (2) and (3).

4.1 Estimators of the dependence functionals

With the aim to estimate Δ, it is natural to plug the estimators (HED, HEF, GK, JKL and JKR) for g and f into expression (4), obtaining the "Integrated" (I) estimator

$$ {}_I\widehat{\Delta} = \int_{\mathcal{S}^2} B\{\widehat{f}(x,y), \widehat{g}(x), \widehat{g}(y)\}\widehat{f}(x,y)dxdy, \tag{38} $$

which can be computed by numerical integration (like in Granger et al, 2004) or well approximated by a summation over a sufficiently fine grid of values. Since (38) is inconvenient for calculation, usually the "Summed" (S) estimator is used:

$$ {}_S\widehat{\Delta} = \frac{1}{n-l}\sum_{t\in S_t} B\{\widehat{f}(X_t, X_{t+l}), \widehat{g}(X_t), \widehat{g}(X_{t+l})\}, \tag{39} $$

where $S_t = \left\{1 \le t \le n-l : \widehat{f}(X_t, X_{t+l}) > 0, \widehat{g}(X_t) > 0, \widehat{g}(X_{t+l}) > 0\right\}$.

If HED is used, then (38) and (39) coincide:

$$ \begin{aligned} {}_I\widehat{\Delta} &= \int_{\mathcal{S}^2} B\{\widehat{f}(x,y), \widehat{g}_1(x), \widehat{g}_2(y)\}\widehat{f}(x,y)dxdy \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k}\int_{C_i\times D_j} B\{\widehat{f}(x,y), \widehat{g}_1(x), \widehat{g}_2(y)\}\widehat{f}(x,y)dxdy \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} B\left\{\frac{\widehat{p}_{ij}}{L_{C_i}L_{D_j}}, \frac{\widehat{p}_{i\cdot}}{L_{C_i}}, \frac{\widehat{p}_{\cdot j}}{L_{D_j}}\right\}\widehat{p}_{ij} \\ &= \frac{1}{n-l}\sum_{i=1}^{k}\sum_{j=1}^{k} B\left\{\frac{\widehat{p}_{ij}}{L_{C_i}L_{D_j}}, \frac{\widehat{p}_{i\cdot}}{L_{C_i}}, \frac{\widehat{p}_{\cdot j}}{L_{D_j}}\right\}\widehat{n}_{ij} \\ &= {}_S\widehat{\Delta}. \end{aligned} $$

Moreover, it is easy to prove that

$$ {}_I\widehat{\Delta} \propto \sum_{i=1}^{k}\sum_{j=1}^{k} B\{\widehat{p}_{ij}, \widehat{p}_{i\cdot}, \widehat{p}_{\cdot j}\}\widehat{p}_{ij}, \tag{40} $$

where "$\propto$" stands for "proportional to". Proportionality becomes equality for the functionals (5) and (6).

The integral (38) is approximated starting from a 100×100 grid of equally spaced values $\{(\tilde{x}_i, \tilde{y}_j); i, j = 1, \dots, 100\}$, obtaining the following estimator:

$$_I\widehat{\Delta} = 10^{-4} \sum_{i=1}^{100} \sum_{j=1}^{100} B\{\widehat{f}(\tilde{x}_i, \tilde{y}_j), \widehat{g}(\tilde{x}_i), \widehat{g}(\tilde{y}_j)\} \widehat{f}(\tilde{x}_i, \tilde{y}_j) \mathbb{1}\left((\tilde{x}_i, \tilde{y}_j) \in S^*\right), \qquad (41)$$

where $(\widehat{f}, \widehat{g})$ are obtained through GK, JKR, or JKL, and

$$S^* = \left\{(\tilde{x}_i, \tilde{y}_j) : \widehat{f}(\tilde{x}_i, \tilde{y}_j) > 0, \widehat{g}(\tilde{x}_i) > 0, \widehat{g}(\tilde{y}_j) > 0\right\}.$$

Concerning the GK estimator, the default settings of the `R` package `sm` are followed:

$$\tilde{x}_i = \left(x_{(1)} - a\right) + (i - 1) \frac{x_{(n)} - x_{(1)} + 2a}{99},$$

with $a = \left(x_{(n)} - x_{(1)}\right)/4$ and $x_{(1)}$ $(x_{(n)})$ denoting the minimum (maximum) observed value. The grid for the y's is exactly the same. For JKR and JKL $\tilde{x}_{(i)} = \tilde{y}_{(i)} = i/101$ are set where $i = 1, \dots, 100$. Note that when GK is used in (41), it results that $S^* \equiv \{(\tilde{x}_i, \tilde{y}_j); i, j = 1, \dots, 100\}$ since, by construction, $\widehat{f}(\tilde{x}_i, \tilde{y}_j)$, $\widehat{g}(\tilde{x}_i)$, and $\widehat{g}(\tilde{y}_j)$ are strictly positive for all $i, j = 1, \dots, 100$. On the contrary using JKL or JKR, which do not provide proper density estimates, it can happen that $S^* \subset \{(\tilde{x}_i, \tilde{y}_j); i, j = 1, \dots, 100\}$.

Concerning the summed estimator (39), both the GK estimators (25) and (26), and the leave-one-out JKL and JKR estimators (29) and (30) are used. When GK is used, it turns out that $S_t \equiv \{1, \dots, n - l\}$. When the leave-one-out JKL or JKR are employed, $S_t \subset \{1, \dots, n - l\}$ may occour.

While (39) has been widely studied in the context of serial independence (see, e.g., Robinson, 1991; Skaug and Tjøstheim, 1996), the estimator (38) was studied only by Granger et al (2004). To our knowledge, no comparison between the performance of these two approaches has been performed.

The copula based dependence functionals are estimated using KC, obtaining:

$$_I\widehat{\Delta}^c = \int_{\mathcal{I}^2} B^c\left\{\widehat{c}(u, v)\right\} \widehat{c}(u, v) du dv. \qquad (42)$$

and

$$_S\widehat{\Delta}^c = \frac{1}{n - l} \sum_{t \in S_t^c} B^c\left\{\widehat{c}\left(X_t, X_{t+l}\right)\right\}, \qquad (43)$$

where $S_t^c = \{1 \leq t \leq (n - l) : \widehat{c}(X_t, X_{t+l}) > 0\}$. The integral in (42) is approximated by the following double summation:

$$_I\widehat{\Delta}^c = 10^{-4} \sum_{i=1}^{100} \sum_{j=1}^{100} B^c\left\{\widehat{c}\left(\frac{i}{100}, \frac{j}{100}\right)\right\} \widehat{c}\left(\frac{i}{100}, \frac{j}{100}\right) \mathbb{1}\left((i, j) \in S^*\right), \qquad (44)$$

where

$$S^* = \left\{(i, j) : \widehat{f}\left(\frac{i}{100}, \frac{j}{100}\right) > 0, \widehat{g}\left(\frac{i}{100}\right) > 0, \widehat{g}\left(\frac{j}{100}\right) > 0\right\}.$$

Note that, even if JKR and HEF can be interpreted as copula density estimators, they are not used in (42) and (43) since they provides estimates with marginals only approximately uniform.

4.2 Computing critical values for single and multiple-lag testing procedures

As observed in Section 2, the greater the value of the dependence functional, the stronger the dependence between the two variables taken into account. So, concerning the single-lag testing problem, intuition suggests that the null hypothesis of serial independence should be rejected for large values of $\widehat{\Delta}$. The following critical function is introduced:

$$\Psi(X_1, \ldots, X_n) = \begin{cases} 1 & \text{if } \widehat{\Delta} > d_\alpha \\ 0 & \text{otherwise} \end{cases}, \tag{45}$$

where d_α represents the critical value at the preassigned level α. Among the various proposals, a widespread approach to define d_α is to use an asymptotic (usually Gaussian) approximation of the test statistic distribution. This approach is followed by Robinson (1991), Hong and White (2005), and Fernandes and Néri (2010), who adopt ${}_S\widehat{\Delta}$ or its "trimmed" version. In particular, to prove the asymptotic normality of ${}_S\widehat{\Delta}$, several regularity conditions have to be assumed. Among them, the compactness of $\mathcal{S}$ plays a crucial role. The latter is usually assured by means of a weight function – usually referred to as *trimming function* – which selects only a compact set $\mathcal{C} \subseteq \mathcal{S}$. This solution is followed, among others, by Fernandes and Néri (2010) and it leads to the trimmed estimator

$${}_w\widehat{\Delta} = \frac{1}{n-l} \sum_{t \in S_t} B\{\widehat{f}(X_t, X_{t+l}), \widehat{g}(X_t), \widehat{g}(X_{t+l})\} w(X_t, X_{t+l}), \tag{46}$$

where the trimming function is usually defined as $w(x, y) = \mathbb{1}((x, y) \in \mathcal{C} \times \mathcal{C})$.

Hong and White (2005) proposed an alternative solution which, as shown in Section 3.3, is based on a preliminary data transformation such as those in (31) or (32). However, as observed by Skaug and Tjøstheim (1993a,b), Skaug (1994), and Diks (2009), the asymptotic Gaussian approximation is generally inappropriate in finite samples because of the slow convergence to the normal distribution. To avoid this problem, a permutation or a bootstrap approach is usually preferred.

The permutation approach exploits the fact that, conditionally on the observed data $x_1, \ldots, x_n$, each of the $n!$ permutations is equally likely under the null hypothesis of serial independence. In detail, let $\widehat{\Delta}_{(0)}$ denote the value assumed by $\widehat{\Delta}$ for the observed data. Analogously, let $\widehat{\Delta}_{(b)}$ denote the dependence functional estimate obtained from a random permutation of the original data, with $b = 1, \ldots, B$. Under the null hypothesis of serial independence, the B values $\widehat{\Delta}_{(1)}, \ldots, \widehat{\Delta}_{(B)}$ are equally likely and an exact p-value is calculated as

$$\widehat{p} = \frac{\#\{\widehat{\Delta}_{(s)} : \widehat{\Delta}_{(s)} > \widehat{\Delta}_{(0)}\,; s = 0, 1, \ldots, B\}}{B+1}. \tag{47}$$

Thus, the null hypothesis is rejected if $\widehat{p} < \alpha$.

Similarly, naive and smoothed bootstrap procedures can be employed drawing B bootstrap samples $(X^*_{1b}, \ldots, X^*_{nb})$, $b = 1, \ldots, B$, from the original data (naive bootstrap) or from the estimated density $\widehat{g}$ (smoothed bootstrap) and computing $\widehat{\Delta}_{(b)}$ for each of the bootstrap samples. The bootstrap p-value is calculated as in (47).

The multiple-lag testing problem (2) can be handled in several ways. The natural one consists in extending the procedure followed for the single-lag testing problem as in Robinson (1991). The dependence functionals defined in Section 2 can be generalized to the p-variate case as $\Delta(f(x_1,\ldots,x_p); g(x_1)\ldots g(x_p))$. Then Δ can be estimated starting from the estimates of g and of the p-variate density f. The resulting estimators can be used to test (2) adopting some asymptotic results or following the permutation/bootstrap approach above described. As noted, among others, in Robinson (1991), this approach is of little practical interest because the quality of the density estimates $\widehat{f}$ deteriorates rapidly as p increases for the so called "curse of dimensionality" (see Bellman, 1961). An alternative procedure, which is often followed in the literature, consists in building a "Portmanteau"-test – hereafter P-test – starting from p single-lag test statistics. Let $\Delta(l)$ be the dependence functional associated to the l-th lag and let $\widehat{\Delta}(l)$ denote its estimator. In formula, the P-test statistic can be defined as

$$\widehat{Q}(p) = \sum_{l=1}^{p} \widehat{\Delta}(l). \tag{48}$$

The asymptotic theory necessary to test hypotheses (2) using $\widehat{Q}(p)$ is developed, for example, in Hong and White (2005) for $\widehat{\Delta}_1$ using JKL density estimates. See also Skaug and Tjøstheim (1993a) for similar results. Since, also in this context, the asymptotic approximations give poor results, usually a permutation/bootstrap approach is adopted. Although the "Portmanteau" approach is one of the most frequently followed, a simpler solution consists in using a Simultaneous test – hereafter S-test – which, through the Bonferroni inequality, leads to the following decision rule: do not reject H_0 if all the p single-lag tests do not reject at level α/p.

Finally, a further approach is illustrated here stemming from the *multiple bandwidth procedure* proposed in Diks and Panchenko (2007). This procedure was used by the authors in order to mitigate the effect of the bandwidth choice on the performance of their serial independence test. Here, this procedure is contextualized by building a *Multiple-lag Permutation* test (hereafter MlP-test):

1. calculate the estimates $\widehat{\Delta}_{(0)}(l)$, $l = 1,\ldots,p$, using the original data;
2. randomly permute B times the data and build the matrix $\underset{B\times p}{\widetilde{\boldsymbol{B}}}$ with elements $\widehat{\Delta}_{(b)}(l)$, $b = 1,\ldots,B$ and $l = 1,\ldots,p$;
3. define the matrix $\underset{(B+1)\times p}{\boldsymbol{B}}$ obtained attaching the vector $\left(\widehat{\Delta}_{(0)}(1),\ldots,\widehat{\Delta}_{(0)}(p)\right)$ as first row to $\widetilde{\boldsymbol{B}}$;
4. transform $\boldsymbol{B}$ into the matrix $\underset{(B+1)\times p}{\boldsymbol{P}}$ of p-values with elements:
$$\widehat{p}_i(l) = \frac{\#\left\{\widehat{\Delta}_{(s)}(l) : \widehat{\Delta}_{(s)}(l) > \widehat{\Delta}_{(i)}(l); s = 0,\ldots,B\right\}}{B+1} \quad i = 0,\ldots,B;$$
5. for each row of $\boldsymbol{P}$ select the smallest p-value:
$$\widehat{T}_i = \min_{l\in\{1,\ldots,p\}} \widehat{p}_i(l) \quad i = 0,\ldots,B;$$

6. use $\widehat{T}$ as test statistic interpreting $\widehat{T}_0$ as its observed value and $\widehat{T}_1, \ldots, \widehat{T}_B$ as the values associated with each permutation. Then, calculate the "overall" p-value
$$\widehat{p}_{MIP} = \frac{\#\left\{\widehat{T}_s(l) : \widehat{T}_s(l) > \widehat{T}_0(l); s = 0, \ldots, B\right\}}{B+1}. \tag{49}$$

5 Design of the Simulation Study

In this section the performance of the serial independence tests, so far considered, is evaluated via an extensive simulation study. The factors outlined in Table 2 are investigated under the well-known and broadly used models specified in Table 3. Apart from models M1 and M12-M14, the strength of the serial dependence, and its relationship with the lags, is governed by one or more parameters ϑ and/or by the value of q (order of the respective model).

Table 2: *Considered factors.*

Identifier	Factors	Possibilities
F1	technique used to compute p-values	(nB) naive Bootstrap (sB) smoothed Bootstrap (P) Permutation
F2	use of trimming functions	yes no
F3	density estimation technique	(HED) Histogram Equi-Distant cells (HEF) Histogram Equi-Frequent cells (GK) Gaussian Kernel (JKL) Jackknife Kernel Logistic (JKR) Jackknife Kernel Rank (KC) Kallenberg's Copula
F4	methodology adopted to compute Δ	(I) Integrated (S) Summed
F5	divergence measure	$\Delta_{1/2}$ Δ_1 Δ_2 Δ_3 Δ_4 Δ_{L_1} Δ_{SD} Δ_{ST} $\Delta_{L_1^*}$ Δ_{2^*}

The dependence structure of each model can be summarized as follows. Model M1 reproduces serial independence and it is taken into account to examine the actual level of the test. On the contrary, models M2-M14 cover a variety of commonly used linear and nonlinear time-series processes which are useful to evaluate and compare the power of the tests. In particular, models M2-M3 are linear, that is, characterized only by serial correlation governed by ϑ_j, $j = 1, \ldots, q$, with M3 being correlated only for the lags for which $\vartheta_j > 0$. Model M4, extensively used in the simulations of Granger and Lin (1994), has zero-correlation but has a quadratic form of dependence in correspondence to the lags for which $\vartheta_j > 0$. Consequently, for models M3-M4, a serial independence test is expected to have a power equal to the level α for the lags j for which $\vartheta_j = 0$. The nonlinear model M5, adopted for example by Fernandes and Néri (2010) and Diks (2009), is a challenging alternative in that it involves a particular form of multiplicative serial dependence in

Table 3: *Adopted models.*

Model	Name	Equation
M1	*i.i.d.*	$X_t = \varepsilon_t$
M2	AR(q)	$X_t = \sum_{j=1}^{q} \vartheta_j X_{t-j} + \varepsilon_t$
M3	MA(q)	$X_t = \sum_{j=1}^{q} \vartheta_j \varepsilon_{t-j} + \varepsilon_t$
M4	Quadratic MA(q)	$X_t = \sum_{j=1}^{q} \vartheta_j \varepsilon_{t-j}^2 + \varepsilon_t$
M5	Multiplicative MA(q)	$X_t = \sum_{j=1}^{q} \vartheta_j \varepsilon_{t-j}\varepsilon_{t-j-1} + \varepsilon_t$
M6	Threshold AR(1)	$X_t = \vartheta X_{t-1}\left[-.5 + .9\ \mathbb{1}\left(X_{t-1} \geq 0\right)\right]$
M7	Bilinear AR(q)	$X_t = \sum_{j=1}^{q} \vartheta_j X_{t-j}\varepsilon_{t-j} + \varepsilon_t$
M8	Sign AR(q)	$X_t = \mathrm{sign}\left(X_{t-q}\right) + (1-\vartheta)\,\varepsilon_t$
M9	Fractional AR(q)	$X_t = \vartheta \left\|X_{t-q}\right\|^{1/2} + \varepsilon_t$
M10	ARCH(q)	$X_t = \sigma_t\varepsilon_t, \quad \sigma_t^2 = .01 + \sum_{j=1}^{q} \vartheta_j X_{t-j}^2$
M11	GARCH(1,q)	$X_t = \sigma_t\varepsilon_t, \quad \sigma_t^2 = .01 + .125X_{t-1}^2 + \sum_{j=1}^{q} \vartheta_j \sigma_{t-1}^2$
M12	E-GARCH(1,1)	$X_t = \sigma_t\varepsilon_t, \quad \log\left(\sigma_t^2\right) = .01 + .7\log\left(\sigma_{t-1}^2\right) - .3\varepsilon_{t-1} + .7\left(\left\|\varepsilon_{t-1}\right\| - E\left\|\varepsilon_{t-1}\right\|\right)$
M13	Threshold GARCH(1,1)	$X_t = \sigma_t\varepsilon_t, \quad \sigma_t^2 = .25 + .6\sigma_{t-1}^2 + .5X_{t-1}^2\mathbb{1}\left(\varepsilon_{t-1} < 0\right) + .2X_{t-1}^2\mathbb{1}\left(\varepsilon_{t-1} \geq 0\right)$
M14	GARCH(1,1)-M	$X_t = .003 + 2\sigma_t^2 + a_t, \quad a_t = \left(.0002 + .13a_{t-1}^2 + .81\sigma_{t-1}^2\right)\varepsilon_t$

correspondence to the lags j and $j+1$ with $\vartheta_j > 0$. Model M6, and models M7-M9, considered by several authors (see, e.g., Hong and White, 2005; Genest et al, 2007; Diks, 2009; Fernandes and Néri, 2010), are respectively a nonlinear variant of the classical AR(1) and of the AR(q) with various decaying memory properties. Models M10-M11 are, respectively, the well-known ARCH and GARCH models commonly employed in financial applications; they, like model M4, are only characterized by a (nonlinear) quadratic form of dependence but, differently from M4, they have a decaying memory structure. Models M12-M14 are variants of M11 but, while M12-M13 are only characterized by nonlinear dependence, M14 adds a further structure of serial correlation (see Tsay, 2002, p. 101).

Since the number of possible combinations among the factors in Table 2 is huge, F1 and F2 (Section 5.1) are investigate before proceeding with a simultaneous analysis on F3-F5 (Section 5.2).

All the simulations are performed using a level $\alpha = 0.05$, a sample size $n = 100$, and $M = 1000$ replications. For each of them, a time series of length 200 is initially generated, but with only the final 100 observations used in order to mitigate the impact of initial values. Moreover, $B = 100$ bootstrap/permutation samples, will always be considered. The entire code is written in the `R` environment and it is available from the authors upon request.

5.1 Screening

This section presents the design and the results of some preliminary experiments related to F1 and F2 in Table 2. First, the performances of naive bootstrap (nB), smoothed bootstrap (sB), and Permutation (P) approaches to compute p-values are compared. Second, using the approach that works best for F1, the effect of considering "trimmed" versions of the dependence functionals is investigated.

Below, the eight models M1-M5, M7-M8, and M11, with standard Gaussian noise ε_t, are considered. An order $q = 1$, and a value $\vartheta_1 = \vartheta = 0.25$, are chosen. Comparison is exclusively focused on the first lag ($p = 1$) and the densities are estimated using the GK technique.

Regarding F1, Table 4 shows some well-known summary statistics computed on simulated rejection rates for each considered model and for each technique chosen to derive p-values (nB, sB and P). Each value in Table 4 is computed on the $2 \cdot 10 = 20$ simulated rejection rates corresponding to all the combinations of F4 and F5.

Summary statistic	F1	M1	M2	M3	M4	M5	M7	M8	M11
Mean	nB	**0.048**	0.223	0.199	0.338	0.088	0.088	**0.965**	0.137
	sB	0.037	0.192	0.190	0.290	0.067	0.069	0.913	0.103
	P	**0.052**	**0.254**	**0.238**	**0.383**	**0.094**	**0.095**	0.948	**0.146**
Median	nB	0.053	0.211	0.181	**0.387**	0.076	0.081	0.991	0.124
	sB	0.038	0.191	0.186	0.280	0.070	0.067	0.992	0.101
	P	**0.049**	**0.256**	**0.238**	0.385	**0.097**	**0.103**	**0.996**	**0.170**
Std. Dev.	nB	0.024	0.083	0.077	0.166	0.051	0.049	0.081	0.082
	sB	0.020	0.100	0.098	0.127	0.034	0.032	0.210	0.057
	P	0.008	0.076	0.077	0.115	0.023	0.020	0.125	0.047
Min	nB	0.003	0.034	0.034	0.036	0.011	0.005	0.649	0.013
	sB	0.002	0.054	0.056	0.081	0.006	0.014	0.264	0.015
	P	0.041	0.121	0.107	0.223	0.058	0.063	0.468	0.070
Max	nB	0.077	0.341	0.320	0.578	0.160	0.153	1.000	0.264
	sB	0.070	0.390	0.388	0.550	0.129	0.119	1.000	0.222
	P	0.072	0.430	0.420	0.573	0.128	0.126	1.000	0.207
Q_1	nB	0.027	0.166	0.142	0.205	0.053	0.053	0.982	0.066
	sB	0.028	0.097	0.099	0.228	0.042	0.048	0.980	0.068
	P	0.047	0.200	0.181	0.257	0.070	0.072	0.988	0.102
Q_3	nB	0.068	0.307	0.266	0.471	0.141	0.139	0.998	0.213
	sB	0.049	0.254	0.257	0.363	0.083	0.091	0.997	0.141
	P	0.060	0.293	0.271	0.464	0.113	0.111	0.998	0.183

Table 4: Summary statistics of the simulated rejection rates obtained in the study for F1. Numbers in bold, only related to mean and median, highlight the approach that works best for each model.

In this scenario, in terms of size (consider model M1) and referring to mean and median, nB and P seem to be the best performers. However, adding the information provided by the other summary statistics – which, roughly speaking, give information about the dispersion of the simulated rejection rates – the permutation approach clearly appears the be the best solution. In terms of power (refer to models M2-M5, M7-M8, and M11), the permutation approach is easily seen to outperform the bootstrap ones with reference to mean and median (see the elements in bold in the last seven columns of Table 4) and this happens for six out of the seven models for both the indicators; furthermore, there is a very slight difference in performance in the two cases in which it does not happen. The other summary statistics confirm this conjecture. In conclusion, the permutation approach is, without doubt, the approach that works best; this is the reason why it will be adopted from now on.

Regarding F2, two different trimming functions are adopted in (46). These two functions, denoted with ω_1 and ω_2 and also considered by Fernandes and Néri (2010, p. 287), are based respectively on the compact sets

$$\mathcal{C}_1 = \{x : |x - \overline{x}| \leq 2s_x\} \qquad \text{and} \qquad \mathcal{C}_2 = \{x : d_1 \leq x \leq d_9\},$$

where $\overline{x}$ and s_x denote the sample mean and sample standard deviation, while d_1 and d_9 represent the first and the ninth decile. In analogy with the previous case, Table 5 displays the summary statistics computed on simulated rejection rates for each considered model and using both ω_1 and ω_2. These results are compared with those obtained through the permutation approach without using the trimming functions.

Summary statistic	F2	M1	M2	M3	M4	M5	M7	M8	M11
Mean	w_1	0.059	0.160	0.196	0.117	0.078	0.986	0.181	0.076
	w_2	0.057	**0.247**	0.259	0.102	0.076	**0.990**	**0.265**	0.072
	P	**0.052**	0.238	**0.383**	**0.146**	**0.095**	0.948	0.254	**0.094**
Median	w_1	0.059	0.164	0.190	0.091	0.069	0.987	0.181	0.068
	w_2	0.053	0.233	0.248	0.095	0.075	0.991	**0.257**	0.070
	P	**0.049**	**0.238**	**0.385**	**0.170**	**0.103**	**0.996**	0.256	**0.097**
Std. Dev.	w_1	0.006	0.014	0.041	0.057	0.023	0.009	0.016	0.020
	w_2	0.008	0.056	0.051	0.036	0.014	0.008	0.056	0.012
	P	0.008	0.077	0.115	0.047	0.020	0.125	0.076	0.023
Min	w_1	0.050	0.131	0.138	0.059	0.058	0.962	0.155	0.055
	w_2	0.047	0.165	0.201	0.064	0.052	0.963	0.180	0.059
	P	0.041	0.107	0.223	0.070	0.063	0.468	0.121	0.058
Max	w_1	0.071	0.183	0.270	0.231	0.130	0.998	0.212	0.125
	w_2	0.069	0.336	0.393	0.196	0.111	0.999	0.353	0.108
	P	0.072	0.420	0.573	0.207	0.126	1.000	0.430	0.128
Q_1	w_1	0.054	0.148	0.168	0.073	0.060	0.983	0.170	0.060
	w_2	0.050	0.209	0.217	0.073	0.064	0.986	0.220	0.063
	P	0.047	0.181	0.257	0.102	0.072	0.988	0.200	0.070
Q_3	w_1	0.062	0.170	0.219	0.173	0.094	0.993	0.193	0.092
	w_2	0.064	0.285	0.282	0.118	0.085	0.995	0.306	0.075
	P	0.060	0.271	0.464	0.183	0.111	0.998	0.293	0.113

Table 5: Summary statistics of the simulated rejection rates obtained in the study for F2. Numbers in bold, only related to mean and median, highlight the approach that works best for each model.

In this scenario, with reference to size comparisons, values of mean and median clearly suggest that the simple permutation approach P outperforms the two approaches based on ω_1 and ω_2; this result remains true also in terms of power. For this reason, in the following, no function ω will be adopted.

5.2 A deep simulation study

This section describes the design of a deeper Monte Carlo simulation study accomplished to provide a practical guide on the choice of the best possibilities for the factors F3-F5 in Table 2. Here, the situation is more complicated than the previous one; thus, we have chosen to change the simulation hierarchical strategy followed in Section 5.1, and we lean towards the simultaneous consideration of all the factors F3-F5. In particular, we consider all the fourteen models in Table 3 and three different kinds of noise ε_t, all with zero-mean and unitary variance: standard normal, standardized skew-t with shape parameter $\alpha = 5$ and $\nu = 5$ degrees of freedom (cfr. Azzalini and Capitanio, 2003, for details on the parameterization), and uniform on $\left[-\sqrt{3}, \sqrt{3}\right]$. Moreover, a wider set of parameters for models M2-M11, specified in Table 6, is taken into account. Once the parametrization is fixed, $p = 5$ lags are considered.

		model parameterizations						
model	ε_t	ϑ_{11}	ϑ_{12}	ϑ_{13}	ϑ_{14}	ϑ_{31}	ϑ_{32}	ϑ_{33}
M2-M5, M7-M11	Std. Gaussian	0.125	0.25	0.5	0.8	0.25	0.5	0.8
	skew-t	0.125	0.25	0.5	0.8			
	uniform	0.125	0.25	0.5	0.8			
M6	Std. Gaussian	0.25	0.5	1				
	skew-t	0.25	0.5	1				
	uniform	0.25	0.5	1				

Table 6: Considered parameterizations, and kinds of noise, for models M2-M11.

6 Simulation results: single-lag procedures

In the first three paragraphs, the simulation results for each kind of noise are presented separately. In the fourth paragraph, final considerations are given.

Remark 1 In Section 2, $\Delta_{ST}^{c} = \Delta_{SD}^{2} = \Delta_{2}^{c} = \Delta_2$ is emphasized. These equivalences suggest that the results obtained using these functionals should be quite similar when KC, JKR and HEF are used. Indeed, as will be seen from Table 7 to Table 12, when KC is adopted, the same rejection rates are obtained with Δ_{ST} and Δ_2. The rejection rates obtained with KC combined with Δ_{SD} are quite different from those obtained with the latter two functionals even if they are theoretically equivalent. The observed differences are due to numerical aspects related to different implementations, (13)-(15), of the functionals. This effect is observed even with the copula density estimators HEF and JKR. When they are used some little differences are observed also between Δ_{SD} and Δ_2. This is due to the fact that, in order to assure the coherence between g and f, the HEF and JKR estimators of Δ_{SD} and Δ_2, are defined by expression (38) and (39) instead of (42) and (43).

6.1 Gaussian noise

Tables 7 and 8 report the mean rejection rates, when the noise is gaussian, for the integrated and summed estimators, respectively. For each combination (F5, F3, model) we calculate the average of the rejection rates obtained from the various parameterizations in Table 6. Each rejection rate used to calculate the averages is obtained by applying the test on the first or third lag depending on the value of q ($q = 1, 3$). For example, the value 0.646, associated with the combination $(\Delta_1, KC, M2)$ in Table 7, is the average of the 7 rejection rates corresponding to the different values of ϑ reported in Table 6; the first four rejection rates are related to the tests on the first lag, the last three are related to the tests on the third lag. While the last columns in Table 7 and 8 report the row averages, the last six rows provide, for each density estimation technique, the averages over the 10 different dependence functionals. In both tables, for each of the $10 \cdot 13 = 130$ combinations (F5, model), the highest mean rejection rate is highlighted in bold (the bold numbers for the total means have an analogous interpretation). A simple inspection shows that no single density estimation technique outperforms the others in all the cases. Nevertheless, we observe that some density estimation techniques work better with certain dependence functionals.

Starting with the results referred to the integrated estimators (Table 7), GK is undoubtedly the better solution for the functionals Δ_1 and $\Delta_{1/2}$ since it provides the highest rejection rate for all the models. In the other cases the situation is more complex. Analyzing the last column of Table 7 it is clear that GK is (in mean) the better technique also for the functionals Δ_2, Δ_3, Δ_{SD}, Δ_{ST}, and Δ_{2^*}. Similarly, JKR outperforms for Δ_4, while KC is the better choice for Δ_{L_1} and $\Delta_{L_1^*}$. However, in general, it is possible to assert that GK has the best overall performance. This consideration is supported by the fact that, among the 60 integrated estimators, ranked in decreasing order with respect to the overall mean rejection rates, the first 5 are composed by GK combined with the following functionals: Δ_1, $\Delta_{1/2}$, Δ_{ST}, Δ_2, and Δ_{2^*}.

Analogous considerations can be made regarding the summed estimator (see Table 8). GK also in this case, combined with the functionals Δ_1, Δ_2, $\Delta_{1/2}$, Δ_{ST}, and Δ_{2^*}, provides (in overall mean) the best results although the performance ordering is different.

In order to obtain information about the factor F4, it is possible to compare the results in Table 7 and Table 8; with reference to the five best functionals:

- Δ_1-integrated works better than Δ_1-summed in 10 models out of 13;
- $\Delta_{1/2}$-integrated works better than $\Delta_{1/2}$-summed in 9 models out of 13;
- Δ_2-integrated works better than Δ_2-summed in 8 models out of 13;
- Δ_{ST}-integrated works better than Δ_{ST}-summed in 13 models out of 13;
- Δ_{2^*}-integrated works better than Δ_{2^*}-summed in 5 models out of 13.

The above considerations are corroborated providing both a cluster and principal component analysis, abbreviated with CA and PCA, respectively. Both procedures are performed starting from the 120×13 data matrix obtained by matching Table 7 and Table 8. The biplot of the PCA and the dendrogram of the CA are given in Fig. 1 and Fig. 2 respectively. In both figures, the different combinations (F5,F3,F4) are simply identified by using the letters I (integrated) and S (summed) for F4 followed by the identifier # in Table 7 and 8 for F5-F3; thus, for example, I48 is related to the triple $(\Delta_{ST}, \mathrm{HED}, \mathrm{I})$.

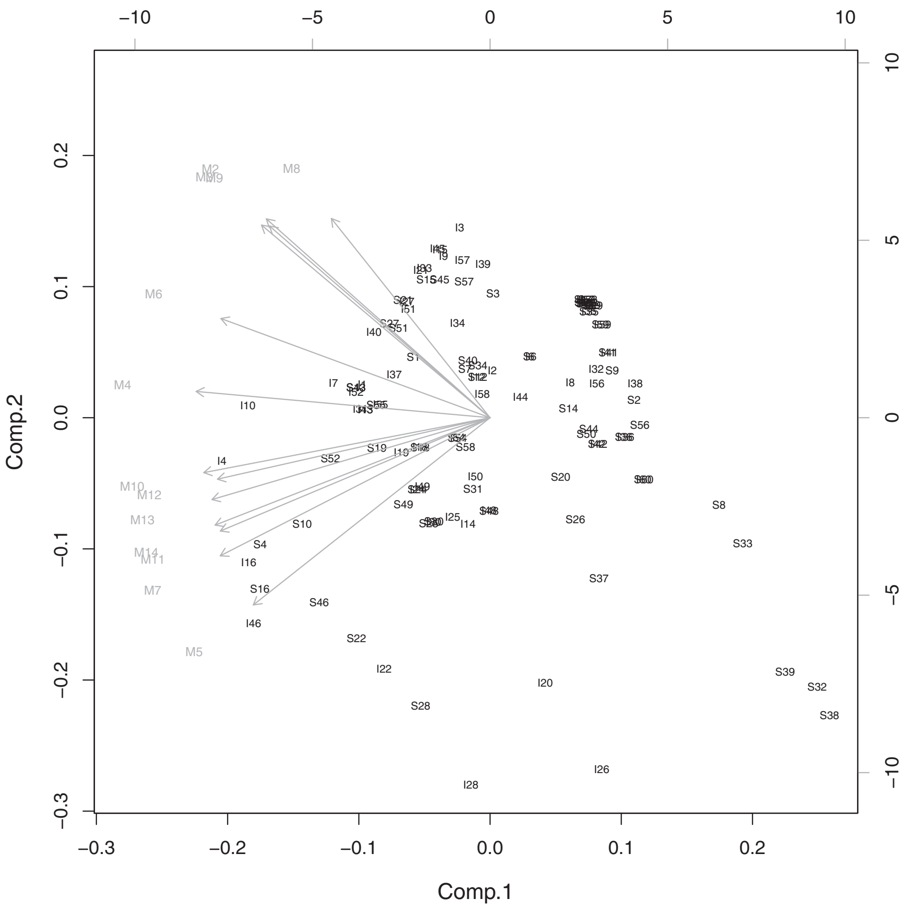

Fig. 1: Biplot for the simulations related to the Gaussian noise.

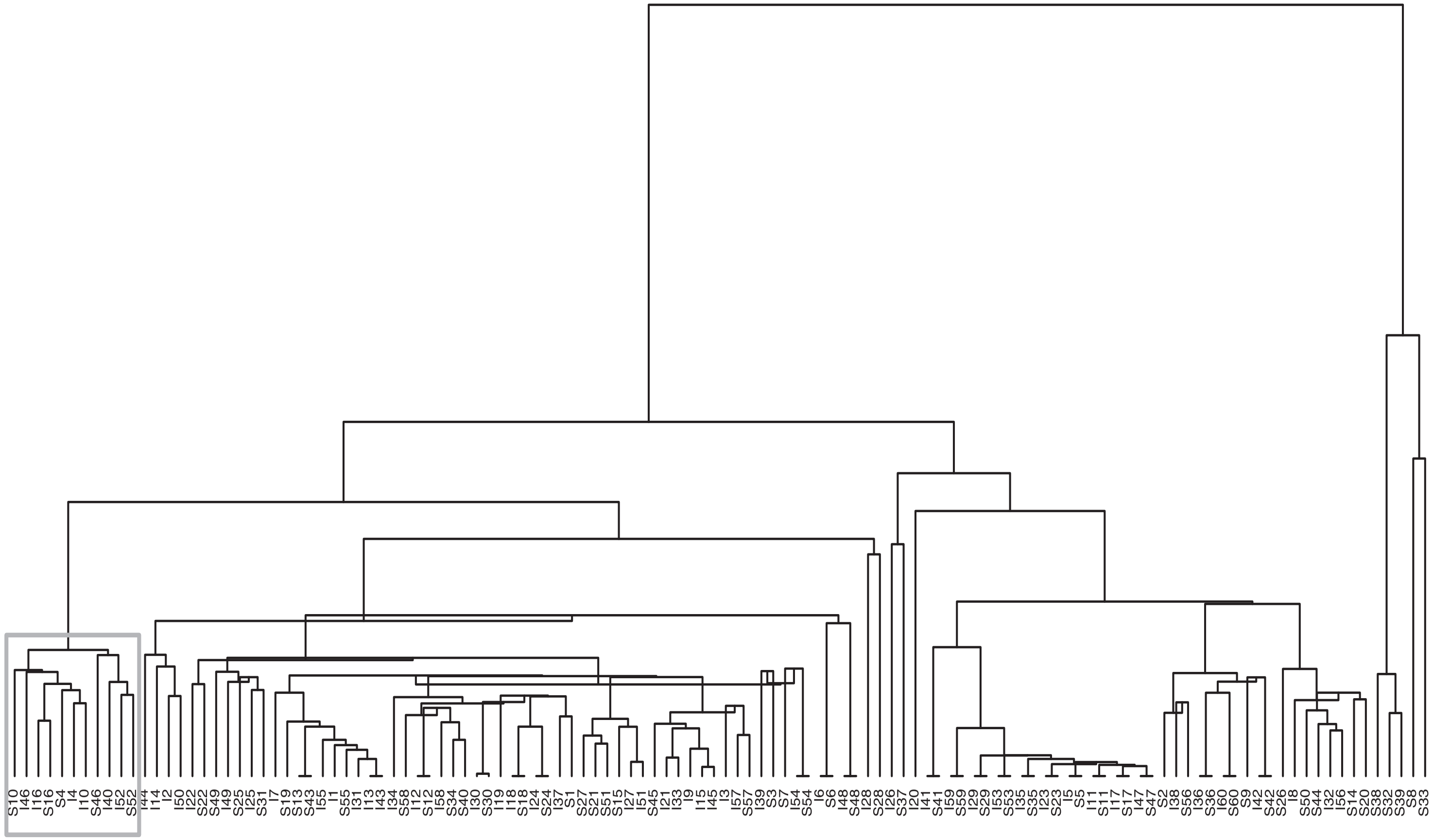

Fig. 2: Dendrogram for the simulations related to the Gaussian noise.

From the dendrogram, it clearly appears that the summed and integrated estimators of the five functionals Δ_1, Δ_2, $\Delta_{1/2}$, Δ_{ST}, and Δ_{2^*}, together with GK, form a well defined group. A similar result turns out from the biplot. The first principal axis can be interpreted as "global performance" of the test. The lower the projection on the first axis, the higher the global performance of a test. Apart from model M4, having a practically null projection on the second axis, all the other models can be split, according to the sign of their projection, into two major groups: (M2, M3, M6, M8, M9) and (M5, M7, M10, M11, M12, M13, M14). The first group of models is composed by linear models and by nonlinear models for which the conditional mean is an autoregressive function of only the series lagged values. The second group consists in nonlinear models and contains all the models with conditional heteroscedasticity. The higher (lower) the projection on the second axis the higher the performance under the first (second) group of models. It is observed that:

- the integrated estimators of functionals Δ_1 and $\Delta_{1/2}$, together with GK, seem to have the highest general performance;
- the integrated estimators of functionals Δ_2 and Δ_{ST} and the summed estimators of functional Δ_1, $\Delta_{1/2}$, Δ_2, and Δ_{ST}, all together with GK, have a good general performance. This is more evident for the second group of models;
- although the integrated and summed estimators of Δ_{2^*}, combined with GK, were detected as possible good choices, the PCA does not highlight this fact.

6.2 Skew-t noise

Tables 9 and 10 provide the results concerning, respectively, the integrated and summed estimators when the noise follows the skew-t distribution. Also in this case, no single density estimation technique outperforms the others.

Regarding the integrated estimators, the last column of Table 9 emphasizes that GK should be considered the better choice for the functionals Δ_1, $\Delta_{1/2}$, Δ_2, Δ_{SD}, Δ_{ST} and Δ_{2^*}. JKR works well with Δ_3 and Δ_4 while KC seems to be the better choice when Δ_{L_1} e $\Delta_{L_1^*}$ are adopted. As shown by the global averages (see the last column and the last six rows of Table 9 and Table 10), the difference between the performance of the "copula-based" density estimation techniques (especially KC and JKR) and GK, becomes lower when the noise follows the skew-t distribution with respect to what happens in the Gaussian case. The relative gain of GK with respect to JKR or KC halves when passing from the Gaussian to the skew-t noise. However, overall, GK seems to remain the best density estimation technique. As for the Gaussian noise, the latter consideration is supported by the fact that, among the 60 integrated estimators, ranked with respect to the mean rejection rate in the last column of the tables, the first 5 are obtained combining GK with (reported in decreasing performance order): Δ_1, $\Delta_{1/2}$, Δ_{ST}, Δ_{2^*}, Δ_2.

Similar considerations can be made analyzing the summed estimators since GK, combined with Δ_1, $\Delta_{1/2}$, Δ_{ST}, Δ_{2^*}, and Δ_2 (reported in decreasing performance order), provides the 5 best solutions.

Moreover, considering jointly the last columns in Table 9 and Table 10, it appears that the best 10 estimators are all constituted by GK combined with both the summed and integrated estimators of Δ_1, $\Delta_{1/2}$, Δ_{ST}, Δ_{2^*} and with

Test	Model Method	#	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	Mean
Δ_1	KC	1	0.646	0.572	0.612	0.146	0.208	0.151	0.800	0.199	0.456	0.111	0.473	0.453	0.186	0.386
	JKL	2	0.618	0.523	0.484	0.113	0.143	0.130	0.790	0.141	0.156	0.060	0.579	0.390	0.087	0.324
	JKR	3	0.679	0.628	0.525	0.075	0.192	0.103	0.798	0.210	0.285	0.074	0.387	0.340	0.098	0.338
	GK	4	**0.703**	**0.668**	**0.727**	**0.210**	**0.220**	**0.211**	**0.810**	**0.225**	**0.592**	**0.140**	**0.774**	**0.647**	**0.240**	**0.474**
	HEF	5	0.605	0.498	0.292	0.068	0.117	0.086	0.783	0.143	0.158	0.061	0.217	0.197	0.075	0.254
	HED	6	0.590	0.476	0.355	0.086	0.141	0.102	0.806	0.140	0.297	0.079	0.373	0.295	0.082	0.294
$\Delta_{1/2}$	KC	7	0.664	0.602	0.650	0.156	0.219	0.162	0.802	0.208	0.481	0.115	0.509	0.480	0.193	0.403
	JKL	8	0.575	0.444	0.344	0.090	0.106	0.108	0.771	0.125	0.110	0.056	0.429	0.316	0.066	0.272
	JKR	9	0.679	0.628	0.530	0.087	0.180	0.114	0.798	0.212	0.316	0.077	0.437	0.370	0.111	0.349
	GK	10	**0.708**	**0.678**	**0.710**	**0.183**	**0.223**	**0.194**	**0.811**	**0.234**	**0.561**	**0.126**	**0.759**	**0.616**	**0.215**	**0.463**
	HEF	11	0.610	0.504	0.299	0.068	0.119	0.087	0.784	0.145	0.161	0.062	0.225	0.201	0.077	0.257
	HED	12	0.610	0.514	0.405	0.102	0.146	0.123	0.806	0.154	0.359	0.086	0.468	0.389	0.104	0.328
Δ_2	KC	13	0.634	0.547	0.613	0.150	**0.200**	0.155	0.804	0.190	0.458	0.113	0.453	0.461	0.195	0.383
	JKL	14	0.565	0.458	0.475	0.171	0.121	0.173	0.765	0.104	0.187	0.082	0.663	0.435	0.105	0.331
	JKR	15	**0.682**	**0.631**	0.547	0.084	0.187	0.116	0.799	**0.212**	0.318	0.077	0.443	0.356	0.111	0.351
	GK	16	0.655	0.595	**0.697**	**0.239**	0.194	**0.211**	**0.810**	0.168	**0.595**	**0.152**	**0.705**	**0.618**	**0.256**	**0.454**
	HEF	17	0.613	0.507	0.297	0.065	0.121	0.084	0.787	0.143	0.164	0.064	0.216	0.203	0.080	0.257
	HED	18	0.620	0.528	0.447	0.134	0.151	0.148	0.777	0.153	0.428	0.104	0.540	0.462	0.162	0.358
Δ_3	KC	19	0.591	0.464	**0.569**	0.152	0.184	0.151	0.791	0.171	0.439	0.111	0.414	0.422	0.187	0.357
	JKL	20	0.447	0.299	0.318	0.184	0.090	0.173	0.531	0.067	0.153	0.075	**0.554**	0.451	0.106	0.265
	JKR	21	**0.684**	**0.628**	0.559	0.093	**0.190**	0.123	**0.800**	**0.213**	0.342	0.083	0.470	0.384	0.129	0.361
	GK	22	0.531	0.387	0.558	**0.216**	0.133	**0.178**	0.727	0.098	**0.514**	**0.145**	0.545	**0.481**	**0.227**	**0.365**
	HEF	23	0.610	0.502	0.287	0.066	0.122	0.083	0.788	0.138	0.157	0.064	0.218	0.196	0.076	0.254
	HED	24	0.609	0.504	0.427	0.152	0.137	0.157	0.802	0.139	0.445	0.112	0.537	0.454	0.176	0.358
Δ_4	KC	25	0.539	0.383	0.512	0.145	0.167	0.142	0.646	0.152	0.406	0.109	0.370	0.374	0.177	0.317
	JKL	26	0.387	0.233	0.228	0.177	0.074	**0.166**	0.280	0.052	0.130	0.072	0.465	**0.449**	0.094	0.216
	JKR	27	**0.682**	**0.623**	**0.567**	0.101	**0.186**	0.130	**0.801**	**0.203**	0.369	0.091	0.503	0.422	0.139	**0.371**
	GK	28	0.422	0.242	0.489	**0.198**	0.119	0.161	0.383	0.080	**0.463**	**0.135**	0.481	0.425	**0.206**	0.293
	HEF	29	0.606	0.491	0.282	0.065	0.120	0.083	0.788	0.137	0.153	0.063	0.209	0.188	0.076	0.251
	HED	30	0.590	0.467	0.405	0.158	0.133	0.152	0.766	0.125	0.442	0.115	**0.506**	0.428	0.179	0.344
Δ_{L_1}	KC	31	0.636	0.552	**0.619**	**0.152**	**0.204**	**0.158**	**0.805**	0.191	**0.462**	**0.113**	0.463	**0.464**	**0.194**	**0.386**
	JKL	32	0.535	0.392	0.362	0.080	0.118	0.101	0.756	0.133	0.064	0.041	0.318	0.348	0.044	0.253
	JKR	33	**0.681**	**0.628**	0.550	0.094	0.184	0.120	0.800	**0.218**	0.334	0.081	0.467	0.384	0.132	0.359
	GK	34	0.620	0.526	0.478	0.083	0.160	0.119	0.796	0.200	0.352	0.079	**0.520**	0.422	0.103	0.343
	HEF	35	0.602	0.489	0.286	0.065	0.118	0.085	0.787	0.139	0.162	0.064	0.220	0.200	0.080	0.254
	HED	36	0.468	0.268	0.217	0.066	0.101	0.088	0.770	0.118	0.196	0.066	0.280	0.230	0.082	0.227
Δ_{SD}	KC	37	0.642	0.561	0.559	**0.136**	**0.196**	**0.143**	0.797	0.190	0.427	**0.101**	0.442	0.410	**0.173**	0.368
	JKL	38	0.518	0.353	0.270	0.072	0.097	0.089	0.738	0.121	0.065	0.040	0.284	0.267	0.042	0.227
	JKR	39	0.651	0.580	0.456	0.078	0.163	0.106	0.793	0.206	0.276	0.073	0.391	0.321	0.099	0.322
	GK	40	**0.665**	**0.603**	**0.578**	0.113	0.186	0.143	**0.803**	**0.221**	**0.433**	0.091	**0.643**	**0.516**	0.141	**0.395**
	HEF	41	0.576	0.444	0.277	0.068	0.115	0.081	0.684	0.132	0.160	0.061	0.218	0.196	0.071	0.237
	HED	42	0.481	0.309	0.239	0.074	0.105	0.100	0.760	0.125	0.219	0.072	0.302	0.269	0.097	0.242
Δ_{ST}	KC	43	0.634	0.547	**0.613**	0.150	**0.200**	0.155	0.804	0.190	0.458	0.113	0.453	0.461	0.195	0.383
	JKL	44	0.577	0.452	0.394	0.100	0.121	0.122	0.787	0.145	0.148	0.061	0.527	0.408	0.100	0.303
	JKR	45	**0.682**	**0.632**	0.549	0.085	0.189	0.118	0.799	**0.215**	0.318	0.078	0.437	0.357	0.111	0.351
	GK	46	0.632	0.534	0.590	**0.216**	0.153	**0.237**	**0.806**	0.165	**0.610**	**0.154**	**0.830**	**0.732**	**0.259**	**0.455**
	HEF	47	0.610	0.504	0.293	0.065	0.119	0.084	0.787	0.141	0.162	0.064	0.221	0.202	0.080	0.256
	HED	48	0.539	0.371	0.329	0.120	0.113	0.147	0.801	0.124	0.387	0.093	0.474	0.434	0.150	0.314
Δ_{2^*}	KC	49	0.561	0.419	0.547	0.150	0.176	0.148	0.736	0.163	0.425	**0.112**	0.393	0.404	**0.186**	0.340
	JKL	50	0.551	0.421	0.545	**0.166**	0.148	**0.171**	0.770	0.130	0.140	0.063	0.606	0.474	0.060	0.327
	JKR	51	**0.680**	**0.617**	0.562	0.103	**0.183**	0.132	0.800	0.201	0.366	0.092	0.500	0.416	0.141	0.369
	GK	52	0.647	0.568	**0.606**	0.131	0.178	0.153	**0.809**	**0.205**	**0.474**	0.102	**0.683**	**0.557**	0.161	**0.406**
	HEF	53	0.613	0.507	0.297	0.065	0.116	0.084	0.787	0.143	0.164	0.064	0.205	0.187	0.080	0.255
	HED	54	0.620	0.528	0.449	0.134	0.114	0.149	0.781	0.153	0.426	0.104	0.360	0.313	0.162	0.330
$\Delta_{L_1^*}$	KC	55	0.630	0.539	**0.593**	**0.145**	**0.197**	**0.149**	**0.803**	0.184	**0.446**	**0.108**	0.446	**0.444**	**0.186**	**0.375**
	JKL	56	0.529	0.379	0.331	0.083	0.106	0.101	0.773	0.132	0.084	0.046	0.333	0.354	0.047	0.253
	JKR	57	**0.668**	**0.599**	0.505	0.082	0.169	0.110	0.798	**0.210**	0.295	0.076	0.416	0.343	0.106	0.337
	GK	58	0.569	0.424	0.406	0.087	0.149	0.118	0.790	0.170	0.363	0.081	**0.541**	0.442	0.101	0.326
	HEF	59	0.590	0.463	0.274	0.067	0.115	0.080	0.786	0.128	0.156	0.062	0.208	0.195	0.076	0.246
	HED	60	0.433	0.216	0.163	0.068	0.088	0.089	0.762	0.103	0.203	0.065	0.256	0.232	0.090	0.213
Mean	KC		0.618	0.519	**0.589**	0.148	**0.195**	0.151	0.779	0.184	0.446	0.111	0.442	0.437	0.187	0.370
	JKL		0.530	0.395	0.375	0.124	0.112	0.133	0.696	0.115	0.124	0.060	0.476	0.389	0.075	0.277
	JKR		**0.677**	**0.619**	0.535	0.088	0.182	0.117	**0.799**	**0.210**	0.322	0.080	0.445	0.369	0.118	0.351
	GK		0.615	0.523	0.584	**0.167**	0.171	**0.172**	0.755	0.177	**0.496**	**0.121**	**0.648**	**0.546**	**0.191**	**0.397**
	HEF		0.604	0.491	0.288	0.066	0.118	0.084	0.776	0.139	0.160	0.063	0.216	0.197	0.077	0.252
	HED		0.556	0.418	0.343	0.109	0.123	0.125	0.783	0.133	0.340	0.089	0.410	0.351	0.128	0.301

Table 7: Mean simulated rejection rates of the integrated estimator for the models M2-M14 with Gaussian noise. For each model, the mean is computed over the different parametrization described in Table 6.

Test	Model Method	#	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	Mean
Δ_1	KC	1	0.642	0.558	0.545	0.123	**0.189**	0.136	0.800	**0.188**	0.404	0.099	0.418	0.383	0.157	0.357
	JKL	2	0.505	0.334	0.335	0.081	0.105	0.091	0.780	0.094	0.056	0.037	0.262	0.281	0.041	0.231
	JKR	3	0.628	0.534	0.503	0.075	0.162	0.097	0.803	0.174	0.300	0.069	0.427	0.325	0.097	0.323
	GK	4	**0.644**	**0.578**	**0.670**	**0.204**	0.184	**0.219**	**0.809**	0.179	**0.590**	**0.138**	**0.784**	**0.674**	**0.231**	**0.454**
	HEF	5	0.605	0.498	0.292	0.068	0.117	0.086	0.783	0.143	0.158	0.061	0.217	0.197	0.075	0.254
	HED	6	0.590	0.476	0.355	0.086	0.141	0.102	0.806	0.140	0.297	0.079	0.373	0.295	0.082	0.294
$\Delta_{1/2}$	KC	7	0.597	0.495	0.485	0.107	0.166	0.118	0.789	**0.176**	0.363	0.092	0.364	0.339	0.142	0.326
	JKL	8	0.352	0.183	0.166	0.069	0.080	0.078	0.655	0.073	0.045	0.040	0.132	0.187	0.041	0.162
	JKR	9	0.513	0.346	0.333	0.063	0.111	0.071	0.791	0.125	0.202	0.057	0.255	0.214	0.076	0.243
	GK	10	**0.631**	**0.550**	**0.621**	**0.184**	**0.171**	**0.204**	**0.808**	0.172	**0.557**	**0.124**	**0.765**	**0.628**	**0.206**	**0.432**
	HEF	11	0.610	0.504	0.299	0.068	0.119	0.087	0.784	0.145	0.161	0.062	0.225	0.201	0.077	0.257
	HED	12	0.610	0.514	0.405	0.102	0.146	0.123	0.806	0.154	0.359	0.086	0.468	0.389	0.104	0.328
Δ_2	KC	13	0.652	0.578	0.633	0.147	**0.204**	0.155	0.806	0.198	0.462	0.111	0.495	0.458	0.189	0.392
	JKL	14	0.543	0.390	0.470	0.103	0.124	0.117	0.771	0.108	0.069	0.045	0.388	0.367	0.054	0.273
	JKR	15	**0.666**	**0.601**	0.577	0.090	0.188	0.115	0.805	**0.206**	0.355	0.078	0.500	0.402	0.127	0.362
	GK	16	0.636	0.561	**0.679**	**0.237**	0.184	**0.209**	**0.809**	0.155	**0.603**	**0.153**	**0.709**	**0.620**	**0.260**	**0.447**
	HEF	17	0.613	0.507	0.297	0.065	0.121	0.084	0.787	0.143	0.164	0.064	0.216	0.203	0.080	0.257
	HED	18	0.620	0.528	0.447	0.134	0.151	0.148	0.777	0.153	0.428	0.104	0.540	0.462	0.162	0.358
Δ_3	KC	19	0.616	0.507	**0.619**	0.156	0.185	0.155	0.798	0.171	0.469	0.113	0.456	0.449	0.200	0.377
	JKL	20	0.508	0.347	0.470	0.120	0.122	0.131	0.730	0.097	0.074	0.051	0.389	0.400	0.082	0.271
	JKR	21	**0.670**	**0.607**	0.593	0.098	**0.190**	0.129	**0.805**	**0.209**	0.377	0.086	0.544	0.428	0.140	0.375
	GK	22	0.564	0.440	0.569	**0.223**	0.148	**0.181**	0.755	0.113	**0.521**	**0.147**	**0.566**	**0.492**	**0.242**	**0.382**
	HEF	23	0.610	0.502	0.287	0.066	0.122	0.083	0.788	0.138	0.157	0.064	0.218	0.196	0.076	0.254
	HED	24	0.609	0.504	0.427	0.152	0.137	0.157	0.802	0.139	0.445	0.112	0.537	0.454	0.176	0.358
Δ_4	KC	25	0.561	0.416	0.566	0.152	0.169	0.146	0.639	0.150	0.437	0.114	0.389	0.391	0.198	0.333
	JKL	26	0.479	0.306	0.451	0.123	0.117	0.134	0.670	0.086	0.077	0.051	0.377	0.397	0.080	0.257
	JKR	27	**0.669**	**0.605**	**0.600**	0.105	**0.192**	0.140	**0.804**	**0.203**	0.397	0.091	**0.570**	**0.444**	0.150	**0.382**
	GK	28	0.498	0.348	0.511	**0.208**	0.129	**0.169**	0.582	0.092	**0.480**	**0.139**	0.507	0.440	**0.229**	0.333
	HEF	29	0.606	0.491	0.282	0.065	0.120	0.083	0.788	0.137	0.153	0.063	0.209	0.188	0.076	0.251
	HED	30	0.590	0.467	0.405	0.158	0.133	0.152	0.766	0.125	0.442	0.115	0.506	0.426	0.179	0.343
Δ_{L_1}	KC	31	0.564	0.408	**0.474**	**0.126**	**0.152**	**0.135**	0.687	0.134	**0.384**	**0.103**	0.363	0.386	**0.154**	0.313
	JKL	32	0.136	0.090	0.061	0.065	0.057	0.064	0.194	0.052	0.026	0.035	0.082	0.076	0.034	0.075
	JKR	33	0.384	0.169	0.093	0.060	0.074	0.064	0.425	0.076	0.072	0.056	0.075	0.085	0.063	0.131
	GK	34	0.586	0.464	0.430	0.082	0.149	0.118	**0.791**	**0.180**	0.357	0.078	**0.540**	**0.437**	0.094	**0.331**
	HEF	35	**0.602**	**0.489**	0.286	0.065	0.118	0.085	0.787	0.139	0.162	0.064	0.220	0.200	0.080	0.254
	HED	36	0.468	0.268	0.217	0.066	0.101	0.088	0.770	0.118	0.196	0.066	0.280	0.230	0.082	0.227
Δ_{SD}	KC	37	0.467	0.299	0.288	**0.100**	0.127	0.112	0.308	0.115	0.244	**0.088**	0.258	0.258	**0.107**	0.213
	JKL	38	0.090	0.078	0.054	0.066	0.057	0.062	0.121	0.053	0.026	0.033	0.075	0.071	0.032	0.063
	JKR	39	0.205	0.137	0.077	0.059	0.073	0.062	0.099	0.075	0.062	0.057	0.070	0.075	0.061	0.086
	GK	40	**0.598**	**0.487**	**0.445**	0.090	**0.148**	**0.122**	**0.789**	**0.188**	**0.358**	0.078	**0.541**	**0.430**	0.102	**0.337**
	HEF	41	0.576	0.444	0.277	0.068	0.115	0.081	0.684	0.132	0.160	0.061	0.218	0.196	0.071	0.237
	HED	42	0.481	0.309	0.239	0.074	0.105	0.100	0.760	0.125	0.219	0.072	0.302	0.269	0.097	0.242
Δ_{ST}	KC	43	0.652	0.578	**0.633**	0.147	**0.204**	0.155	**0.806**	0.198	0.462	0.111	0.495	0.458	0.189	0.392
	JKL	44	0.518	0.351	0.317	0.086	0.089	0.103	0.781	0.120	0.113	0.055	0.393	0.377	0.077	0.260
	JKR	45	**0.662**	**0.593**	0.559	0.088	0.179	0.114	0.803	**0.204**	0.342	0.076	0.473	0.384	0.120	0.354
	GK	46	0.593	0.465	0.541	**0.187**	0.139	**0.216**	0.802	0.152	**0.564**	**0.130**	**0.788**	**0.673**	**0.215**	**0.420**
	HEF	47	0.610	0.504	0.293	0.065	0.119	0.084	0.787	0.141	0.162	0.064	0.221	0.202	0.080	0.256
	HED	48	0.539	0.371	0.329	0.120	0.113	0.147	0.801	0.124	0.387	0.093	0.474	0.434	0.150	0.314
Δ_{2^*}	KC	49	0.581	0.448	0.590	0.157	0.174	0.153	0.700	0.157	0.461	**0.116**	0.419	0.434	**0.199**	0.353
	JKL	50	0.501	0.326	0.422	0.102	0.115	0.111	0.756	0.116	0.064	0.045	0.336	0.404	0.054	0.258
	JKR	51	**0.663**	**0.595**	0.583	0.106	**0.185**	0.139	0.805	**0.202**	0.383	0.091	0.543	0.427	0.153	0.375
	GK	52	0.636	0.553	**0.624**	**0.161**	0.173	**0.170**	**0.809**	0.190	**0.511**	0.113	**0.706**	**0.594**	0.193	**0.418**
	HEF	53	0.613	0.507	0.297	0.065	0.116	0.084	0.787	0.143	0.164	0.064	0.205	0.187	0.080	0.255
	HED	54	0.620	0.528	0.449	0.134	0.114	0.149	0.781	0.153	0.426	0.104	0.360	0.313	0.162	0.330
$\Delta_{L_1^*}$	KC	55	0.638	0.547	**0.605**	**0.147**	**0.195**	**0.152**	**0.804**	0.182	**0.454**	**0.107**	0.471	0.455	**0.179**	**0.380**
	JKL	56	0.473	0.294	0.262	0.071	0.092	0.086	0.750	0.111	0.067	0.046	0.280	0.318	0.054	0.224
	JKR	57	**0.652**	**0.572**	0.506	0.085	0.168	0.109	0.803	**0.201**	0.297	0.079	0.423	0.350	0.111	0.335
	GK	58	0.558	0.407	0.419	0.101	0.135	0.135	0.792	0.158	0.403	0.086	**0.597**	**0.489**	0.115	0.338
	HEF	59	0.590	0.463	0.274	0.067	0.115	0.080	0.786	0.128	0.156	0.062	0.208	0.195	0.076	0.246
	HED	60	0.433	0.216	0.163	0.068	0.088	0.089	0.762	0.103	0.203	0.065	0.256	0.232	0.090	0.213
Mean	KC		0.597	0.483	0.544	0.136	**0.177**	0.142	0.714	0.167	0.414	0.106	0.413	0.401	0.171	0.343
	JKL		0.411	0.270	0.301	0.089	0.096	0.098	0.621	0.091	0.062	0.044	0.271	0.288	0.055	0.207
	JKR		0.571	0.476	0.443	0.083	0.152	0.104	0.694	**0.168**	0.279	0.074	0.388	0.313	0.110	0.297
	GK		0.594	0.485	**0.551**	**0.168**	0.156	**0.174**	0.775	0.158	**0.494**	**0.119**	**0.650**	**0.548**	**0.189**	**0.389**
	HEF		**0.604**	**0.491**	0.288	0.066	0.118	0.084	0.776	0.139	0.160	0.063	0.216	0.197	0.077	0.252
	HED		0.556	0.418	0.343	0.109	0.123	0.125	**0.783**	0.133	0.340	0.089	0.410	0.350	0.128	0.301

Table 8: Mean simulated rejection rates of the summed estimator for the models M2-M14 with Gaussian noise. For each model, the mean is computed over the different parametrization described in Table 6.

Δ_2-integrated and Δ_{SD}-integrated. The latter combinations coincide, in practice, with the union of the combinations obtained separately for the integrated and summed estimators.

Referring to the best summed and integrated versions of Δ_1, $\Delta_{1/2}$, Δ_{ST}, Δ_{2^*}, and Δ_2, it follows that:

- Δ_1-integrated works better than Δ_1-summed in 6 models out of 13;
- $\Delta_{1/2}$-integrated works better than $\Delta_{1/2}$-summed in 10 models out of 13;
- Δ_2-integrated works better than Δ_2-summed in 12 models out of 13;
- Δ_{ST}-integrated works better than Δ_{ST}-summed in 12 models out of 13;
- Δ_{2^*}-integrated works better than Δ_{2^*}-summed in 7 models out of 13.

Also here, all the previous observations are confirmed performing the PCA (see Fig. 3) and the CA (see Fig. 4) on the data of Table 9 and Table 10. The dendrogram emphasizes that the summed and integrated estimators of the functionals Δ_1, $\Delta_{1/2}$, Δ_{ST}, Δ_{2^*}, and Δ_2, together with GK, form a well defined group. A similar result is shown by the biplot of the PCA. Moreover, the latter emphasizes that the high performance of the aforementioned combinations is particularly relevant under the models M2, M3, M4, M4, M10, and M13. No connection with the two groups observed in the Gaussian case seems to exist.

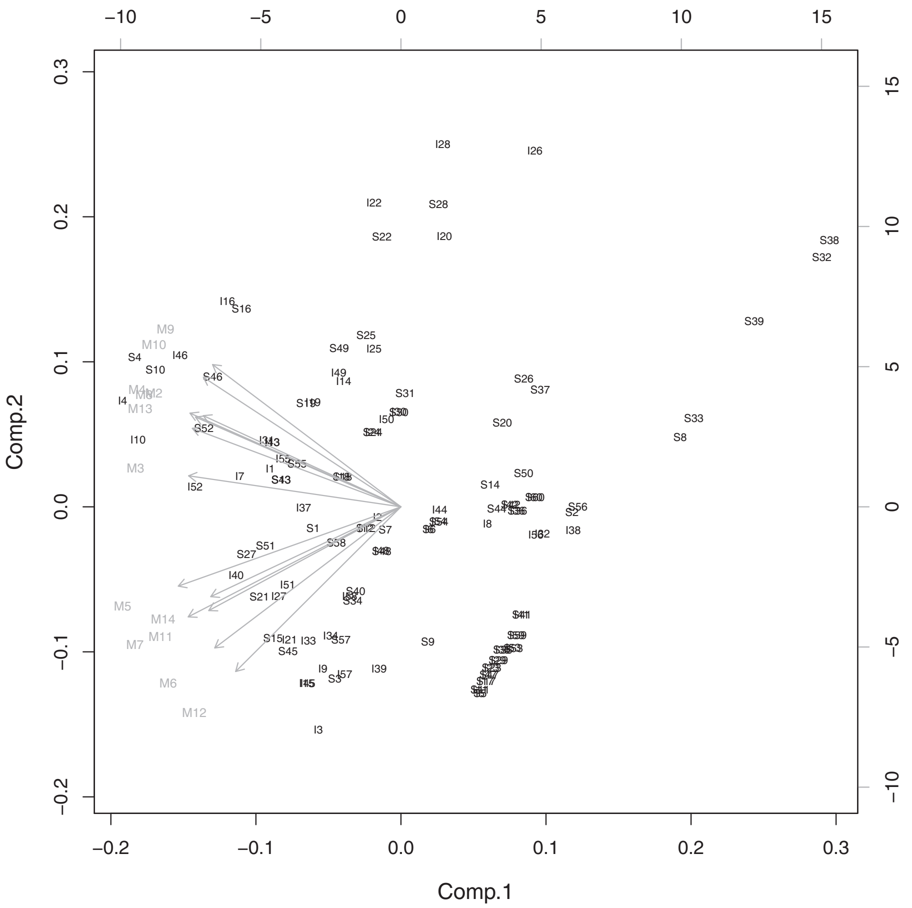

Fig. 3: Biplot for the simulations related to the skew-t noise.

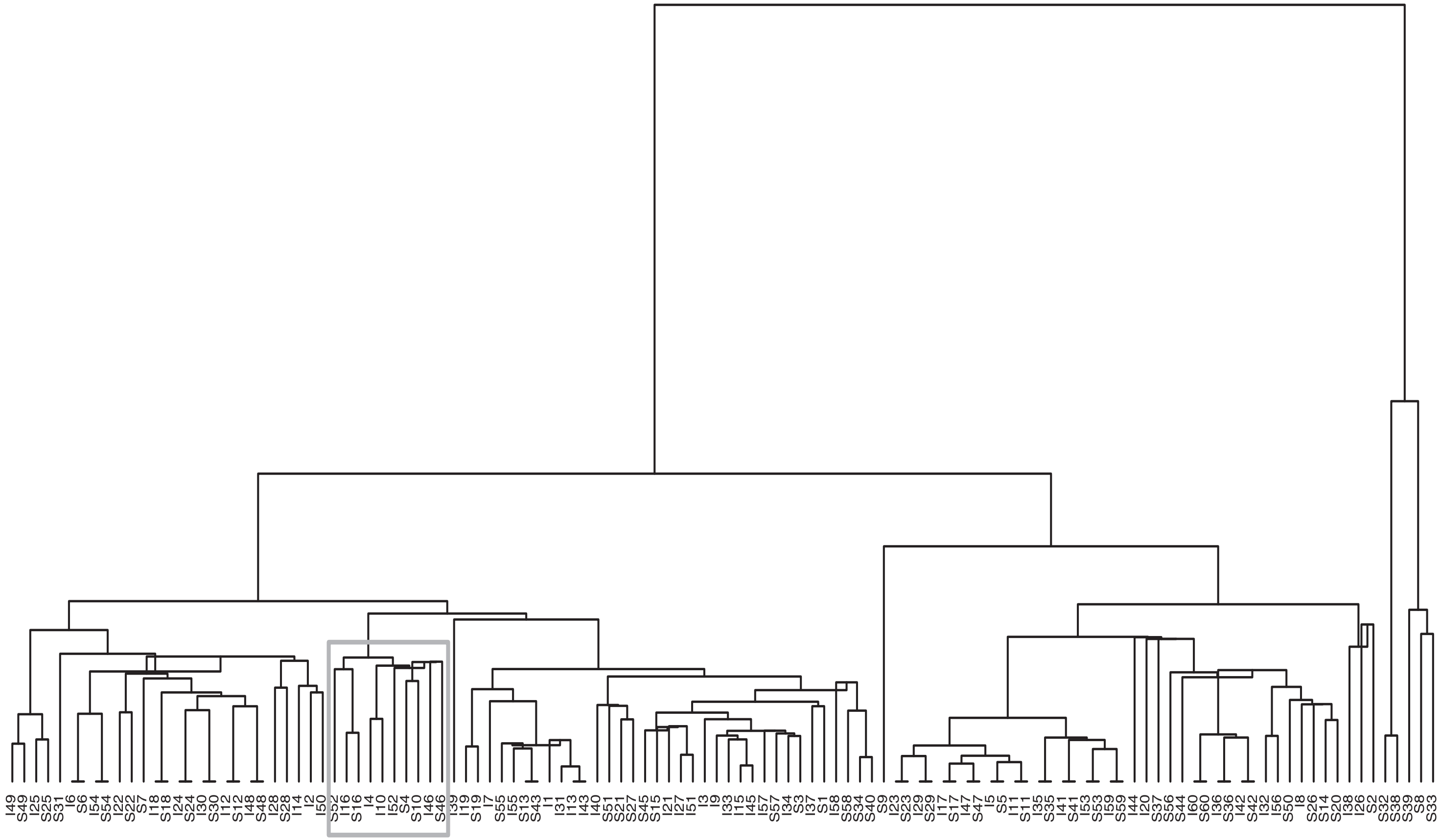

Fig. 4: Dendrogram for the simulations related to the skew-t noise.

Test	Model Method	#	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	Mean
Δ_1	KC	1	0.633	0.597	0.814	0.218	0.278	0.215	0.379	0.280	0.444	0.925	0.371	0.370	0.205	0.441
	JKL	2	0.648	0.587	0.565	0.223	0.254	0.222	0.371	0.283	0.108	0.294	0.428	0.334	0.050	0.336
	JKR	3	**0.704**	0.668	0.559	0.144	0.317	0.158	0.399	**0.397**	0.265	**0.964**	0.319	0.256	0.106	0.404
	GK	4	0.694	**0.670**	**0.835**	**0.346**	**0.349**	**0.339**	**0.440**	0.337	**0.595**	0.958	**0.484**	**0.494**	**0.252**	**0.523**
	HEF	5	0.633	0.537	0.341	0.098	0.214	0.095	0.351	0.288	0.143	0.758	0.177	0.143	0.075	0.296
	HED	6	0.608	0.505	0.611	0.167	0.209	0.174	0.419	0.215	0.278	0.585	0.155	0.195	0.110	0.325
$\Delta_{1/2}$	KC	7	0.656	0.621	0.809	0.229	0.299	0.230	0.383	0.301	0.455	0.948	0.408	0.399	0.216	0.458
	JKL	8	0.600	0.499	0.442	0.143	0.189	0.155	0.311	0.231	0.070	0.264	0.320	0.247	0.051	0.271
	JKR	9	0.681	0.652	0.536	0.157	0.272	0.170	0.401	**0.359**	0.290	**0.962**	0.360	0.279	0.119	0.403
	GK	10	**0.702**	**0.673**	**0.817**	**0.329**	**0.354**	**0.325**	**0.440**	0.348	**0.564**	0.955	**0.461**	**0.462**	**0.236**	**0.513**
	HEF	11	0.633	0.535	0.357	0.096	0.212	0.095	0.352	0.284	0.144	0.765	0.171	0.150	0.073	0.297
	HED	12	0.633	0.571	0.693	0.206	0.243	0.208	0.424	0.246	0.328	0.701	0.191	0.227	0.124	0.369
Δ_2	KC	13	0.621	0.580	**0.817**	0.223	0.262	0.218	0.393	0.260	0.454	0.916	0.373	0.376	0.218	0.439
	JKL	14	0.629	0.555	0.634	0.319	0.235	0.293	0.375	0.236	0.124	0.288	**0.450**	**0.433**	0.048	0.355
	JKR	15	**0.695**	**0.663**	0.571	0.157	**0.292**	0.174	0.411	**0.378**	0.299	**0.956**	0.356	0.287	0.123	0.412
	GK	16	0.607	0.576	0.812	**0.324**	0.242	**0.304**	**0.432**	0.205	**0.584**	0.842	0.373	0.428	**0.254**	**0.460**
	HEF	17	0.622	0.528	0.365	0.096	0.210	0.093	0.356	0.268	0.140	0.755	0.172	0.146	0.068	0.294
	HED	18	0.622	0.574	0.758	0.233	0.246	0.230	0.429	0.219	0.372	0.736	0.213	0.242	0.143	0.386
Δ_3	KC	19	0.587	0.527	**0.807**	0.215	0.230	0.213	0.383	0.215	0.446	0.860	0.336	0.349	0.209	0.414
	JKL	20	0.500	0.390	0.603	**0.311**	0.160	**0.284**	0.318	0.127	0.102	0.202	0.344	**0.423**	0.048	0.293
	JKR	21	**0.693**	**0.656**	0.587	0.171	**0.278**	0.189	**0.419**	**0.362**	0.332	**0.956**	**0.388**	0.333	0.141	**0.423**
	GK	22	0.494	0.432	0.717	0.281	0.162	0.257	0.285	0.116	**0.533**	0.672	0.293	0.339	**0.219**	0.369
	HEF	23	0.614	0.522	0.363	0.095	0.203	0.094	0.357	0.259	0.138	0.735	0.161	0.149	0.075	0.289
	HED	24	0.585	0.538	0.756	0.230	0.223	0.225	0.412	0.182	0.362	0.678	0.188	0.230	0.149	0.366
Δ_4	KC	25	0.540	0.465	**0.794**	0.205	0.195	0.196	0.328	0.168	0.417	0.769	0.292	0.319	0.194	0.375
	JKL	26	0.385	0.252	0.491	**0.289**	0.110	**0.262**	0.241	0.073	0.090	0.149	0.307	**0.380**	0.044	0.236
	JKR	27	**0.686**	**0.649**	0.593	0.191	**0.257**	0.202	**0.427**	**0.344**	0.366	**0.955**	**0.417**	0.370	0.148	**0.431**
	GK	28	0.422	0.358	0.661	0.263	0.149	0.241	0.152	0.097	**0.507**	0.612	0.273	0.302	**0.207**	0.326
	HEF	29	0.606	0.517	0.362	0.094	0.196	0.091	0.356	0.251	0.133	0.719	0.161	0.141	0.078	0.285
	HED	30	0.562	0.511	0.751	0.219	0.209	0.219	0.384	0.165	0.350	0.645	0.179	0.207	0.147	0.350
Δ_{L_1}	KC	31	0.622	0.584	**0.819**	**0.227**	0.264	**0.221**	0.396	0.262	**0.459**	0.922	**0.379**	**0.382**	**0.220**	**0.443**
	JKL	32	0.563	0.451	0.443	0.105	0.167	0.122	0.279	0.219	0.041	0.271	0.236	0.211	0.029	0.241
	JKR	33	**0.679**	**0.648**	0.567	0.164	0.264	0.182	**0.409**	**0.354**	0.315	**0.961**	0.378	0.308	0.136	0.413
	GK	34	0.660	0.601	0.533	0.160	**0.272**	0.182	0.393	0.347	0.310	0.950	0.361	0.286	0.114	0.398
	HEF	35	0.595	0.497	0.345	0.094	0.197	0.091	0.356	0.239	0.140	0.722	0.177	0.150	0.067	0.282
	HED	36	0.555	0.416	0.428	0.127	0.133	0.133	0.360	0.184	0.211	0.538	0.175	0.173	0.080	0.270
Δ_{SD}	KC	37	0.630	0.586	**0.774**	0.191	0.278	0.200	0.361	0.279	0.387	0.942	0.345	0.329	**0.170**	0.421
	JKL	38	0.563	0.432	0.339	0.093	0.151	0.105	0.238	0.205	0.040	0.236	0.241	0.184	0.033	0.220
	JKR	39	0.646	0.603	0.477	0.125	0.245	0.138	0.379	0.326	0.235	0.937	0.331	0.240	0.107	0.368
	GK	40	**0.697**	**0.651**	0.646	**0.225**	**0.311**	**0.241**	**0.417**	**0.363**	**0.418**	**0.969**	**0.461**	**0.368**	0.155	**0.455**
	HEF	41	0.574	0.462	0.334	0.092	0.187	0.091	0.335	0.214	0.136	0.683	0.174	0.144	0.072	0.269
	HED	42	0.551	0.433	0.444	0.134	0.143	0.142	0.342	0.187	0.220	0.568	0.168	0.181	0.075	0.276
Δ_{ST}	KC	43	0.621	0.580	**0.817**	0.223	0.262	0.218	0.393	0.260	0.454	0.916	0.373	0.376	0.218	0.439
	JKL	44	0.622	0.545	0.532	0.166	0.207	0.191	0.344	0.236	0.093	0.294	0.333	0.296	0.062	0.301
	JKR	45	**0.694**	**0.664**	0.578	0.158	**0.294**	0.176	0.408	**0.379**	0.297	0.956	0.345	0.283	0.121	0.412
	GK	46	0.647	0.613	0.701	**0.320**	0.266	**0.337**	**0.423**	0.280	**0.536**	**0.990**	**0.495**	**0.471**	**0.269**	**0.488**
	HEF	47	0.618	0.519	0.352	0.096	0.205	0.094	0.357	0.263	0.140	0.756	0.177	0.149	0.068	0.292
	HED	48	0.647	0.574	0.619	0.179	0.231	0.192	0.392	0.255	0.301	0.719	0.209	0.234	0.124	0.360
Δ_{2^*}	KC	49	0.566	0.497	**0.801**	0.213	0.208	0.206	0.368	0.191	0.441	0.815	0.317	0.337	**0.208**	0.397
	JKL	50	0.612	0.526	0.664	0.268	0.219	0.253	0.372	0.234	0.101	0.293	0.376	0.378	0.038	0.333
	JKR	51	**0.680**	**0.637**	0.590	0.189	0.247	0.204	0.423	0.333	0.363	0.953	0.405	0.364	0.148	0.426
	GK	52	0.674	0.631	0.768	**0.278**	**0.311**	**0.274**	**0.442**	**0.346**	**0.514**	**0.959**	**0.475**	**0.415**	0.185	**0.482**
	HEF	53	0.598	0.495	0.348	0.090	0.185	0.089	0.359	0.231	0.129	0.685	0.153	0.138	0.070	0.275
	HED	54	0.587	0.494	0.585	0.162	0.170	0.166	0.423	0.205	0.284	0.665	0.197	0.199	0.096	0.326
$\Delta_{L_1^*}$	KC	55	0.616	0.573	**0.813**	**0.212**	0.266	**0.215**	0.392	0.263	**0.435**	0.923	0.363	**0.363**	**0.197**	**0.433**
	JKL	56	0.571	0.450	0.421	0.112	0.163	0.126	0.304	0.211	0.052	0.289	0.245	0.213	0.031	0.245
	JKR	57	**0.667**	**0.627**	0.539	0.142	**0.266**	0.152	**0.401**	**0.348**	0.268	0.946	0.341	0.258	0.113	0.390
	GK	58	0.633	0.554	0.499	0.156	0.251	0.183	0.364	0.329	0.307	**0.960**	**0.367**	0.299	0.124	0.386
	HEF	59	0.592	0.475	0.338	0.093	0.184	0.089	0.353	0.223	0.128	0.695	0.165	0.139	0.071	0.272
	HED	60	0.536	0.396	0.430	0.113	0.126	0.127	0.324	0.180	0.209	0.554	0.168	0.164	0.080	0.262
Mean	KC		0.609	0.561	**0.806**	0.215	0.254	0.213	0.377	0.248	0.439	0.893	0.356	0.360	**0.206**	0.426
	JKL		0.569	0.469	0.513	0.203	0.185	0.201	0.315	0.205	0.082	0.258	0.328	0.310	0.043	0.283
	JKR		**0.682**	**0.647**	0.560	0.160	**0.273**	0.174	**0.408**	**0.358**	0.303	**0.954**	0.364	0.298	0.126	0.408
	GK		0.623	0.576	0.699	**0.268**	0.267	**0.268**	0.379	0.277	**0.487**	0.886	**0.404**	**0.386**	0.202	**0.440**
	HEF		0.608	0.509	0.350	0.094	0.199	0.092	0.353	0.252	0.137	0.727	0.169	0.145	0.072	0.285
	HED		0.588	0.501	0.607	0.177	0.193	0.182	0.391	0.204	0.291	0.639	0.184	0.205	0.113	0.329

Table 9: Mean simulated rejection rates of the integrated estimator for the models M2-M14 with Skew-t noise. For each model, the mean is computed over the different parametrization described in Table 6.

Test	Model Method	#	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	Mean
Δ_1	KC	1	0.632	0.588	0.741	0.189	0.266	0.189	0.377	0.297	0.367	0.892	0.329	0.310	0.175	0.412
	JKL	2	0.526	0.327	0.347	0.110	0.146	0.123	0.341	0.177	0.043	0.350	0.190	0.187	0.027	0.223
	JKR	3	0.666	0.589	0.529	0.154	0.297	0.160	0.413	**0.362**	0.307	**0.951**	0.346	0.255	0.097	0.394
	GK	4	**0.677**	**0.628**	**0.772**	**0.347**	**0.317**	**0.343**	**0.441**	0.315	**0.597**	0.889	**0.547**	**0.515**	**0.261**	**0.511**
	HEF	5	0.633	0.537	0.341	0.098	0.214	0.095	0.351	0.288	0.143	0.758	0.177	0.143	0.075	0.296
	HED	6	0.608	0.505	0.611	0.167	0.209	0.174	0.419	0.215	0.278	0.585	0.155	0.195	0.110	0.325
$\Delta_{1/2}$	KC	7	0.611	0.520	0.662	0.160	0.248	0.165	0.328	0.279	0.314	0.754	0.266	0.244	0.148	0.361
	JKL	8	0.395	0.165	0.189	0.082	0.101	0.091	0.277	0.120	0.028	0.251	0.130	0.109	0.024	0.151
	JKR	9	0.612	0.415	0.388	0.134	0.259	0.131	0.397	**0.311**	0.257	0.850	0.257	0.197	0.084	0.330
	GK	10	**0.671**	**0.606**	**0.700**	**0.325**	**0.297**	**0.324**	**0.436**	0.308	**0.561**	**0.916**	**0.569**	**0.503**	**0.269**	**0.499**
	HEF	11	0.633	0.535	0.357	0.096	0.212	0.095	0.352	0.284	0.144	0.765	0.171	0.150	0.073	0.297
	HED	12	0.633	0.571	0.693	0.206	0.243	0.208	0.424	0.246	0.328	0.701	0.191	0.227	0.124	0.369
Δ_2	KC	13	0.627	0.586	0.803	0.212	0.260	0.209	0.408	0.273	0.423	0.926	0.366	0.361	0.192	0.434
	JKL	14	0.559	0.423	0.541	0.160	0.185	0.176	0.347	0.212	0.057	0.360	0.246	0.282	0.028	0.275
	JKR	15	**0.685**	**0.645**	0.616	0.186	**0.306**	0.198	0.427	**0.381**	0.367	**0.944**	**0.408**	0.328	0.132	0.432
	GK	16	0.603	0.566	**0.809**	**0.315**	0.243	**0.300**	0.422	0.204	**0.562**	0.833	0.344	**0.408**	**0.252**	**0.451**
	HEF	17	0.622	0.528	0.365	0.096	0.210	0.093	0.356	0.268	0.140	0.755	0.172	0.146	0.068	0.294
	HED	18	0.622	0.574	0.758	0.233	0.246	0.230	**0.429**	0.219	0.372	0.736	0.213	0.242	0.143	0.386
Δ_3	KC	19	0.593	0.541	**0.821**	0.216	0.227	0.215	0.397	0.208	0.453	0.875	0.342	**0.360**	0.205	0.419
	JKL	20	0.528	0.398	0.576	0.180	0.170	0.191	0.324	0.178	0.057	0.339	0.222	0.289	0.033	0.268
	JKR	21	**0.684**	**0.643**	0.635	0.202	**0.294**	0.214	**0.434**	**0.370**	0.401	**0.896**	**0.427**	0.358	0.152	**0.439**
	GK	22	0.502	0.452	0.731	**0.269**	0.176	**0.250**	0.272	0.129	**0.496**	0.684	0.262	0.309	**0.210**	0.365
	HEF	23	0.614	0.522	0.363	0.095	0.203	0.094	0.357	0.259	0.138	0.735	0.161	0.149	0.075	0.289
	HED	24	0.585	0.538	0.756	0.230	0.223	0.225	0.412	0.182	0.362	0.678	0.188	0.230	0.149	0.366
Δ_4	KC	25	0.542	0.478	**0.814**	0.204	0.196	0.202	0.321	0.158	0.442	0.770	0.303	0.325	**0.203**	0.381
	JKL	26	0.495	0.362	0.570	0.180	0.158	0.190	0.294	0.143	0.057	0.331	0.220	0.286	0.034	0.255
	JKR	27	**0.681**	**0.639**	0.645	0.217	**0.282**	**0.231**	**0.437**	**0.357**	0.432	**0.875**	**0.449**	**0.399**	0.163	**0.447**
	GK	28	0.459	0.400	0.688	**0.250**	0.163	0.230	0.170	0.115	**0.477**	0.629	0.245	0.281	0.188	0.330
	HEF	29	0.606	0.517	0.362	0.094	0.196	0.091	0.356	0.251	0.133	0.719	0.161	0.141	0.078	0.285
	HED	30	0.562	0.511	0.751	0.219	0.209	0.219	0.384	0.165	0.350	0.645	0.179	0.207	0.147	0.350
Δ_{L_1}	KC	31	0.544	0.461	**0.689**	**0.176**	0.172	0.184	0.352	0.151	**0.356**	0.799	0.278	0.292	**0.168**	0.356
	JKL	32	0.111	0.076	0.072	0.058	0.061	0.062	0.044	0.059	0.017	0.054	0.066	0.059	0.026	0.059
	JKR	33	0.382	0.210	0.149	0.066	0.060	0.074	0.248	0.075	0.074	0.378	0.093	0.081	0.075	0.151
	GK	34	**0.634**	**0.565**	0.525	0.161	**0.255**	**0.188**	**0.384**	**0.334**	0.301	**0.864**	**0.331**	**0.293**	0.111	**0.380**
	HEF	35	0.595	0.497	0.345	0.094	0.197	0.091	0.356	0.239	0.140	0.722	0.177	0.150	0.067	0.282
	HED	36	0.555	0.416	0.428	0.127	0.133	0.133	0.360	0.184	0.211	0.538	0.175	0.173	0.080	0.270
Δ_{SD}	KC	37	0.460	0.343	0.427	0.125	0.142	0.133	0.220	0.128	0.232	0.657	0.217	0.208	**0.123**	0.263
	JKL	38	0.072	0.073	0.065	0.065	0.061	0.059	0.036	0.059	0.016	0.037	0.060	0.062	0.024	0.053
	JKR	39	0.234	0.173	0.115	0.063	0.058	0.068	0.078	0.071	0.064	0.334	0.087	0.070	0.071	0.114
	GK	40	**0.636**	**0.565**	**0.513**	**0.169**	**0.253**	**0.193**	**0.375**	**0.327**	**0.297**	**0.855**	**0.324**	**0.290**	0.113	**0.378**
	HEF	41	0.574	0.462	0.334	0.092	0.187	0.091	0.335	0.214	0.136	0.683	0.174	0.144	0.072	0.269
	HED	42	0.551	0.433	0.444	0.134	0.143	0.142	0.342	0.187	0.220	0.568	0.168	0.181	0.075	0.276
Δ_{ST}	KC	43	0.627	0.586	**0.803**	0.212	0.260	0.209	0.408	0.273	0.423	0.926	0.366	0.361	0.192	0.434
	JKL	44	0.581	0.464	0.486	0.134	0.164	0.156	0.329	0.183	0.075	0.491	0.259	0.250	0.048	0.278
	JKR	45	**0.676**	**0.639**	0.604	0.183	**0.302**	0.188	**0.423**	**0.378**	0.350	**0.972**	0.384	0.303	0.121	0.425
	GK	46	0.639	0.585	0.651	**0.290**	0.257	**0.319**	0.415	0.285	**0.495**	0.917	**0.483**	**0.458**	**0.249**	**0.465**
	HEF	47	0.618	0.519	0.352	0.096	0.205	0.094	0.357	0.263	0.140	0.756	0.177	0.149	0.068	0.292
	HED	48	0.647	0.574	0.619	0.179	0.231	0.192	0.392	0.255	0.301	0.719	0.209	0.234	0.124	0.360
Δ_{2^*}	KC	49	0.558	0.496	**0.816**	0.213	0.205	0.212	0.359	0.168	0.457	0.811	0.326	0.348	0.208	0.398
	JKL	50	0.537	0.394	0.547	0.142	0.159	0.162	0.317	0.180	0.051	0.440	0.213	0.243	0.020	0.262
	JKR	51	**0.666**	**0.618**	0.633	0.210	0.258	0.224	0.438	**0.329**	0.410	**0.947**	**0.424**	0.384	0.166	0.439
	GK	52	0.653	0.609	0.792	**0.294**	**0.301**	**0.294**	**0.439**	0.317	**0.533**	0.873	0.405	**0.416**	**0.216**	**0.472**
	HEF	53	0.598	0.495	0.348	0.090	0.185	0.089	0.359	0.231	0.129	0.685	0.153	0.138	0.070	0.275
	HED	54	0.587	0.494	0.585	0.162	0.170	0.166	0.423	0.205	0.284	0.665	0.197	0.199	0.096	0.326
$\Delta_{L_1^*}$	KC	55	0.611	0.563	**0.798**	**0.204**	0.249	0.209	0.405	0.249	**0.414**	0.926	**0.361**	**0.349**	**0.189**	**0.425**
	JKL	56	0.529	0.365	0.406	0.094	0.133	0.111	0.271	0.166	0.044	0.473	0.191	0.175	0.035	0.230
	JKR	57	**0.653**	**0.601**	0.563	0.150	**0.249**	0.160	**0.418**	**0.331**	0.297	**0.950**	0.350	0.278	0.119	0.394
	GK	58	0.627	0.547	0.535	0.180	0.246	**0.213**	0.372	0.319	0.334	0.870	**0.361**	0.325	0.144	0.390
	HEF	59	0.592	0.475	0.338	0.093	0.184	0.089	0.353	0.223	0.128	0.695	0.165	0.139	0.071	0.272
	HED	60	0.536	0.396	0.430	0.113	0.126	0.127	0.324	0.180	0.209	0.554	0.168	0.164	0.080	0.262
Mean	KC		0.580	0.516	**0.737**	0.191	0.223	0.193	0.357	0.218	0.388	**0.834**	0.315	0.316	0.180	0.388
	JKL		0.433	0.305	0.380	0.120	0.134	0.132	0.258	0.148	0.044	0.312	0.180	0.194	0.030	0.205
	JKR		0.594	0.517	0.488	0.156	0.237	0.165	0.371	**0.296**	0.296	0.809	0.323	0.265	0.118	0.356
	GK		**0.610**	**0.552**	0.672	**0.260**	**0.251**	**0.265**	0.373	0.265	**0.465**	0.833	**0.387**	**0.380**	**0.201**	**0.424**
	HEF		0.608	0.509	0.350	0.094	0.199	0.092	0.353	0.252	0.137	0.727	0.169	0.145	0.072	0.285
	HED		0.588	0.501	0.607	0.177	0.193	0.182	**0.391**	0.204	0.291	0.639	0.184	0.205	0.113	0.329

Table 10: Mean simulated rejection rates of the summed estimator for the models M2-M14 with Skew-t noise. For each model, the mean is computed over the different parametrization described in Table 6.

6.3 Uniform noise

Tables 11 and 12 report the results concerning, respectively, the integrated and summed estimators when the noise is uniform. As in the first two cases, no single density estimation technique outperforms the others although the results are markedly different. The overall averages in (last column-last 6 lines) tables 11 and 12 emphasize that the density estimation technique with the highest overall performance is KC, not GK. This fact confirms the improvement in the performance of the dependence functional estimator obtained with the copula based density estimators, already observed for the skew-t noises.

Considering the integrated estimators, the last column of Table 11 emphasizes that GK should be considered the best choice for Δ_1, Δ_2, Δ_3. KC results the most efficient solution with $\Delta_{1/2}$, Δ_{SD}, Δ_{L_1}, Δ_{SD}, Δ_{ST}, and $\Delta_{L_1^*}$, while JKR works well with Δ_4 and Δ_{2^*}. It is worthwhile to note that, even if KC has the highest overall performance, the 5 best combinations between F3 and F5, among the 60 concerning the integrated estimators, are: Δ_2-GK, Δ_1-GK, $\Delta_{1/2}$-KC, $\Delta_{1/2}$-GK e Δ_3-GK. Among these combinations, the first 2 and the last 2 (in order of performance) are connected with GK. This result highlights that GK still remains very powerful, compared with KC, also under uniform noise.

In the previous two subsections, a substantial similarity is observed between the results obtained for summed and integrated estimators. In the uniform noise-case the results are more confused. From Table 12 it turns out that KC is the best density estimation technique for the functionals Δ_1, Δ_{L_1}, Δ_1, Δ_{ST}, and $\Delta_{L_1^*}$ while JKR works well with Δ_{2^*}. The methodology GK outperforms with $\Delta_{1/2}$, Δ_2, Δ_3, Δ_4, and Δ_{SD}. Moreover, the 5 best combinations of factor F3 and F5, among the 60 concerning the summed estimators, are: Δ_2-GK, Δ_2-KC, Δ_{ST}-KC, Δ_3-GK e Δ_3-KC. The last 5 combinations differ substantially from those obtained for the integrated estimators. The only common combinations are Δ_2-GK and Δ_3-GK:

- Δ_2-integrated works better than Δ_2-summed for 12 models out of 13;
- Δ_3-integrated works better than Δ_3-summed for 7 models out of 13.

All the previous observations are confirmed again performing the PCA and the CA on the data of tables 11 and 12. The results of both analyses are reported in figures 5 and 6 respectively.

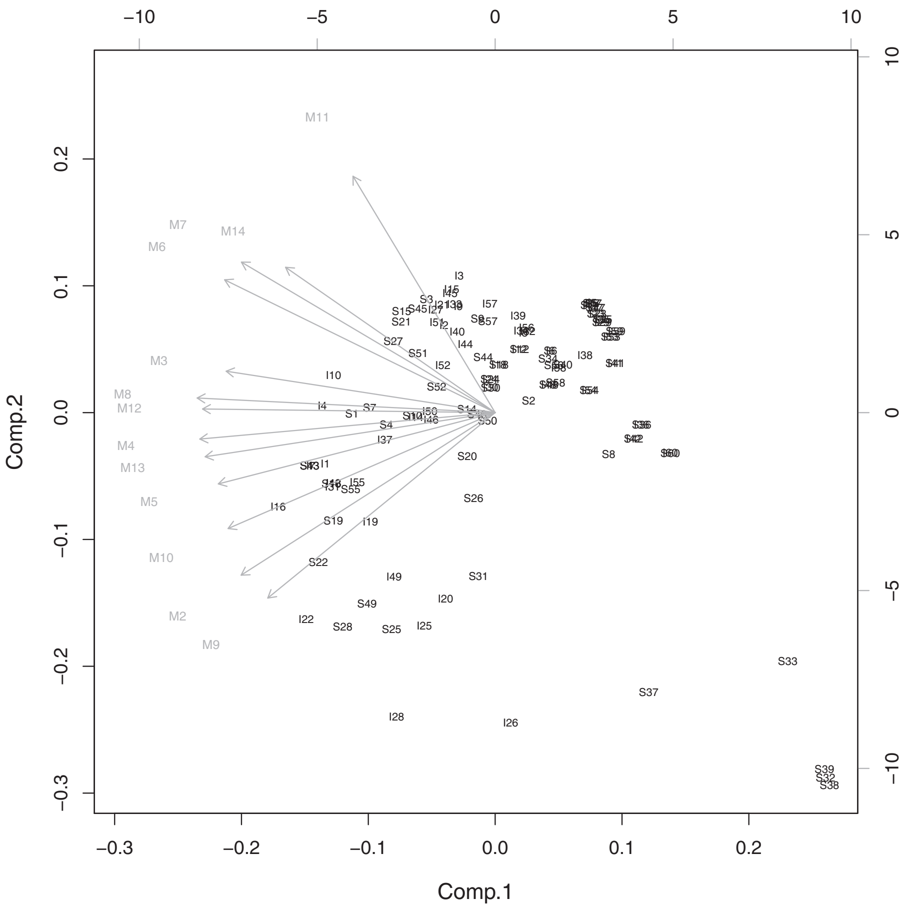

Fig. 5: Biplot for the simulations related to the uniform noise.

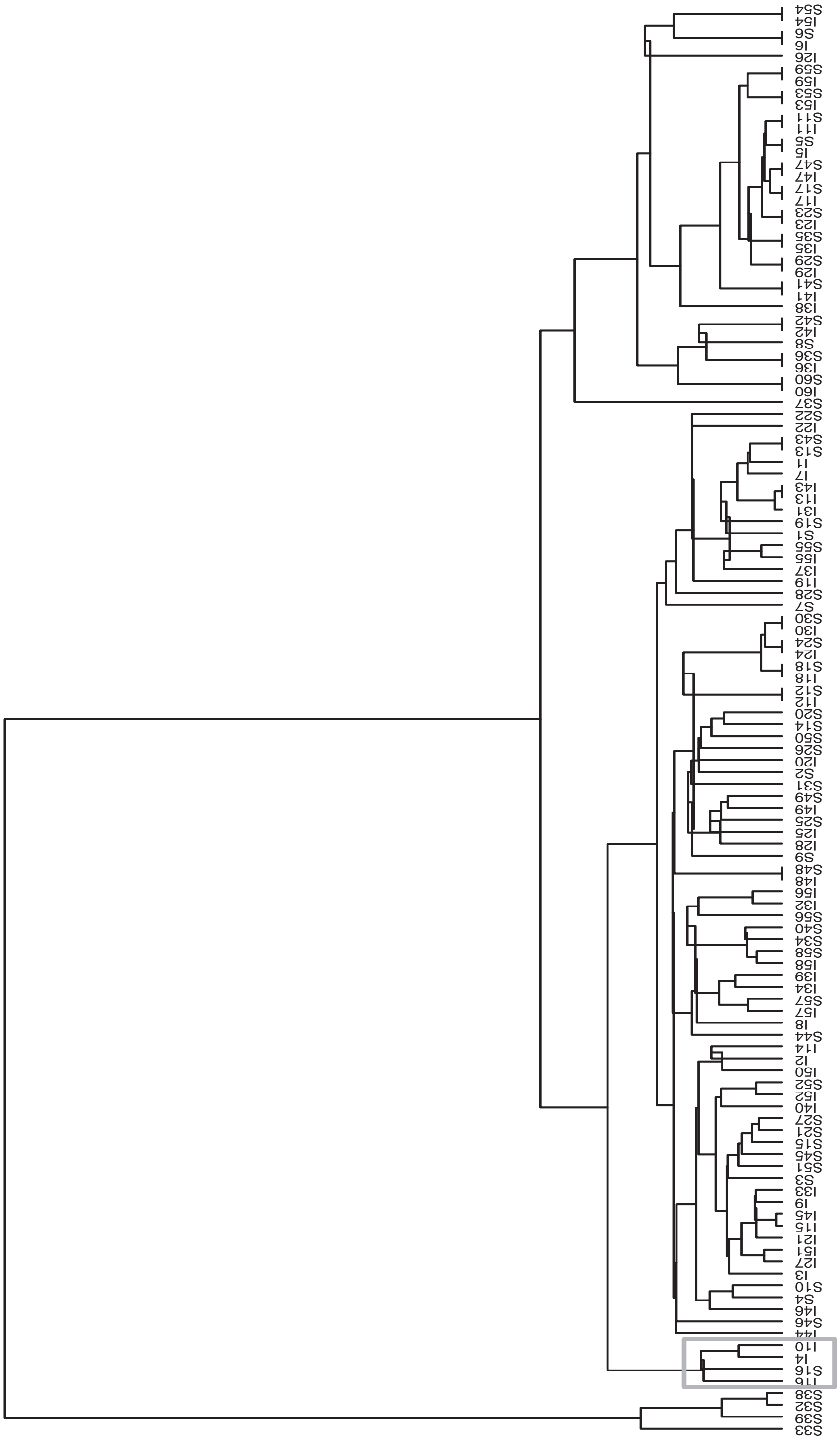

Fig. 6: Dendrogram for the simulations related to the uniform noise.

Test	Model / Method	#	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	Mean
Δ_1	KC	1	0.673	0.621	0.383	**0.153**	**0.274**	0.138	0.525	0.270	**0.592**	0.908	0.682	0.681	**0.264**	0.474
	JKL	2	0.639	0.576	0.301	0.100	0.163	0.112	**0.531**	0.222	0.339	0.896	0.812	0.452	0.126	0.405
	JKR	3	0.641	0.609	0.330	0.069	0.163	0.093	**0.531**	0.209	0.393	0.951	0.626	0.463	0.100	0.398
	GK	4	**0.676**	**0.642**	**0.385**	0.147	0.207	**0.166**	0.529	**0.272**	0.561	**0.959**	**0.914**	**0.750**	0.173	**0.491**
	HEF	5	0.554	0.495	0.194	0.062	0.101	0.075	0.530	0.137	0.219	0.715	0.303	0.263	0.060	0.285
	HED	6	0.577	0.529	0.223	0.070	0.107	0.093	0.527	0.145	0.321	0.431	0.557	0.398	0.074	0.312
$\Delta_{1/2}$	KC	7	0.671	0.632	**0.396**	**0.157**	**0.272**	0.148	0.529	0.269	**0.617**	0.937	0.728	0.712	**0.264**	**0.487**
	JKL	8	0.573	0.471	0.205	0.082	0.126	0.090	0.527	0.169	0.285	0.873	0.678	0.336	0.128	0.349
	JKR	9	0.613	0.575	0.307	0.073	0.150	0.100	**0.530**	0.197	0.418	**0.964**	0.665	0.496	0.133	0.401
	GK	10	**0.682**	**0.657**	0.388	0.126	0.208	**0.155**	0.528	**0.280**	0.546	0.936	**0.906**	**0.713**	0.157	0.483
	HEF	11	0.551	0.490	0.192	0.064	0.102	0.072	0.528	0.142	0.222	0.726	0.316	0.272	0.064	0.288
	HED	12	0.576	0.522	0.235	0.075	0.115	0.102	0.528	0.153	0.366	0.596	0.639	0.485	0.088	0.345
Δ_2	KC	13	0.656	0.579	0.366	0.160	**0.272**	0.139	0.526	**0.268**	0.587	0.896	0.668	0.681	**0.264**	0.466
	JKL	14	0.628	0.543	0.303	0.158	0.168	0.144	0.520	0.220	0.377	0.881	0.828	0.477	0.156	0.416
	JKR	15	0.633	0.600	0.330	0.068	0.160	0.098	0.530	0.200	0.418	**0.959**	0.665	0.505	0.120	0.406
	GK	16	**0.686**	**0.639**	**0.411**	**0.220**	0.220	**0.180**	**0.532**	0.267	**0.613**	0.940	**0.920**	**0.785**	0.263	**0.514**
	HEF	17	0.548	0.474	0.191	0.059	0.105	0.072	0.529	0.146	0.220	0.736	0.317	0.263	0.061	0.286
	HED	18	0.569	0.501	0.234	0.089	0.120	0.108	0.527	0.154	0.390	0.702	0.695	0.562	0.105	0.366
Δ_3	KC	19	0.622	0.488	0.325	0.163	**0.258**	0.134	0.514	**0.253**	0.560	0.806	0.597	0.650	0.261	0.433
	JKL	20	0.554	0.393	0.279	0.209	0.172	0.165	0.449	0.198	0.364	0.648	0.624	0.493	0.193	0.365
	JKR	21	0.633	**0.596**	0.333	0.074	0.162	0.099	0.529	0.201	0.435	**0.954**	0.689	0.537	0.139	0.414
	GK	22	**0.657**	0.537	**0.379**	**0.252**	0.230	**0.182**	0.519	0.224	**0.629**	0.755	**0.792**	**0.688**	**0.329**	**0.475**
	HEF	23	0.540	0.454	0.184	0.059	0.103	0.073	**0.530**	0.144	0.225	0.712	0.311	0.261	0.065	0.281
	HED	24	0.568	0.496	0.236	0.102	0.120	0.109	0.528	0.155	0.397	0.717	0.710	0.595	0.115	0.373
Δ_4	KC	25	0.580	0.386	0.290	0.163	**0.238**	0.128	0.367	**0.227**	0.519	0.681	0.542	0.595	0.258	0.383
	JKL	26	0.465	0.276	0.206	0.214	0.140	0.165	0.337	0.151	0.339	0.475	0.510	0.478	0.208	0.305
	JKR	27	**0.633**	**0.589**	**0.343**	0.073	0.161	0.100	0.528	0.200	0.455	**0.949**	**0.717**	0.567	0.142	**0.420**
	GK	28	0.559	0.357	0.308	**0.249**	0.188	**0.174**	0.439	0.162	**0.576**	0.586	0.664	**0.600**	**0.319**	0.398
	HEF	29	0.535	0.438	0.178	0.058	0.105	0.074	**0.529**	0.139	0.218	0.684	0.309	0.256	0.064	0.276
	HED	30	0.566	0.490	0.238	0.109	0.120	0.112	0.527	0.156	0.396	0.700	0.707	0.580	0.111	0.370
Δ_{L_1}	KC	31	**0.654**	**0.579**	**0.365**	**0.161**	**0.272**	**0.141**	0.526	**0.264**	**0.588**	0.902	0.668	**0.686**	**0.268**	**0.467**
	JKL	32	0.576	0.463	0.237	0.085	0.143	0.099	0.521	0.163	0.192	0.874	0.554	0.433	0.081	0.340
	JKR	33	0.616	0.573	0.315	0.069	0.151	0.100	**0.530**	0.193	0.431	**0.963**	**0.679**	0.523	0.132	0.406
	GK	34	0.548	0.433	0.199	0.072	0.114	0.099	0.520	0.162	0.378	0.959	0.638	0.447	0.091	0.358
	HEF	35	0.517	0.427	0.182	0.060	0.102	0.070	0.528	0.142	0.219	0.725	0.325	0.262	0.061	0.278
	HED	36	0.425	0.278	0.148	0.063	0.106	0.075	0.525	0.125	0.212	0.423	0.295	0.244	0.082	0.231
Δ_{SD}	KC	37	**0.626**	**0.542**	**0.314**	**0.136**	**0.225**	**0.127**	0.528	**0.231**	**0.551**	0.947	0.637	**0.617**	**0.226**	**0.439**
	JKL	38	0.497	0.367	0.180	0.073	0.111	0.082	0.510	0.136	0.195	0.839	0.493	0.294	0.091	0.298
	JKR	39	0.539	0.482	0.227	0.062	0.112	0.095	**0.528**	0.156	0.382	**0.959**	0.602	0.447	0.107	0.361
	GK	40	0.594	0.522	0.259	0.083	0.144	0.117	0.527	0.195	0.445	**0.959**	**0.762**	0.553	0.109	0.405
	HEF	41	0.473	0.377	0.172	0.063	0.096	0.070	0.491	0.133	0.224	0.726	0.312	0.273	0.072	0.268
	HED	42	0.364	0.268	0.154	0.064	0.099	0.080	0.521	0.121	0.237	0.513	0.349	0.302	0.095	0.243
Δ_{ST}	KC	43	**0.656**	0.579	**0.366**	**0.160**	**0.272**	0.139	0.526	**0.268**	**0.587**	0.896	0.668	0.681	**0.264**	**0.466**
	JKL	44	0.611	0.524	0.258	0.095	0.166	0.106	0.529	0.208	0.289	0.924	0.683	0.523	0.144	0.389
	JKR	45	0.632	**0.600**	0.330	0.069	0.164	0.100	**0.530**	0.201	0.417	0.958	0.662	0.501	0.122	0.407
	GK	46	0.568	0.451	0.223	0.116	0.118	**0.164**	0.526	0.162	0.502	**0.997**	**0.854**	**0.718**	0.154	0.427
	HEF	47	0.545	0.462	0.186	0.061	0.102	0.071	0.528	0.142	0.220	0.731	0.320	0.263	0.062	0.284
	HED	48	0.507	0.364	0.180	0.077	0.107	0.102	0.529	0.132	0.361	0.744	0.579	0.485	0.096	0.328
Δ_{2^*}	KC	49	0.601	0.435	0.310	0.164	**0.254**	0.133	0.443	**0.240**	**0.545**	0.745	0.573	**0.623**	**0.259**	0.410
	JKL	50	0.615	0.513	0.303	**0.169**	0.172	**0.148**	0.526	0.197	0.302	0.888	0.796	0.559	0.100	0.407
	JKR	51	**0.626**	**0.575**	**0.337**	0.073	0.164	0.103	0.526	0.194	0.456	0.947	0.708	0.571	0.153	**0.418**
	GK	52	0.597	0.490	0.257	0.098	0.148	0.124	0.525	0.189	0.471	**0.972**	**0.804**	0.606	0.135	0.417
	HEF	53	0.513	0.408	0.171	0.059	0.101	0.074	**0.530**	0.137	0.210	0.665	0.289	0.254	0.068	0.268
	HED	54	0.502	0.368	0.173	0.070	0.110	0.084	0.529	0.138	0.288	0.502	0.456	0.357	0.083	0.281
$\Delta_{L_1^*}$	KC	55	**0.637**	**0.536**	**0.338**	**0.154**	**0.255**	**0.135**	0.526	**0.252**	**0.565**	0.906	**0.640**	**0.659**	**0.258**	**0.451**
	JKL	56	0.569	0.454	0.225	0.083	0.141	0.097	0.529	0.171	0.203	0.902	0.549	0.446	0.086	0.343
	JKR	57	0.586	0.525	0.275	0.067	0.132	0.096	**0.530**	0.172	0.395	0.957	0.634	0.469	0.110	0.381
	GK	58	0.481	0.312	0.159	0.072	0.102	0.097	0.518	0.131	0.361	**0.969**	0.580	0.426	0.091	0.331
	HEF	59	0.498	0.394	0.168	0.057	0.097	0.069	0.529	0.133	0.216	0.704	0.293	0.259	0.063	0.268
	HED	60	0.373	0.223	0.127	0.066	0.096	0.072	0.525	0.112	0.188	0.345	0.248	0.205	0.078	0.204
Mean	KC		**0.637**	0.538	**0.345**	**0.157**	**0.259**	0.136	0.501	**0.254**	**0.571**	0.862	0.640	**0.659**	**0.259**	**0.448**
	JKL		0.573	0.458	0.250	0.127	0.150	0.121	0.498	0.183	0.288	0.820	0.653	0.449	0.131	0.362
	JKR		0.615	**0.572**	0.313	0.070	0.152	0.098	**0.529**	0.192	0.420	**0.956**	0.665	0.508	0.126	0.401
	GK		0.605	0.504	0.297	0.143	0.168	**0.146**	0.516	0.204	0.508	0.903	**0.783**	0.629	0.182	0.430
	HEF		0.527	0.442	0.181	0.060	0.101	0.072	0.525	0.139	0.219	0.712	0.310	0.263	0.064	0.278
	HED		0.503	0.404	0.195	0.078	0.110	0.094	0.527	0.139	0.315	0.567	0.524	0.421	0.093	0.305

Table 11: Mean simulated rejection rates of the integrated estimator for the models M2-M14 with Uniform noise. For each model, the mean is computed over the different parametrization described in Table 6.

Test	Model Method	#	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	Mean
Δ_1	KC	1	**0.685**	**0.639**	**0.358**	0.132	**0.260**	0.129	0.528	**0.279**	**0.552**	0.876	0.646	0.623	**0.211**	**0.455**
	JKL	2	0.579	0.452	0.239	0.090	0.146	0.096	0.527	0.175	0.261	0.440	0.627	0.416	0.119	0.320
	JKR	3	0.648	0.611	0.349	0.081	0.195	0.096	**0.532**	0.238	0.445	0.917	0.723	0.507	0.114	0.420
	GK	4	0.611	0.527	0.290	**0.140**	0.153	**0.163**	0.525	0.205	0.536	**0.976**	**0.878**	**0.715**	0.165	0.452
	HEF	5	0.554	0.495	0.194	0.062	0.101	0.075	0.530	0.137	0.219	0.715	0.303	0.263	0.060	0.285
	HED	6	0.577	0.529	0.223	0.070	0.107	0.093	0.527	0.145	0.321	0.431	0.557	0.398	0.074	0.312
$\Delta_{1/2}$	KC	7	**0.696**	**0.646**	**0.363**	0.124	**0.267**	0.118	0.528	**0.286**	**0.530**	0.711	0.607	0.576	**0.186**	0.434
	JKL	8	0.521	0.340	0.182	0.075	0.127	0.089	0.509	0.139	0.197	0.162	0.415	0.303	0.095	0.243
	JKR	9	0.632	0.563	0.306	0.076	0.182	0.084	**0.528**	0.222	0.395	0.740	0.580	0.401	0.103	0.370
	GK	10	0.592	0.497	0.265	**0.128**	0.142	**0.151**	0.520	0.193	0.516	**0.963**	**0.858**	**0.684**	0.156	**0.436**
	HEF	11	0.551	0.490	0.192	0.064	0.102	0.072	0.528	0.142	0.222	0.726	0.316	0.272	0.064	0.288
	HED	12	0.576	0.522	0.235	0.075	0.115	0.102	0.528	0.153	0.366	0.596	0.639	0.485	0.088	0.345
Δ_2	KC	13	**0.685**	**0.628**	**0.389**	0.156	**0.287**	0.143	0.528	**0.288**	**0.603**	0.919	0.700	0.702	**0.265**	0.484
	JKL	14	0.611	0.503	0.295	0.113	0.172	0.112	0.530	0.215	0.288	0.573	0.710	0.496	0.151	0.367
	JKR	15	0.653	0.620	0.370	0.086	0.204	0.106	**0.532**	0.241	0.476	**0.960**	0.764	0.577	0.131	0.440
	GK	16	0.647	0.568	0.344	**0.200**	0.183	**0.168**	0.525	0.229	0.581	0.959	**0.910**	**0.770**	0.218	**0.485**
	HEF	17	0.548	0.474	0.191	0.059	0.105	0.072	0.529	0.146	0.220	0.736	0.317	0.263	0.061	0.286
	HED	18	0.569	0.501	0.234	0.089	0.120	0.108	0.527	0.154	0.390	0.702	0.695	0.562	0.105	0.366
Δ_3	KC	19	0.655	0.549	0.359	0.172	**0.271**	0.145	0.522	**0.269**	0.596	0.837	0.640	0.681	**0.285**	0.460
	JKL	20	0.599	0.478	0.299	0.127	0.175	0.122	0.529	0.211	0.274	0.470	0.684	0.502	0.172	0.357
	JKR	21	0.644	**0.610**	**0.369**	0.088	0.197	0.106	0.530	0.235	0.481	**0.953**	0.786	0.599	0.141	0.441
	GK	22	**0.664**	0.563	0.364	**0.233**	0.211	**0.166**	0.504	0.232	**0.600**	0.830	**0.844**	**0.724**	**0.285**	**0.478**
	HEF	23	0.540	0.454	0.184	0.059	0.103	0.073	**0.530**	0.144	0.225	0.712	0.311	0.261	0.065	0.281
	HED	24	0.568	0.496	0.236	0.102	0.120	0.109	0.528	0.155	0.397	0.717	0.710	0.595	0.115	0.373
Δ_4	KC	25	0.616	0.440	0.318	0.173	**0.257**	0.140	0.383	**0.246**	0.556	0.680	0.526	0.608	0.279	0.402
	JKL	26	0.577	0.448	0.292	0.137	0.186	0.123	0.520	0.205	0.261	0.392	0.640	0.512	0.188	0.345
	JKR	27	0.648	**0.598**	**0.371**	0.090	0.204	0.112	**0.529**	0.233	0.491	**0.944**	**0.796**	0.611	0.164	0.445
	GK	28	**0.652**	0.512	0.350	**0.237**	0.220	**0.167**	0.459	0.219	**0.591**	0.690	0.748	**0.652**	**0.301**	**0.446**
	HEF	29	0.535	0.438	0.178	0.058	0.105	0.074	**0.529**	0.139	0.218	0.684	0.309	0.256	0.064	0.276
	HED	30	0.566	0.490	0.238	0.109	0.120	0.112	0.527	0.156	0.396	0.700	0.707	0.580	0.111	0.370
Δ_{L_1}	KC	31	**0.529**	0.344	**0.234**	**0.144**	**0.179**	**0.118**	0.397	**0.172**	**0.480**	0.789	0.487	**0.542**	**0.232**	**0.357**
	JKL	32	0.064	0.056	0.051	0.057	0.055	0.059	0.037	0.055	0.048	0.079	0.079	0.058	0.062	0.058
	JKR	33	0.216	0.086	0.056	0.058	0.053	0.059	0.184	0.058	0.083	0.178	0.092	0.111	0.056	0.099
	GK	34	0.514	0.353	0.177	0.072	0.113	0.096	0.513	0.137	0.358	**0.961**	**0.571**	0.418	0.097	0.337
	HEF	35	0.517	**0.427**	0.182	0.060	0.102	0.070	**0.528**	0.142	0.219	0.725	0.325	0.262	0.061	0.278
	HED	36	0.425	0.278	0.148	0.063	0.106	0.075	0.525	0.125	0.212	0.423	0.295	0.244	0.082	0.231
Δ_{SD}	KC	37	0.222	0.168	0.143	**0.117**	**0.108**	0.087	0.175	0.102	0.281	0.732	0.300	0.335	**0.150**	0.224
	JKL	38	0.040	0.052	0.049	0.058	0.053	0.060	0.036	0.057	0.046	0.071	0.066	0.057	0.060	0.054
	JKR	39	0.028	0.056	0.051	0.056	0.053	0.057	0.045	0.055	0.061	0.157	0.070	0.090	0.057	0.064
	GK	40	**0.487**	0.339	0.163	0.067	0.098	**0.097**	0.506	0.128	**0.349**	**0.938**	**0.578**	**0.403**	0.089	**0.326**
	HEF	41	0.473	**0.377**	**0.172**	0.063	0.096	0.070	0.491	**0.133**	0.224	0.726	0.312	0.273	0.072	0.268
	HED	42	0.364	0.268	0.154	0.064	0.099	0.080	**0.521**	0.121	0.237	0.513	0.349	0.302	0.095	0.243
Δ_{ST}	KC	43	**0.685**	**0.628**	**0.389**	**0.156**	**0.287**	0.143	0.528	**0.288**	**0.603**	0.919	0.700	**0.702**	**0.265**	**0.484**
	JKL	44	0.585	0.460	0.248	0.089	0.170	0.099	**0.531**	0.213	0.262	0.903	0.585	0.557	0.135	0.372
	JKR	45	0.642	0.602	0.349	0.078	0.193	0.106	**0.531**	0.231	0.463	0.954	0.741	0.561	0.125	0.429
	GK	46	0.537	0.363	0.192	0.098	0.112	**0.145**	0.516	0.140	0.466	**0.994**	**0.791**	0.645	0.131	0.395
	HEF	47	0.545	0.462	0.186	0.061	0.102	0.071	0.528	0.142	0.220	0.731	0.320	0.263	0.062	0.284
	HED	48	0.507	0.364	0.180	0.077	0.107	0.102	0.529	0.132	0.361	0.744	0.579	0.485	0.096	0.328
Δ_{2^*}	KC	49	**0.633**	0.468	0.336	**0.178**	**0.264**	**0.144**	0.421	**0.252**	**0.577**	0.749	0.569	**0.650**	**0.285**	0.425
	JKL	50	0.575	0.437	0.268	0.118	0.167	0.121	0.517	0.189	0.218	0.724	0.618	0.542	0.131	0.356
	JKR	51	0.625	**0.559**	**0.339**	0.089	0.179	0.112	0.529	0.211	0.471	0.923	0.762	0.591	0.168	**0.427**
	GK	52	0.589	0.478	0.249	0.111	0.157	0.128	0.520	0.187	0.486	**0.979**	**0.810**	0.631	0.137	0.420
	HEF	53	0.513	0.408	0.171	0.059	0.101	0.074	**0.530**	0.137	0.210	0.665	0.289	0.254	0.068	0.268
	HED	54	0.502	0.368	0.173	0.070	0.110	0.084	0.529	0.138	0.288	0.502	0.456	0.357	0.083	0.281
$\Delta_{L_1^*}$	KC	55	**0.641**	**0.532**	**0.344**	**0.159**	**0.253**	**0.140**	0.528	**0.252**	**0.573**	0.920	0.642	**0.677**	**0.266**	**0.456**
	JKL	56	0.524	0.381	0.195	0.080	0.138	0.090	0.507	0.177	0.164	0.860	0.440	0.466	0.090	0.316
	JKR	57	0.580	0.504	0.266	0.073	0.137	0.097	**0.529**	0.173	0.402	0.936	**0.647**	0.497	0.114	0.381
	GK	58	0.482	0.293	0.162	0.073	0.103	0.102	0.509	0.125	0.368	**0.978**	0.588	0.445	0.099	0.333
	HEF	59	0.498	0.394	0.168	0.057	0.097	0.069	**0.529**	0.133	0.216	0.704	0.293	0.259	0.063	0.268
	HED	60	0.373	0.223	0.127	0.066	0.096	0.072	0.525	0.112	0.188	0.345	0.248	0.205	0.078	0.204
Mean	KC		**0.605**	**0.504**	**0.323**	**0.151**	**0.243**	0.131	0.454	**0.243**	**0.535**	0.813	0.582	**0.610**	**0.242**	**0.418**
	JKL		0.467	0.361	0.212	0.094	0.139	0.097	0.424	0.163	0.202	0.467	0.486	0.391	0.120	0.279
	JKR		0.532	0.481	0.283	0.077	0.160	0.093	0.447	0.190	0.377	0.766	0.596	0.455	0.117	0.352
	GK		0.577	0.449	0.255	0.136	0.149	**0.138**	0.510	0.179	0.485	**0.927**	**0.758**	0.609	0.168	0.411
	HEF		0.527	0.442	0.181	0.060	0.101	0.072	0.525	0.139	0.219	0.712	0.310	0.263	0.064	0.278
	HED		0.503	0.404	0.195	0.078	0.110	0.094	**0.527**	0.139	0.315	0.567	0.524	0.421	0.093	0.305

Table 12: Mean simulated rejection rates of the summed estimator for the models M2-M14 with Uniform noise. For each model, the mean is computed over the different parametrization described in Table 6.

6.4 General considerations

In order to identify which are the best combinations of factors F3-F5, the results obtained in the previous three subsections can be intersected. Since the results obtained under Gaussian and skew-t noise are quite similar, the uniform noise setting plays a very important role. The best combinations in the uniform-noise case are related to GK combined with the integrated and summed estimators of functional Δ_2. These combinations are contained in the ranking of the 10 best solutions also for the Gaussian and skew-t noises, but in the latter cases they are not among the first positions. Analogously, the other 4 best combinations related to summed estimators under uniform noise are not in the ranking for the Gaussian and skew-t noise. The same observation can be made for the combinations $\Delta_{1/2}$-KC-integrated and Δ_3-GK-integrated obtained under uniform noise. The remaining combinations are Δ_1-GK-integrated and $\Delta_{1/2}$-GK-integrated. Both perform well under all the kind of noise but, undoubtedly, the best is Δ_1-GK-Integrated which is the best solution under Gaussian and skew-t noise and occupies second place under uniform noise. So, in the applications the use of this latter combination is advised. Table 13 gives the power obtained with Δ_1-GK-integrated under M1-M11 for all the parameterizations in Table 6. The simulated rejection rates related to M12 and M13 are not reported here since they are already reported in tables 7-12.

The results of our simulation study seem not to be coherent with Fernandes and Néri (2010). They point out that the power of the test based on the summed estimators of Δ_q, tends to be increasing in q. Our simulations are in contrast with that result since we observe that the power is roughly decreasing in q. This difference is due to the use of the trimming function and simulation results, not reported here for the sake of brevity, confirms this issue. Not by chance, the results obtained under the uniform noise (which, in a sense, is trimmed by definition) turn out to be more consistent with the results of Fernandes and Néri (2010).

Another interesting observation is that the test obtained with the copula-based density estimators seems to be more robust with respect to changes in the distribution of noise. However, this robustness seems not to be sufficient in order to make the test based on KC the most efficient.

7 Simulation results: multiple-lag procedures

This section aims to compare the multiple-lag procedures, P-test, MlP-test, and S-test, outlined in Section 4.2. Table 14 reports their simulated rejection rates related to the integrated-GK estimator of Δ_1 under Gaussian noise. Only the latter test is adopted since, as illustrated in the previous section, it can be considered the most powerful. Moreover, the results concerning the parametrization ϑ_{31}, ϑ_{32}, and ϑ_{33} and those concerning the skew-t and uniform noise are not reported since they show a very similar behavior. The values $p = 2, \dots, 5$ are taken into account.

In order to identify the best procedure, it is necessary to understand what the desirable features of a multiple-lag test are. First, its actual level should be equal to the chosen nominal level. Second, it is reasonable to assert that a good multiple-lag procedure should have, roughly speaking, a "persistent" power. For example, consider the first-lag dependence induced by a simple MA(1). In this case, the

Model	M1		M2	M3	M4	M5	M6	M7	M8	M9	M10	M11
Noise		Parameter										
Gaussian	0.041	ϑ_{11}	0.133	0.106	0.215	0.067	0.077	0.073	1.000	0.075	0.185	0.175
		ϑ_{12}	0.405	0.401	0.573	0.108	0.146	0.110	0.999	0.088	0.362	0.174
		ϑ_{13}	0.981	0.905	0.925	0.271	0.436	0.307	0.890	0.160	0.734	0.188
		ϑ_{14}	1.000	0.985	0.947	0.444		0.403	0.051	0.507	0.905	0.233
		ϑ_{31}	0.427	0.395	0.577	0.078		0.091	1.000	0.077	0.331	0.072
		ϑ_{32}	0.972	0.899	0.902	0.197		0.219	0.974	0.174	0.703	0.071
		ϑ_{33}	1.000	0.984	0.950	0.302		0.275	0.754	0.492	0.923	0.069
skew-t	0.065	ϑ_{11}	0.167	0.163	0.543	0.074	0.093	0.087	0.916	0.084	0.256	0.969
		ϑ_{12}	0.613	0.549	0.829	0.184	0.217	0.175	0.575	0.094	0.507	0.955
		ϑ_{13}	0.997	0.977	0.981	0.447	0.737	0.487	0.098	0.338	0.726	0.950
		ϑ_{14}	1.000	0.991	0.985	0.679		0.607	0.170	0.832	0.891	0.958
uniform	0.055	ϑ_{11}	0.131	0.148	0.108	0.077	0.077	0.076	0.999	0.073	0.109	0.956
		ϑ_{12}	0.580	0.486	0.274	0.088	0.096	0.095	1.000	0.092	0.299	0.954
		ϑ_{13}	0.992	0.958	0.579	0.145	0.449	0.208	0.062	0.220	0.851	0.959
		ϑ_{14}	1.000	0.977	0.580	0.276		0.284	0.056	0.703	0.984	0.965

Table 13: Simulated rejection rates for the integrated-GK estimator of $\varDelta_1$ along the different parametrizations described in Table 6 and the different kind of noise. In the second column the simulated level are also given.

variables X_t and X_{t+l}, for $l > 1$, are independent; consequently, a serial independence test performed on the single lag l should have a low power (approximately equal to α) if $l > 1$ and a high power if $l = 1$. In this scenario, it is preferable that the power of a multiple lag test (which incorporates the first lag) does not decrease when p increases.

From an initial glance at models M2-M14 in Table 14, the S-test procedure appears to be the best one in terms of permanence and strength of power across the considered values of p (see the global means in the last row of Table 14). Nevertheless, a more accurate analysis highlights interesting considerations. The P-test for $p = 2$ has the highest simulated rejection rates, with respect to the MlP-test and the S-test, in 29 cases out of the 43 considered. This result is confirmed for $p = 3$ (the P-test has the higher power in 27 cases out of 43). For $p = 4$ and $p = 5$ the procedure that works beter seems to be the S-test which has the highest power in 38 and 40 cases out of 43, respectively. However, taking into consideration also the size of the multiple tests (see the results in the first row of Table 14 related to M1), it is clear that the level of the S-test increases when p increases. Then, the higher power of the S-test, for $p = 4$ and $p = 5$, could be due to the increase in the actual level, and not to an effective better performance. This is the reason why the use of the S-test is not advised. Similarly, the slightly lower power of the MlP-test, for $p = 4$ and $p = 5$, could be due to lower size-values. Other simulations, whose results are not reported here for brevity, performed with different kinds of noise (Gaussian, uniform and skew-t) and with p assuming values up to 10, confirm that: the P-test is the most stable in terms of size, the MlP-test has a high variability, and the S-test, as already mentioned, shows a size which increases when p increases. Since the performance of the P-test and the MlP-test seems to be similar in terms of power (see Table 14), the considerations about the size suggest the use of the P-test. This is also supported by other authors, such as Hong and White (2005) and Skaug and Tjøstheim (1993a) among others, which

			P-test				MlP-test				S-test			
	p	**1**	2	3	4	5	2	3	4	5	2	3	4	5
Model	Parameter													
M1		*0.041*	*0.050*	*0.052*	*0.049*	*0.055*	*0.047*	*0.048*	*0.036*	*0.039*	*0.047*	*0.048*	*0.063*	*0.081*
M2	ϑ_{11}	0.133	0.097	0.088	0.085	0.089	0.095	0.083	0.067	0.066	0.095	0.083	0.108	0.129
	ϑ_{12}	0.405	0.313	0.257	0.233	0.223	0.340	0.292	0.229	0.212	0.339	0.291	0.300	0.315
	ϑ_{13}	0.981	0.951	0.915	0.890	0.872	0.963	0.948	0.916	0.906	0.963	0.948	0.949	0.949
	ϑ_{14}	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Mean	**0.630**	**0.590**	**0.565**	**0.552**	**0.546**	**0.600**	**0.581**	**0.553**	**0.546**	**0.599**	**0.581**	**0.589**	**0.598**
M3	ϑ_{11}	0.106	0.102	0.096	0.092	0.110	0.093	0.087	0.070	0.075	0.093	0.087	0.106	0.131
	ϑ_{12}	0.401	0.289	0.233	0.217	0.203	0.327	0.276	0.217	0.207	0.327	0.276	0.302	0.315
	ϑ_{13}	0.905	0.821	0.736	0.696	0.647	0.851	0.807	0.730	0.719	0.851	0.807	0.810	0.817
	ϑ_{14}	0.985	0.978	0.946	0.904	0.867	0.974	0.961	0.939	0.931	0.974	0.961	0.963	0.964
	Mean	**0.599**	**0.548**	**0.503**	**0.477**	**0.457**	**0.561**	**0.533**	**0.489**	**0.483**	**0.561**	**0.533**	**0.545**	**0.557**
M4	ϑ_{11}	0.215	0.161	0.131	0.122	0.110	0.157	0.137	0.100	0.099	0.157	0.137	0.154	0.174
	ϑ_{12}	0.573	0.394	0.339	0.280	0.245	0.444	0.373	0.301	0.278	0.444	0.373	0.382	0.392
	ϑ_{13}	0.925	0.802	0.708	0.617	0.528	0.873	0.828	0.752	0.730	0.873	0.828	0.829	0.831
	ϑ_{14}	0.947	0.842	0.736	0.623	0.547	0.898	0.854	0.786	0.763	0.898	0.854	0.857	0.858
	Mean	**0.665**	**0.550**	**0.479**	**0.411**	**0.358**	**0.593**	**0.548**	**0.485**	**0.468**	**0.593**	**0.548**	**0.556**	**0.564**
M5	ϑ_{11}	0.067	0.069	0.061	0.049	0.065	0.071	0.056	0.039	0.042	0.071	0.056	0.072	0.091
	ϑ_{12}	0.108	0.114	0.099	0.088	0.088	0.096	0.086	0.062	0.071	0.096	0.086	0.101	0.123
	ϑ_{13}	0.271	0.293	0.256	0.221	0.202	0.254	0.210	0.165	0.146	0.254	0.210	0.223	0.232
	ϑ_{14}	0.444	0.510	0.404	0.358	0.314	0.439	0.391	0.317	0.304	0.438	0.390	0.398	0.409
	Mean	**0.223**	**0.247**	**0.205**	**0.179**	**0.167**	**0.215**	**0.186**	**0.146**	**0.141**	**0.215**	**0.186**	**0.199**	**0.214**
M6	ϑ_{11}	0.077	0.082	0.079	0.083	0.085	0.076	0.071	0.052	0.059	0.075	0.071	0.094	0.114
	ϑ_{12}	0.146	0.106	0.104	0.104	0.086	0.101	0.089	0.077	0.070	0.100	0.089	0.112	0.127
	ϑ_{13}	0.436	0.331	0.274	0.245	0.229	0.347	0.302	0.244	0.234	0.347	0.302	0.322	0.340
	Mean	**0.220**	**0.173**	**0.152**	**0.144**	**0.133**	**0.175**	**0.154**	**0.124**	**0.121**	**0.174**	**0.154**	**0.176**	**0.194**
M7	ϑ_{11}	0.073	0.080	0.070	0.063	0.068	0.090	0.085	0.055	0.057	0.090	0.084	0.098	0.123
	ϑ_{12}	0.110	0.133	0.118	0.112	0.096	0.122	0.109	0.078	0.071	0.122	0.109	0.130	0.141
	ϑ_{13}	0.307	0.445	0.376	0.366	0.351	0.377	0.354	0.292	0.266	0.377	0.354	0.368	0.379
	ϑ_{14}	0.403	0.818	0.783	0.769	0.743	0.794	0.765	0.712	0.700	0.794	0.765	0.773	0.776
	Mean	**0.223**	**0.369**	**0.337**	**0.328**	**0.315**	**0.346**	**0.328**	**0.284**	**0.274**	**0.346**	**0.328**	**0.342**	**0.355**
M8	ϑ_{11}	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	ϑ_{12}	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.998	0.998	0.999	0.999	0.999	0.999
	ϑ_{13}	0.890	0.885	0.887	0.886	0.885	0.891	0.887	0.883	0.883	0.891	0.887	0.890	0.890
	ϑ_{14}	0.051	0.063	0.058	0.060	0.053	0.063	0.049	0.039	0.041	0.063	0.049	0.066	0.080
	Mean	**0.735**	**0.737**	**0.736**	**0.736**	**0.734**	**0.738**	**0.734**	**0.730**	**0.731**	**0.738**	**0.734**	**0.739**	**0.742**
M9	ϑ_{11}	0.075	0.071	0.079	0.074	0.067	0.065	0.067	0.056	0.056	0.065	0.067	0.090	0.107
	ϑ_{12}	0.088	0.076	0.061	0.077	0.071	0.073	0.062	0.054	0.051	0.072	0.062	0.085	0.101
	ϑ_{13}	0.160	0.122	0.108	0.107	0.094	0.109	0.104	0.080	0.086	0.109	0.103	0.121	0.140
	ϑ_{14}	0.507	0.404	0.351	0.308	0.276	0.420	0.364	0.299	0.286	0.419	0.364	0.379	0.393
	Mean	**0.208**	**0.168**	**0.150**	**0.142**	**0.127**	**0.167**	**0.149**	**0.122**	**0.120**	**0.166**	**0.149**	**0.169**	**0.185**
M10	ϑ_{11}	0.185	0.151	0.124	0.104	0.108	0.145	0.122	0.091	0.089	0.145	0.122	0.134	0.152
	ϑ_{12}	0.362	0.298	0.243	0.227	0.207	0.307	0.252	0.202	0.179	0.307	0.252	0.266	0.274
	ϑ_{13}	0.734	0.646	0.588	0.528	0.469	0.668	0.597	0.506	0.494	0.668	0.597	0.605	0.611
	ϑ_{14}	0.905	0.869	0.815	0.783	0.736	0.864	0.833	0.785	0.766	0.864	0.833	0.834	0.836
	Mean	**0.547**	**0.491**	**0.443**	**0.411**	**0.380**	**0.496**	**0.451**	**0.396**	**0.382**	**0.496**	**0.451**	**0.460**	**0.468**
M11	ϑ_{11}	0.175	0.160	0.132	0.119	0.118	0.147	0.119	0.094	0.100	0.147	0.118	0.128	0.148
	ϑ_{12}	0.174	0.175	0.167	0.141	0.128	0.146	0.132	0.091	0.087	0.145	0.132	0.142	0.151
	ϑ_{13}	0.188	0.188	0.191	0.181	0.181	0.166	0.154	0.117	0.106	0.166	0.154	0.168	0.186
	ϑ_{14}	0.233	0.280	0.318	0.333	0.353	0.247	0.250	0.219	0.212	0.247	0.250	0.280	0.308
	Mean	**0.193**	**0.201**	**0.202**	**0.194**	**0.195**	**0.177**	**0.164**	**0.130**	**0.126**	**0.176**	**0.164**	**0.180**	**0.198**
M12		0.240	0.274	0.302	0.324	0.341	0.238	0.259	0.229	0.226	0.238	0.259	0.300	0.332
M13		0.774	0.805	0.787	0.762	0.719	0.754	0.733	0.650	0.624	0.754	0.733	0.746	0.755
M14		0.647	0.730	0.770	0.770	0.753	0.672	0.664	0.639	0.633	0.672	0.664	0.692	0.714
	Mean on M2-M14	**0.438**	**0.427**	**0.400**	**0.381**	**0.365**	**0.423**	**0.400**	**0.363**	**0.355**	**0.423**	**0.400**	**0.414**	**0.426**

Table 14: Simulated rejection rates for the integrated-GK estimator of Δ_1 along the the different parametrisations described in Table 6 and the Gaussian noise.

select the P-test as natural extension of some single-lag procedures to the multiple lag context.

8 Conclusions

This paper focuses on the class of nonparametric serial independence tests based on divergence measures between the estimated joint density and the product of the estimated marginal densities. For this class, a detailed review is provided. The copula version of all the considered divergence functionals is also presented.

The existing tests essentially differ in the choices made for the following five aspects: technique used to estimate the densities, divergence measure, methodology (summed or integrated) used to compute the functional estimates, use of trimming functions, and approach (bootstrap or permutation) considered to obtain p-values. In the literature different combinations of these factors are considered in order to obtain a particular test, although an exhaustive comparison of performance among them does not exist. In these terms, the performed simulation study on a wide class of linear and nonlinear models provides some useful guidelines. In summary, the permutation approach is the best method to obtain p-values and the use of trimming functions is not needed. Also, the integrated estimator of the Kullback-Leibler functional $\varDelta_1$, combined with the Gaussian kernel density estimation techniques, provides the best global performance. From simulations, the “Portmanteau” approach to extend the results obtained on the single lags to more than one, reveals to be the best choice since it preserves size across lags.

References

Ahmad I, Li Q (1997) Testing independence by nonparametric kernel method. Statistics & Probability Letters 34(2):201–210

Azzalini A, Capitanio A (2003) Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 65(2):367–389

Bagnato L, Punzo A (2010) On the Use of χ^2-Test to Check Serial Independence. Statistica & Applicazioni VIII(1):57–74

Bagnato L, Punzo A (2011) Checking serial independence of residuals from a nonlinear model (to appear). In: Buza K, Schmidt L (eds) Studies in Classification, Data Analysis, and Knowledge Organization, Springer, Berlin

Bellman R (1961) Adaptive control processes. Princeton University Press, Princeton, N.J.

Brock W, Hsieh D, LeBaron B (1991) Nonlinear dynamics, chaos, and instability: statistical theory and economic evidence. MIT Press, Cambridge

Chan N, Tran L (1992) Nonparametric tests for serial dependence. Journal of Time Series Analysis 13(1):19–28

Darrat A, Zhong M (2000) On Testing the Random-Walk Hypothesis: A Model-Comparison Approach. Financial Review 35(3):105–124

Delgado M (1996) Testing serial independence using the sample distribution function. Journal of Time Series Analysis 17(3):271–285

Diks C (2009) Nonparametric tests for independence. In: Meyers RA (ed) Encyclopedia of Complexity and Systems Science, Springer, New York, pp 6252–6271
Diks C, Panchenko V (2007) Nonparametric tests for serial independence based on quadratic forms. Statistica Sinica 17(1):81–89
Dionísio A, Menezes R, Mendes D (2006) Entropy-based independence test. Nonlinear Dynamics 44(1):351–357
Dufour J, Roy R (1985) Some robust exact results on sample autocorrelations and tests of randomness. Journal of Econometrics 29(3):257–273
Durbin J, Watson G (1950) Testing for serial correlation in least squares regression. I. Biometrika 37(3–4):409–428
Durbin J, Watson G (1951) Testing for serial correlation in least squares regression. II. Biometrika 38(1–2):159–177
Fernandes M, Néri B (2010) Nonparametric entropy-based tests of independence between stochastic processes. Econometric Reviews 29(3):276–306
Genest C, Verret F (2005) Locally most powerful rank tests of independence for copula models. Journal of Nonparametric Statistics 17(5):521–539
Genest C, Quessy JF, Rémillard B (2002) Tests of serial independence based on Kendall's process. Canadian Journal of Statistics 30(3):441–461
Genest C, Ghoudi K, Remillard B (2007) Rank-based extensions of the Brock, Dechert, and Scheinkman test. Journal of the American Statistical Association 102(480):1363–1376
Granger C, Andersen A (1978) An introduction to bilinear time series models. Vandenhoeck & Ruprecht, Göttingen
Granger C, Lin J (1994) Using the mutual information coefficient to identify lags in nonlinear models. Journal of Time Series Analysis 15(4):371–384
Granger C, Teräsvirta T (1993) Modelling nonlinear economic relationships. Oxford University Press, New York
Granger C, Maasoumi E, Racine J (2004) A dependence metric for possibly nonlinear processes. Journal of Time Series Analysis 25(5):649–669
Hall P, Wolff R (1995) On the strength of dependence of a time series generated by a chaotic map. Journal of time series analysis 16(6):571–583
Hallin M, Mélard G (1988) Rank-based tests for randomness against first-order serial dependence. Journal of the American Statistical Association 83(404):1117–1128
Härdle W (1992) Applied nonparametric regression. Cambridge University Press
Hong Y (1999) Hypothesis testing in time series via the empirical characteristic function: A generalized spectral density approach. Journal of the American Statistical Association 94(448):1201–1220
Hong Y, White H (2005) Asymptotic distribution theory for nonparametric entropy measures of serial dependence. Econometrica 73(3):837–901
Hull J (1999) Options, futures, and other derivative securities, 4th edn. Prentice Hall, New York
Kallenberg W (2009) Estimating copula densities, using model selection techniques. Insurance: mathematics and economics 45(2):209–223
Kendall MG, Stuart A (1966) The Advanced Theory of Statistics, vol 3. Charles Griffin & Company Limited, London
King M (1987) Testing for autocorrelation in linear regression models: a survey. In: King M, Giles D (eds) Specification analysis in the linear model: in honour of Donald Cochrane, Routledge Kegan & Paul, London

Ljung G, Box G (1978) On a measure of lack of fit in time series models. Biometrika 65(2):297–303
Lo A (2000) Finance: A selective survey. Journal of the American Statistical Association 95(450):629–635
Maasoumi E, Racine J (2002) Entropy and predictability of stock market returns. Journal of Econometrics 107(1–2):291–312
Moran P (1948) Some theorems on time series II. The significance of the serial correlation coefficient. Biometrika 35(3–4):255–260
Mortara G (1922) Lezioni di statistica. Società tipografica "Leonardo da Vinci", Città di Castello
Nelsen R (2006) An introduction to copulas. Springer
Priestley M (1988) Non-Linear and Non-stationary Time Series Analysis. Academic Press, London
Racine J, Maasoumi E (2007) A versatile and robust metric entropy test of time-reversibility, and other hypotheses. Journal of Econometrics 138(2):547–567
Rice J (1984) Boundary modifications for kernel regression. Communications in statistics Theory and methods 13(7):893–900
Robinson P (1991) Consistent nonparametric entropy-based testing. The Review of Economic Studies 58(3):437–453
Rosenblatt B, Wahlen BE (1992) A nonparametric measure of independence under a hypothesis of independent components. Statistics & Probability Letters 15(3):245–252
Rosenblatt M (1975) A quadratic measure of deviation of two-dimensional density estimates and a test of independence. The Annals of Statistics 3(1):1–14
Silverman BW (1986) Density Estimation for Statistics and Data Analysis. Chapman & Hall Ltd
Skaug H (1994) On the quality of the normal approximation in nonparamentric functional estimation. Tech. rep., University of Bergen, Bergen
Skaug H, Tjøstheim D (1993a) Measures of distance between densities with application to testing for serial independence. Tech. rep., University of Bergen, Bergen
Skaug HJ, Tjøstheim D (1993b) Nonparametric tests for serial independence. In: Subba Rao T (ed) Developments in Time Series Analysis: The Priestley Birthday Volume, Chapman & Hall, London, pp 207–229
Skaug HJ, Tjøstheim D (1996) Measures of distance between densities with application to testing for serial independence. In: Robinson P, Rosenblatt M (eds) Time Series Analysis in Memory of E.J. Hannan, Springer, New York, pp 363–377
Subba Rao T, Gabr M (1980) A test for linearity of stationary time series. Journal of time series analysis 1(2):145–158
Subba Rao T, Gabr M (1984) An introduction to bispectral analysis and bilinear time series models, Lecture Notes in Statistics, vol 24. Springer, New York
Szegö G (1959) Orthogonal polynomials. Revised ed. American Mathematical Society (AMS)
Tjøstheim D (1996) Measures of dependence and tests of independence. Statistics 28(3):249–284
Tong H (1990) Nonlinear Time Series: A Dynamic System Approach. Clarendon Press, Oxford

Tsallis C (1988) Possible generalization of Boltzmann-Gibbs statistics. Journal of statistical physics 52(1):479–487

Tsallis C (1998) Generalized entropy-based criterion for consistent testing. Physical Review E 58(2):1442–1445

Tsay R (2002) Analysis of financial time series. John Wiley & Sons, New Jersey

Wald A, Wolfowitz J (1943) An exact test for randomness in the non-parametric case based on serial correlation. The Annals of Mathematical Statistics 14(4):378–388