Fashion cycle dynamics induced by agents’ heterogeneity
for generic bell-shaped attractiveness functions

Ahmad Naimzada and Marina Pireddu

QUERY SHEET

This page lists questions we have about your paper. The numbers displayed at left are hyperlinked to the location of the query in your paper.

The title and author names are listed on this sheet as they will be published, both on your paper and on the Table of Contents. Please review and ensure the information is correct and advise us if any changes need to be made. In addition, please review your paper as a whole for typographical and essential corrections.

Your PDF proof has been enabled so that you can comment on the proof directly using Adobe Acrobat. For further information on marking corrections using Acrobat, please visit <http://journalauthors.tandf.co.uk/production/acrobat.asp>; <https://authorservices.taylorandfrancis.com/how-to-correct-proofs-with-adobe/>

The CrossRef database (www.crossref.org/) has been used to validate the references. Changes resulting from mismatches are tracked in red font.

AUTHOR QUERIES

QUERY NO.	QUERY DETAILS
Q1	The disclosure statement has been inserted. Please correct if this is inaccurate.
Q2	The CrossRef database (www.crossref.org/) has been used to validate the references. Mismatches between the original manuscript and CrossRef are tracked in red font. Please provide a revision if the change is incorrect. Do not comment on correct changes
Q3	The resolution of figure 1-6 is too low. Please resupply the figures in a resolution of at least 300 dpi.

Fashion cycle dynamics induced by agents' heterogeneity for generic bell-shaped attractiveness functions

Ahmad Naimzada^a and Marina Pireddu^b

^aDept. of Economics, Management and Statistics, University of Milano – Bicocca, Milano, Italy; ^bDept. of Mathematics and its Applications, University of Milano – Bicocca, Milano, Italy

ABSTRACT

We extend the exchange economy evolutionary discrete-time model with heterogeneous agents introduced in 'Fashion cycle dynamics in a model with endogenous discrete evolution of heterogeneous preferences', by A. Naimzada and M. Pireddu, appeared in *Chaos: An Interdisciplinary Journal of Nonlinear Science* 28, 055907 (2018), by considering generic bell-shaped attractiveness functions for the two groups of agents, increasing for low visibility levels, but decreasing when the visibility of the group exceeds a given threshold value, due to a congestion effect. The reproduction level of a group is related to its attractiveness degree, which depends on its social visibility level, determined by the consumption choices of the agents in that group. We investigate the asymptotic heterogeneity among agents, finding that the global dynamics may differ according to the chosen functional form for the attractiveness. Nonetheless, the model is still able to reproduce the recurrent dynamic behaviour typical of the fashion cycle.

ARTICLE HISTORY

Received 17 November 2018
Accepted 26 May 2019

KEYWORDS

Heterogeneous agents; evolution; fashion cycle; bifurcations; multistability; complex dynamics

2010 MATHEMATICS SUBJECT CLASSIFICATIONS

37N40; 39A60; 91B55

1. Introduction

In human societies, individuals often wish to distinguish themselves from others. On the other hand, following fashion makes people feel accepted and socially integrated, answering to their innate tendency for conformity. This produces imitation of others, that is, the so called bandwagon effect. However, when too many individuals choose the same commodities, then those become less desirable, because they confer minor distinction, and need to be replaced by new ones. Such phenomenon is known as snob effect. According to [45], those two opposite forces, imitation and distinction, drive the fashion cycle, i.e. the oscillatory behaviour of the variable describing the consumed or purchased amount of a certain good, characterized by booms and busts.

In the past decades, some dynamical models have been proposed to give a formal representation of the fashion cycle (see, for instance, [1,8,14–16,21,27,34,35,42,48,49]). More precisely, such works may be split into two different branches. Indeed, in addition to dealing with the market mechanism, which is common to all of them, the authors adopt either an

evolutionary perspective, like in [27,35], or a viewpoint based on the alternation between bandwagon and snob behaviours, like in [15,16].

Differently from those papers, we present an evolutive general equilibrium model, in which we may investigate the combined effects of three ingredients: the price formation mechanism, the social interaction mechanism, according to which consumption choices produce visibility, and the population share updating mechanism, based on the relative attractiveness of the different lifestyles.

In more detail, we here extend the exchange economy evolutionary discrete-time model with heterogeneous agents introduced in [40], where just one particular formulation of the attractiveness was taken into account, by considering generic bell-shaped attractiveness functions for the two groups of agents, increasing for low visibility levels, but decreasing when the visibility of the group exceeds a given threshold value, due to a congestion effect. In such more general context we study the asymptotic heterogeneity among agents, investigating how the local and global analysis performed in [40] is modified.

We recall that exchange economy evolutionary models with agents heterogeneous in the structure of preferences have been considered in [13,37–39,41], too. However, unlike those works, we deal with a setting in which the share updating rule is based on sociological, rather than biological or merely economic, aspects. Namely, the reproduction level of a group is here related to its attractiveness degree, which depends on its social visibility level, determined by the consumption choices of the agents in that group.

As concerns the agents' heterogeneity, in the model we assume a constitutive heterogeneity, the weights assigned to the two consumption goods in the Cobb-Douglas utility functions being not coinciding across groups. Along the paper we call the difference between such weights degree of heterogeneity and we study the model asymptotic heterogeneity, i.e. we discuss the possible dynamics arising when choosing initial conditions characterized by the coexistence between heterogeneous agents, in view of understanding whether the initial heterogeneity eventually disappears and, in case it persists, which new equilibria or more complex attractors emerge.

A paper which bears some resemblance to our work, in regard to the topic and the outcomes, is [3]. Indeed, the common focus is on cultural transmission, for us in the form of transmission of the preference structure due to social imitation, while [3] deals with the problem of cultural transmission of knowledge, habits and preferences from parents to their offspring. As concerns the outcomes, [3] analyzes what are the conditions on the transmission mechanisms which induce heterogeneity in the long run stationary distribution of preferences in the population, whereas we do also investigate whether the coexistence of different groups can be observed along non-stationary orbits. In particular, we find sufficient conditions for the presence of phenomena of dynamic coexistence which are oscillatory in nature, both regular or not, in terms of the heterogeneity degree. We stress that another paper with an approach similar to ours and in which chaotic dynamics are detected, too, is [5], where, however, the authors consider an economy with production and study whether market selection favours profit maximizing firms. We also recall that evolutionary frameworks with binary choices at a collective level have been considered, for instance, in [11,18]. On the other hand, differently from those papers, the binary choice here occurs between preference structures that characterize the two groups of agents, which are embedded in a general equilibrium framework where a price mechanism operates. Two further works which bear some resemblance to our setting are [28,36]. Namely, in [28] a

class of pursuit problems is proposed, in which a decision-maker reacts in an adaptive manner to the choices of the other agent, which tries to run away, in view of reducing the distance between them. In [36], this time in an opinion dynamics framework, agents look for a mediation between their positions in an adaptive perspective. Thus, both in [28,36] a sort of bandwagon behaviour can be observed, while the snob attitude can be witnessed in [28] only. Moreover, non-constant pursuit strategies in [28] and the endogenous reactivity assumption in [36] lead to nonlinear systems, in which chaotic dynamics can emerge. Finally, we stress that, rather than considering stable preferences as we do, it is possible to deal with preference adjusting consumers, like in [22–26,33], where cycles are found in a model with agents having globally interdependent preferences. The setting proposed in those works has been extended in [19,20], to deal with the case in which the interdependence among preferences is local in nature, and in [32], where the authors propose a stochastic version of the original framework. We recall that consumption choices of agents disposed on a network and having locally interdependent preferences have been considered in [10], too.

Before summarizing the results and comparing them with those found in [40], we explain in more detail our share updating mechanism, which, like in [40], is described via the discrete exponential replicator rule (see e.g. [7,31,43]), in which the reproduction level of a group is based on a comparison between the attractiveness degree of the two structures of preferences. The attractiveness of a group depends on its social visibility level, which is determined as a linear combination of the amount of the two goods consumed by the agents in that group, multiplied by commodity-specific visibility factors. In particular, the attractiveness of a group is increasing with its visibility level when the latter is low, and this describes the bandwagon regime, characterized by an imitative behaviour. However, when the visibility of a group exceeds a given threshold value, a congestion effect originates. The attractiveness of that group then becomes a decreasing function of its visibility level, and this describes the snob regime, characterized by a predominating wish for distinction. In other words, our model may be interpreted as a trade-off between a coordination game and a minority game. A similar phenomenon has been recently described by Bischi et al. in [2] where, in an evolutionary framework with binary choices, a unimodal reaction function puts together a threshold effect, inducing imitation, and a minority game, when too many agents make the same choice.

As concerns the results we obtain investigating how the local and global analysis performed in [40] changes when dealing with a generic bell-shaped attractiveness function for the two groups of agents, it holds that, although the same findings on existence and results on local stability of the equilibria closely related to those obtained in [40] still hold true when considering a generic bell-shaped attractiveness function, the global dynamics may differ according to the chosen functional form for the attractiveness. Namely, performing a bifurcation analysis on varying the parameter representing the heterogeneity degree in the preference structure of groups, in [40] we found that one of the nontrivial equilibria, characterized by the coexistence between the agents of the two groups, loses stability via a subcritical flip bifurcation and that, far from that equilibrium, period-two cycles may arise via fold bifurcations of the map governing the dynamics, also when the internal equilibrium is locally stable. On the other hand, considering now two different formulations for attractiveness, i.e. the Gaussian map and a parabolic function, in view of illustrating the theoretical results that we obtain in the present work, we observe different

outcomes. Indeed, while with the Gaussian formulation we find again a destabilization of the nontrivial equilibrium via a subcritical flip bifurcation, with the parabolic formulation the same nontrivial equilibrium loses stability via a supercritical flip bifurcation, followed by a cascade of period-doubling bifurcations leading to chaos. Nonetheless, like in [40], in both cases we obtain interesting multistability phenomena, involving equilibria, as well as periodic or chaotic attractors. Namely, the main difference between the detected scenarios lies in the fact that subcritical flip bifurcations suddenly lead to strong oscillations in the agents' consumption choices and in the population shares, while supercritical flip bifurcations are initially followed by small oscillations, which then grow for increasing values of the heterogeneity parameter, until we observe large oscillations, similar to those which immediately follow a subcritical flip bifurcation. Another dissimilarity between our results and the findings in [40] lies in the fact that, unlike that setting, where in relation to the system oscillatory behaviour we encountered a *mixed visibility framework* characterized by the lack of persistence of a given ordering among the visibilities of the two groups of agents and the threshold visibility level, in the case of the Gaussian formulation for attractiveness we here obtain that both in relation to dynamics converging toward a trivial steady state and in relation to the system oscillatory behaviour we are in *weak visibility frameworks*, in which the visibility levels for the two groups of agents lie below the threshold visibility level. In either the stationary and the oscillatory frameworks we observe a reduction of the share of the group with the lower attractiveness degree, caused by a social visibility level more distant from the threshold visibility level. However, while for the converging scenario there is a fixed ordering among the visibilities, which leads to the extinction of the group with the lower attractiveness degree, in regard to the oscillatory non-convergent scenario, still remaining in a weak visibility framework, we witness a continual alternation in the relative position of the visibilities of the two groups.

In particular, in the latter scenario the dynamic coexistence between groups and the oscillatory nature of the consumption activities over the two goods for the agents belonging to the two groups display the recurrent behaviour typical of the fashion cycle, with booms and busts. More generally, in our model fashion cycle dynamics are generated by an intermediate heterogeneity degree between groups. Indeed, both with the Gaussian and the parabolic formulations for attractiveness, when the level of agents' heterogeneity is too small, the only equilibria are the trivial ones, in which just one of the two kinds of agents is present, and this excludes the possibility of any sort of non-convergent dynamics. When increasing the heterogeneity degree parameter, three nontrivial equilibria emerge via transcritical bifurcations and this allows for interesting behaviours. Namely, such nontrivial equilibria may lose stability, giving rise to periodic and chaotic dynamics. However, when the dissimilarity between the two preference structures becomes excessive, no oscillatory behaviours occur anymore, as the chaotic attractor disappears due to a contact bifurcation with one of the nontrivial equilibria, and thus the system, according to the chosen initial condition, asymptotically converges toward one or the other of the trivial steady states, characterized by the presence of a unique group of agents. We stress that similar conclusions are valid also for the setting analyzed in [40].

In regard to the coexistence phenomena that we find involving equilibria, as well as periodic and chaotic attractors, we remark that multistability may be considered as a source of richness for the framework under analysis because, other parameters being equal, i.e. under the same institutional, cultural and social conditions, it allows to explain different

trajectories and evolutionary paths. The initial conditions, leading to the various attractors, represent indeed a summary of the past history, which in the presence of multistability phenomena does matter in determining the evolution of the system. Such property, in the literature on complex systems, is also called ‘path-dependence’ (see [6]). Moreover, in the specific context we deal with, the presence of multiple (stationary, periodic or chaotic) attractors well represents the variety of historical experiences across different countries in relation to the approach they adopt towards consumption choices and fashion.

The remainder of the paper is organized as follows. In Section 2 we present our model and we analyze the existence and local stability of its equilibria for generic attractiveness functions. In Section 3 we perform a qualitative bifurcation analysis, showing the possible dynamic behaviours for our system when considering a Gaussian or a parabolic formulation for the attractiveness, and we interpret the main scenarios from a sociological and economic viewpoint. In Section 4 we briefly discuss our results and we describe possible extensions of the model.

2. The model

Let us consider an exchange economy with a continuum of agents, which may be of type α or of type β . There are two consumption goods, x and y , and agent preferences are described by Cobb-Douglas utility functions, i.e. $U_i(x, y) = x^i y^{1-i}$, for $i \in \{\alpha, \beta\}$, with $0 < \beta < \alpha < 1$. The quantity of good x (y) consumed by an agent of type $i \in \{\alpha, \beta\}$ is denoted by x_i (y_i). Both kinds of agents have the same endowments of the two goods, denoted respectively by w_x and w_y . The analysis is performed in terms of the relative price $p_t = p_{y,t}/p_{x,t}$, where $p_{x,t}$ and $p_{y,t}$ are the prices at time t for goods x and y , respectively. The size of the population of kind α (β) at time t is denoted by A_t (B_t). The normalized variable $a_t = A_t/(A_t + B_t)$ represents the population fraction composed by the agents of type α and $b_t = 1 - a_t = B_t/(A_t + B_t)$ represents the population fraction composed by the agents of type β .

Before presenting Definition 2.1 of market equilibrium, we will refer to in the remainder of the paper, we explain the timing of the agents’ actions in our model. At the beginning of time t , while population shares are determined on the basis of the previous period’s choices, agents solve their maximization problem, making the optimal consumption choices, and simultaneously the equilibrium price is determined by market clearing condition, also on the basis of the population shares at time t . The optimal consumption choices allow to compute the visibilities of the two groups, and consequently, their attractiveness. In the passage from time t to time $t+1$ the population shares of period $t+1$ are determined by the updating rule based on the relative group attractiveness and the process goes on.

Definition 2.1: Given the economy and the population share a_t , a market equilibrium at time t is a vector $(p_t^*, x_{i,t}^*, y_{i,t}^*)$, with $i \in \{\alpha, \beta\}$, such that:

- every kind of agent chooses a utility-maximizing consumption bundle, given p_t^* ;
- the markets for the two goods clear.

Simple computations show that, solving the consumer maximization problems for agents of type α and β and using a market clearing condition, the market equilibrium price

is given by

$$p_t^* = \frac{[1 - (a_t\alpha + (1 - a_t)\beta)]w_x}{(a_t\alpha + (1 - a_t)\beta)w_y}$$

and the consumer equilibrium quantities of the two goods for an agent of type $i \in \{\alpha, \beta\}$ are

$$\begin{aligned} x_{i,t}^* &= i(w_x + p_t^* w_y) = \frac{iw_x}{a_t\alpha + (1 - a_t)\beta}, \\ y_{i,t}^* &= (1 - i) \left(\frac{w_x}{p_t^*} + w_y \right) = \frac{(1 - i)w_y}{1 - (a_t\alpha + (1 - a_t)\beta)}. \end{aligned} \quad (1)$$

See [13,37] for further mathematical details.

Definition 2.1 describes the notion of market equilibrium, which encompasses both the agents' choice of utility-maximizing consumption bundles and the market clearing conditions. We stress that such concept differs from the dynamic notion of equilibrium, i.e. of market stationary equilibrium. Indeed, by the latter we mean a market equilibrium in which shares, and consequently prices and optimal consumption choices, are constant over time. Accordingly, we introduce the following definition:

Definition 2.2: Given the economy, the vector (a^*, p^*, x_i^*, y_i^*) , $i \in \{\alpha, \beta\}$, is a market stationary equilibrium if a^* is constant and if, given a^* , (p^*, x_i^*, y_i^*) , $i \in \{\alpha, \beta\}$, is a market equilibrium for every t .

We remark that, in order not to overburden notation, in Definition 2.2 we use the symbol $*$ even for the shares, although a^* is not part of the market equilibrium vector introduced in Definition 2.1. We also notice that in Definition 2.1 there are time subscripts, missing in Definition 2.2, as the latter describes a stationary, time-unvarying, situation.

For the sake of brevity, we shall identify market stationary equilibria just with the population share a , since it determines all other equilibrium components.

The market stationary equilibria, at which for every t the population shares, and thus also the market equilibrium price and the consumer equilibrium quantities, are constant, will be called trivial if they are not characterized by the coexistence between the two groups of agents, and nontrivial otherwise.

Let us now illustrate the population share evolutionary mechanism, based on a social payoff.

In a social interaction framework, consumption choices produce visibility. Since, in general, the various goods induce different visibility levels, we introduce the positive parameters v_x and v_y describing the degree of visibility that each agent derives from the consumption of a unit of commodity x and of commodity y , respectively. Given such assumptions, we define the social visibility level $V_{i,t}$ of an agent of type $i \in \{\alpha, \beta\}$ at time t as a linear combination of the units $x_{i,t}$ and $y_{i,t}$ of goods x and y he consumes, weighted respectively with v_x and v_y , i.e. $V_{i,t} = v_x x_{i,t} + v_y y_{i,t}$. In particular, recalling (1), at the market

equilibrium it holds that

$$V_{i,t}^* = \frac{i v_x w_x}{a_t \alpha + (1 - a_t) \beta} + \frac{(1 - i) v_y w_y}{1 - a_t \alpha - (1 - a_t) \beta}. \quad (2)$$

For ease of notation, since we will consider visibility values just in correspondence to the market equilibrium, we will denote $V_{i,t}^*$ simply by $V_{i,t}$. The same remark applies to the attractiveness functions introduced in Definition 2.3, which we will denote by $\mathcal{A}_{i,t}$, rather than by $\mathcal{A}_{i,t}^*$.

Agents' consumption choices, deriving by the underlying preference structures, give rise to different attractiveness degrees for the various preference structures. Indeed, in a social interaction setting, the attractiveness of a preference structure depends on its visibility in a non-monotone manner: the social attractiveness of a preference structure is increasing in its visibility as long as the latter is not excessive, and then such dependence becomes decreasing.

Introducing the attractiveness $\mathcal{A}_{i,t}$ of group i as a function of $V_{i,t}$, $i \in \{\alpha, \beta\}$, suitable hypotheses on the map are that it is bell-shaped, increasing with the visibility level up to a certain threshold value $\widehat{V}$, above which it becomes decreasing in a symmetric manner with respect to $\widehat{V}$. Hence, $\mathcal{A}_{i,t}$ is for us any differentiable and strictly decreasing function of $d(\widehat{V}, V_{i,t}) = |\widehat{V} - V_{i,t}|$, where d denotes the Euclidean distance. In order to avoid differentiability issues, we will deal with functions depending on $d^2(\widehat{V}, V_{i,t}) = (\widehat{V} - V_{i,t})^2$. For ease of notation, in what follows we will sometimes denote $d^2(\widehat{V}, V_{i,t})$ simply by d_i^2 . We stress that it would be possible to view the attractiveness as an index, normalizing such variable and assuming that $\mathcal{A}_{i,t}$ varies in the interval $(-1, 1)$, so that negative values of the attractiveness could be interpreted as a repulsion, positive values of $\mathcal{A}_{i,t}$ would describe an attraction, and a null attractiveness would represent indifference toward a certain lifestyle. However, since such normalization would complicate the expression of attractiveness, we do not impose any restriction on the values that $\mathcal{A}_{i,t}$ may assume. We recall that the formulation for attractiveness considered in [40], i.e.

$$\mathcal{A}_{i,t}(V_{i,t}; \sigma, \widehat{V}) = \frac{1}{1 + \sigma(\widehat{V} - V_{i,t})^2}, \quad (3)$$

satisfies the conditions in Definition 2.3 and thus the results we will derive in the present section generalize those obtained in [40].

Definition 2.3: Recalling the definition of $V_{i,t}$ in (2), for $i \in \{\alpha, \beta\}$, any function

$$\mathcal{A}_{i,t} : (0, +\infty)^3 \rightarrow (-\infty, \mathcal{A}_{i,t}(\widehat{V})], \quad \mathcal{A}_{i,t}(V_{i,t}; \sigma, \widehat{V}) = f(\sigma d^2(\widehat{V}, V_{i,t})), \quad (4)$$

with $f : [0, +\infty) \rightarrow (-\infty, f(0)]$ differentiable and strictly decreasing, will be called attractiveness of group i at time t .

We remark that, in order to simplify computations and, most importantly, in view of making the share updating rule depending just on a comparison between the distance of the visibility levels for the two groups from the threshold value $\widehat{V}$, we will assume that the same map f in (4) describes the attractiveness of both groups α and β .

As concerns the parameters, we notice that, by (4), the attractiveness depends in an explicit manner only on σ . The dependence of $\mathcal{A}_{i,t}$ on the other model parameters is indirect, and occurs due to the presence of $V_{i,t}$. The parameter σ in (4) describes the sensitivity of the attractiveness $\mathcal{A}_{i,t}$, for $i \in \{\alpha, \beta\}$, to the distance between the visibility level $V_{i,t}$ and the threshold level $\widehat{V}$, below which increasing values of the visibility $V_{i,t}$ produce progressively larger attractiveness values, while for visibility levels above $\widehat{V}$ the attractiveness becomes decreasing in the visibility, due to a congestion effect. In particular, a visibility level coinciding with $\widehat{V}$ allows maximizing the attractiveness degree. We stress that when $\sigma \rightarrow 0$ agents are insensitive also to a large distance between $V_{i,t}$ and $\widehat{V}$, and $\mathcal{A}_{i,t}$ tends towards its maximum value $\mathcal{A}_{i,t}(\widehat{V})$. When instead $\sigma \rightarrow +\infty$, even a small, but still positive, distance between $V_{i,t}$ and $\widehat{V}$ leads to a very low attractiveness level because of an excessive sensitivity to such distance.

The attractiveness functions satisfying the assumptions in Definition 2.3 are suitable to describe, in relation to the preference structure $i \in \{\alpha, \beta\}$, both the bandwagon behaviour, which occurs as long as $V_{i,t} < \widehat{V}$, and the snob behaviour, which occurs when $V_{i,t} > \widehat{V}$. Indeed, according to [45], two contrasting tendencies operate in determining people behaviour towards fashion. On the one hand, to follow fashion makes people feel accepted and socially integrated, answering to their innate tendency for conformity. This generates imitation of others, i.e. the so called bandwagon behaviour. Such phenomenon is reproduced by $\mathcal{A}_{i,t}$ as long as $V_{i,t}$ is smaller than $\widehat{V}$, when increasing values for $V_{i,t}$ imply higher and higher attractiveness degrees $\mathcal{A}_{i,t}$. On the other hand, when the visibility of a group becomes excessive, a congestion effect arises and the people wish to distinguish themselves predominates. That phenomenon is known as snob behaviour and it is reproduced by $\mathcal{A}_{i,t}$ when $V_{i,t}$ is larger than $\widehat{V}$, because in this regime an increase in the visibility level $V_{i,t}$ leads to a decrease in $\mathcal{A}_{i,t}$. Namely, according to [46], on the basis of the empirical literature, too, in the context of luxury and prestige-seeking consumption agents may oscillate between snob and bandwagon behaviours. Put differently, our model stresses may be interpreted as a trade-off between a coordination game and a minority game, similar to the one described in [2]. Such two opposite forces, imitation and distinction, drive the fashion cycle, which for us emerges at the aggregate level as a continual oscillation of the attractiveness and of the shares of the two groups, corresponding to the different preference structures, while on the individual level it is characterized by oscillatory consumption choices, presenting booms and busts, over the two goods for the agents belonging to the two groups. We shall find evidence for it in Section 3 (see Figures 2(C) and 6).

The share of agents which adopt a given preference structure in the next period depends on the present attractiveness levels of the two preference structures. More precisely, following [7, 31, 43], in order to avoid negativity issues in the population fractions, we do consider the discrete exponential replicator mechanism to formalize the share updating rule, so that the evolution of the fraction a_t of agents of type α is described by the discrete choice model

$$\begin{aligned} a_{t+1} &= \frac{a_t \exp(\mu \mathcal{A}_{\alpha,t})}{a_t \exp(\mu \mathcal{A}_{\alpha,t}) + (1 - a_t) \exp(\mu \mathcal{A}_{\beta,t})} \\ &= \frac{a_t}{a_t + (1 - a_t) \exp(\mu (\mathcal{A}_{\beta,t} - \mathcal{A}_{\alpha,t}))}, \end{aligned} \quad (5)$$

where μ is a positive parameter measuring the sensitivity of the share formation mechanism to the preference structures attractiveness levels. When we will need to make explicit the dependence, for $i \in \{\alpha, \beta\}$, of $\mathcal{A}_{i,t}$ and of $V_{i,t}$ on a_t , we will also write $\mathcal{A}_{i,t}(a_t)$ and $V_{i,t}(a_t)$, respectively, or, at the market stationary equilibria, simply $\mathcal{A}_i(a)$ and $V_i(a)$ (see for instance (7)).

By (4), we obtain

$$a_{t+1} = g(a_t), \quad (6)$$

where the one-dimensional map $g : [0, 1] \rightarrow \mathbb{R}$ is defined as

$$g(a) = \frac{a}{a + (1-a) \exp \left(\mu \left(f(\sigma(\widehat{V} - V_\beta(a))^2) - f(\sigma(\widehat{V} - V_\alpha(a))^2) \right) \right)}. \quad (7)$$

We stress that if $\mu \rightarrow 0$, independently of V_α and V_β , then $a_{t+1} = a_t$, for all t , and thus there is no evolution, the population shares remain unchanged with respect to the initial ones, as agents are insensitive to the attractiveness degrees of the preference structures; when instead $\mu \rightarrow +\infty$, agents are extremely sensitive to social attractiveness of the preference structures and they instantaneously move towards the ‘best’ one, i.e. the one which gave a visibility level closer to $\widehat{V}$.

Let us then start our analysis by deriving the expressions of the market stationary equilibria for (6).

Proposition 2.4: *Equation (6) admits at most five market stationary equilibria. In addition to the always existing trivial equilibria $a = 0$ and $a = 1$, under suitable conditions also $a = a^*$, $a = a_1^*$ and $a = a_2^*$ exist, with*

$$a^* = \frac{(1-\beta)v_x w_x - \beta v_y w_y}{(\alpha-\beta)(v_x w_x + v_y w_y)}, \quad (8)$$

$$a_{1,2}^* = \frac{2\widehat{V}(1-2\beta) + v_x w_x(\alpha+\beta) - v_y w_y(2-\alpha-\beta) \pm \sqrt{\xi}}{4\widehat{V}(\alpha-\beta)}, \quad (9)$$

where

$$\begin{aligned} \xi = & 4\widehat{V}^2 + (v_x w_x(\alpha+\beta) - v_y w_y(2-\alpha-\beta))^2 \\ & - 4\widehat{V}(v_x w_x(\alpha+\beta) + v_y w_y(2-\alpha-\beta)), \end{aligned}$$

as long as they are real and belong to $(0, 1)$.

Proof: The conclusion immediately follows by observing that $a = 0$, $a = 1$, $a = a^*$ in (8), and $a = a_{1,2}^*$ in (9) are all the solutions to the fixed-point equation $g(a) = a$, with g as in (7). Namely, the nontrivial equilibria for equation (6) are found as solutions to the equation $\mathcal{A}_\beta = f(\sigma d^2(\widehat{V}, V_{\beta,t})) = f(\sigma d^2(\widehat{V}, V_{\alpha,t})) = \mathcal{A}_\alpha$. Since f is strictly decreasing, all solutions have to satisfy $d(\widehat{V}, V_{\beta,t}) = d(\widehat{V}, V_{\alpha,t})$, i.e. $V_\alpha = V_\beta$ or $(V_\alpha + V_\beta)/2 = \widehat{V}$. Straightforward computations show that the former condition is fulfilled just by a^* in (8), while the latter is satisfied just by $a_{1,2}^*$ in (9). This concludes the proof. ■

We notice that the statement of the previous result coincides with Proposition II.1 in [40], since the more general formulation we are here considering does affect neither the

Figure 1. In (A) the bifurcation diagram of g for f_1 in (19) with $\Delta \in (0, 0.9)$. We denote in orange $a = 0$, in cyan $a = 1$, in red $a = a^*$, in green the attractor surrounding it, in blue $a = a_1^*$ and $a = a_2^*$. Solid (dashed) curves refer to stable (unstable) equilibria and cycles. In (B) the bifurcation diagram of g for f_1 in (19) with $\Delta \in (0.514, 0.520)$. The magenta points are generated by the initial condition $a_0 = 0.55$, while the green points are generated by the initial condition $a_0 = 0.657$.

number, nor the position of the equilibria. In particular, as shown in that paper, a^* belongs to $(0, 1)$ if and only if

$$\frac{v_x w_x}{v_y w_y} \in \left(\frac{\beta}{1 - \beta}, \frac{\alpha}{1 - \alpha} \right). \quad (10)$$

Moreover, as observed therein, the condition $(V_\alpha + V_\beta)/2 = \widehat{V}$, identifying $a_{1,2}^*$ in (9), means that $\widehat{V}$ is the midpoint between V_α and V_β , with V_α and V_β lying at the same distance from $\widehat{V}$, but on its opposite sides.

When introducing the heterogeneity degree between groups $\Delta = \alpha - \beta$, measuring the difference in the preference structures, it is possible to rewrite (10) as

$$\frac{v_x w_x}{v_y w_y} \in \left(\frac{\beta}{1 - \beta}, \frac{\beta + \Delta}{1 - \beta - \Delta} \right). \quad (11)$$

Since the lower bound does not depend on Δ and the upper bound is increasing in Δ , we can infer that for large values of the heterogeneity degree it becomes easier to have $a^* \in (0, 1)$, as confirmed by Figure 1(A) and by Figure 5. In particular, since (11) is not satisfied for $\Delta = 0$, by continuity it follows that for small values of Δ we have $a^* \notin (0, 1)$. Indeed, as we will see in Section 3, a^* enters the interval $(0, 1)$ through a transcritical bifurcation occurring at $a = 1$ when

$$\frac{v_x w_x}{v_y w_y} = \frac{\beta + \Delta}{1 - \beta - \Delta},$$

i.e. for

$$\Delta = \widetilde{\Delta} = 1 - \beta - \frac{v_y w_y}{v_x w_x + v_y w_y} \quad (12)$$

(cf. again Figure 1(A) and Figure 5). We remark that transcritical bifurcations occur also when a_1^* and a_2^* enter the interval $(0, 1)$. We will return back to this point in the next section.

In regard to a_1^* and a_2^* , we stress that in general we do not know how they are placed with respect to a^* and how they depend on Δ . The dependence of a^* on Δ is instead described in the following result:

Corollary 2.5: *The nontrivial stationary equilibrium a^* in (8) is decreasing in Δ .*

Proof: Rewriting a^* in terms of Δ as

$$a^* = \frac{(1 - \beta)v_x w_x - \beta v_y w_y}{\Delta(v_x w_x + v_y w_y)}, \quad (13)$$

the desired conclusion immediately follows. ■

A confirmation of Corollary 2.5 can be found e.g. in Figure 1(A). Such decreasing behaviour of a^* with respect to Δ may be explained in terms of the agents' pressure on the price formation mechanism. Namely, if Δ were raised, since β would not change and $\alpha = \beta + \Delta$ would increase, the preference of the agents of type α for commodity x would become stronger. Hence, the individual demand of the agents of type α for commodity x would raise and, if the population shares were fixed, also the aggregate demand for commodity x would raise. This would make the corresponding price p_x increase and, as a consequence, agents of type α would consume a lower amount of commodity x . However, a decrease in x_α^* would induce a lower visibility level for the agents of type α , violating the visibility balance condition $V_\alpha = V_\beta$ characterizing $a = a^*$. Hence, if we want to remain in the steady state $a = a^*$, the share of agents of type α has to decrease, so that the aggregate demand for commodity x falls and consequently the price of good x tends to decrease, compensating the opposite price effect produced by the increase in Δ .

We notice that, due to (10), the parameters v_x and v_y cannot vanish. Nonetheless, the case of polarized values for v_x and v_y , in which both of them are positive but one is much larger than the other, is allowed and well approximates those frameworks in which visibility and attractiveness are produced by the consumption of a single good. In Section 3 we will consider polarized values for v_x and v_y , even if our results hold true also in the case of more balanced values for such parameters.

In the next proposition we analytically investigate the local stability of the trivial and nontrivial stationary equilibria for map g in (7).

Proposition 2.6: *The equilibrium $a = 0$ is locally asymptotically stable for map g in (7) if $(v_x w_x + v_y w_y - \widehat{V})^2 < (\alpha v_x w_x / \beta + (1 - \alpha) v_y w_y / (1 - \beta) - \widehat{V})^2$.*

The equilibrium $a = 1$ is locally asymptotically stable for map g if $(v_x w_x + v_y w_y - \widehat{V})^2 < (\beta v_x w_x / \alpha + (1 - \beta) v_y w_y / (1 - \alpha) - \widehat{V})^2$.

When $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ do not depend on μ , the equilibrium $a = a^$ in (8) is locally asymptotically stable for map g if (10) is satisfied, if $v_x w_x + v_y w_y < \widehat{V}$ and $\mu < \mu^*$, with*

$$\mu^* = \frac{v_x w_x v_y w_y}{\sigma((1 - \beta)v_x w_x - \beta v_y w_y)(\alpha v_y w_y - (1 - \alpha)v_x w_x) \frac{\partial \mathcal{A}_\alpha(a)}{\partial (\sigma d_\alpha^2)} \Big|_{a=a^*}}. \quad (14)$$

$$(v_x w_x + v_y w_y - \widehat{V})(v_x w_x + v_y w_y)$$

In particular, a flip bifurcation occurs at $a = a^$ if $\mu = \mu^*$.*

If $v_x w_x + v_y w_y > \widehat{V}$, then $a^* \in (0, 1)$ is unstable.

When $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ do not depend on μ , for $j \in \{1, 2\}$, the equilibrium $a = a_j^*$ in (9) is locally asymptotically stable for map g in (7) when it belongs to $(0, 1)$ if $(\widehat{V} - V_j(a_j^*))(V'_\alpha(a_j^*) + V'_\beta(a_j^*)) < 0$ and $\mu < \mu_j^*$, with

$$\mu_j^* = \frac{1}{\sigma a_j^* (1 - a_j^*) \frac{\partial \mathcal{A}_\alpha(a)}{\partial (\sigma d_\alpha^2)} \Big|_{a=a_j^*} (\widehat{V} - V_\alpha(a_j^*)) (V'_\alpha(a_j^*) + V'_\beta(a_j^*))}, \quad (15)$$

where

$$V_\alpha(a_j^*) = \frac{\alpha v_x w_x}{a_j^* \alpha + (1 - a_j^*) \beta} + \frac{(1 - \alpha) v_y w_y}{1 - a_j^* \alpha - (1 - a_j^*) \beta} \quad (16)$$

and

$$V'_\alpha(a_j^*) + V'_\beta(a_j^*) = (\alpha - \beta) \left(\frac{(2 - \alpha - \beta) v_y w_y}{(1 - a_j^* \alpha - (1 - a_j^*) \beta)^2} - \frac{(\alpha + \beta) v_x w_x}{(a_j^* \alpha + (1 - a_j^*) \beta)^2} \right). \quad (17)$$

In particular, a flip bifurcation occurs at $a = a_j^*$ if $\mu = \mu_j^*$.

If $(\widehat{V} - V_\alpha(a_j^*))(V'_\alpha(a_j^*) + V'_\beta(a_j^*)) > 0$, then $a_j^* \in (0, 1)$ is unstable.

Proof: Since $g'(0) = 1/(\exp(\mu(\mathcal{A}_\beta(0) - \mathcal{A}_\alpha(0))))$, the condition $g'(0) > -1$ is always fulfilled, while $g'(0) < 1$ is fulfilled for $\mathcal{A}_\beta(0) > \mathcal{A}_\alpha(0)$. Independently of f , exploiting the strictly decreasing behaviour, for $i \in \{\alpha, \beta\}$, of $\mathcal{A}_i$ with respect to $\sigma d^2(\widehat{V}, V_i)$, we then find $(V_\alpha(0) - \widehat{V})^2 > (V_\beta(0) - \widehat{V})^2$. As $V_\alpha(0) = \alpha v_x w_x / \beta + (1 - \alpha) v_y w_y / (1 - \beta)$ and $V_\beta(0) = v_x w_x + v_y w_y$, we obtain the desired conclusion about the local asymptotic stability of $a = 0$ for map g .

As concerns the local asymptotic stability of $a = 1$ for map g , we find $g'(1) = \exp(\mu(\mathcal{A}_\beta(0) - \mathcal{A}_\alpha(0)))$. Hence, the condition $g'(1) > -1$ is always fulfilled, while $g'(1) < 1$ is fulfilled for $\mathcal{A}_\beta(0) < \mathcal{A}_\alpha(0)$. Independently of f , exploiting again the strictly decreasing behaviour, for $i \in \{\alpha, \beta\}$, of $\mathcal{A}_i$ with respect to $d^2(\widehat{V}, V_i)$, we then find $(V_\alpha(1) - \widehat{V})^2 < (V_\beta(1) - \widehat{V})^2$. As $V_\alpha(1) = v_x w_x + v_y w_y$ and $V_\beta(1) = \beta v_x w_x / \alpha + (1 - \beta) v_y w_y / (1 - \alpha)$, we obtain the desired conclusion about the local asymptotic stability of $a = 1$ for map g .

In regard to a^* , recalling that $V_\alpha(a^*) = V_\beta(a^*)$, we find that $g'(a^*) = 1 - 2\mu\sigma a^*(1 - a^*)(\partial \mathcal{A}_\alpha(a) / \partial (\sigma d_\alpha^2)) \Big|_{a=a^*} (V_\beta(a^*) - \widehat{V})(V'_\beta(a^*) - V'_\alpha(a^*))$. Direct computations show that

$$a^*(1 - a^*) = \frac{((1 - \beta) v_x w_x - \beta v_y w_y) (\alpha v_y w_y - (1 - \alpha) v_x w_x)}{(\alpha - \beta)^2 (v_x w_x + v_y w_y)^2},$$

$V_\alpha(a^*) = V_\beta(a^*) = v_x w_x + v_y w_y$ and $V'_\beta(a^*) - V'_\alpha(a^*) = (\alpha - \beta)^2 (v_x w_x + v_y w_y)^3 / v_x w_x v_y w_y > 0$. Since $(\partial \mathcal{A}_\alpha(a) / \partial (\sigma d_\alpha^2)) \Big|_{a=a^*}$ is negative, then $a = a^* \in (0, 1)$ in (8) is unstable for map g if $V_\beta(a^*) - \widehat{V} > 0$. If instead $V_\beta(a^*) - \widehat{V} < 0$, and $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ do not depend on μ , then $g'(a^*) < 1$ is always fulfilled, while $g'(a^*) > -1$ is fulfilled for $\mu < \mu^*$, with μ^* as in (14), obtaining the desired conclusion about the local asymptotic stability of $a = a^* \in (0, 1)$ for map g , too.

The condition for the flip bifurcation follows by setting $g'(a^*) = -1$.

Finally, in regard to a_j^* , for $j \in \{1, 2\}$, recalling that $V_\beta(a_j^*) - \widehat{V} = \widehat{V} - V_\alpha(a_j^*)$, we find that $g'(a_j^*) = 1 - 2\mu\sigma a_j^*(1 - a_j^*)(\partial \mathcal{A}_\alpha(a)/\partial(\sigma d_\alpha^2))|_{a=a_j^*}(\widehat{V} - V_\alpha(a_j^*))(V'_\alpha(a_j^*) + V'_\beta(a_j^*))$. Direct computations show that at a_j^* , for $j \in \{1, 2\}$, (16) and (17) are fulfilled. Since $(\partial \mathcal{A}_\alpha(a)/\partial(\sigma d_\alpha^2))|_{a=a_j^*}$ is negative, then $a = a_j^* \in (0, 1)$ is unstable for map g if $(\widehat{V} - V_\alpha(a_j^*))(V'_\alpha(a_j^*) + V'_\beta(a_j^*)) > 0$. If instead $(\widehat{V} - V_\alpha(a_j^*))(V'_\alpha(a_j^*) + V'_\beta(a_j^*)) < 0$, then $g'(a_j^*) < 1$ is always fulfilled, while, if $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ do not depend on μ , $g'(a_j^*) > -1$ is fulfilled for $\mu < \mu_j^*$, with μ_j^* as in (15). The condition for the flip bifurcation follows by setting $g'(a_j^*) = -1$. This concludes the proof. ■

We remark that, when $a^* \in (0, 1)$, simple computations show that $V_\beta(0) = v_x w_x + v_y w_y < \alpha v_x w_x / \beta + (1 - \alpha) v_y w_y / (1 - \beta) = V_\alpha(0)$ and, for $a = 1$, $V_\alpha(1) = v_x w_x + v_y w_y < \beta v_x w_x / \alpha + (1 - \beta) v_y w_y / (1 - \alpha) = V_\beta(1)$. Thus, according to Proposition 2.6, in order to have stability both of $a = 0$ and $a = 1$ for map g in (7), we need $\widehat{V}$ to be closer, at the two endpoints, to the smaller between V_α and V_β .

Moreover, by Proposition 2.6 we can conclude that in our setting the parameter μ has a destabilizing role on the nontrivial equilibria.

We stress that, since the stability conditions for $a = 0$ and $a = 1$ are not influenced by the functional form of the groups' attractiveness, they coincide with the stability conditions derived in [40, Proposition II.2]. On the other hand, the value of parameter μ for which the flip bifurcation occurs at $a = a^*$ depends on $\partial \mathcal{A}_\alpha(a)/(\partial(\sigma d_\alpha^2))|_{a=a^*}$. Hence, although also in [40] we found that $a = a^*$ loses stability via a flip bifurcation, the value of μ^* in (14) varies with the formulation of the attractiveness. We finally notice that in [40, Proposition II.2] the stability conditions for $a_{1,2}^*$ were not investigated due to the heavy computations involved when dealing with the expression of $a_{1,2}^*$ in (9). Nonetheless, those stability conditions would follow from (15)–(17) when considering $\mathcal{A}_\alpha$ as in (3).

Since in Section 3 we will consider the heterogeneity degree Δ as bifurcation parameter, in the next result we derive the corresponding stability conditions. Namely, the stability conditions found in Proposition 2.6 for $a = 0, a = 1$ and $a = a^*$ may be rewritten in terms of Δ as stated in Corollary 2.7, which highlights the destabilizing role of the heterogeneity degree on $a = a^* \in (0, 1)$. Due to the heavy expressions of $a_{1,2}^*$ in (9), we do not make explicit the dependence on Δ in (15)–(17).

Corollary 2.7: *The equilibrium $a = 0$ is locally asymptotically stable for map g in (7) if $(v_x w_x + v_y w_y - \widehat{V})^2 < (v_x w_x + v_y w_y + \Delta(v_x w_x / \beta - v_y w_y / (1 - \beta)) - \widehat{V})^2$.*

The equilibrium $a = 1$ is locally asymptotically stable for map g if $(v_x w_x + v_y w_y - \widehat{V})^2 < (v_x w_x + v_y w_y + \Delta(v_y w_y / (1 - \beta) - v_x w_x / (\beta + \Delta)) - \widehat{V})^2$.

When the map f characterizing $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ depends on Δ only through the visibilities, the equilibrium $a = a^ \in (0, 1)$ in (8) is locally asymptotically stable for map g in (7) if (10) is satisfied, if $v_x w_x + v_y w_y < \widehat{V}$ and $\Delta < \Delta^*$, with*

$$\Delta^* = \frac{v_x w_x v_y w_y}{\mu\sigma((1 - \beta)v_x w_x - \beta v_y w_y) \frac{\partial \mathcal{A}_\beta(a)}{\partial(\sigma d_\beta^2)}|_{a=a^*} (v_x w_x + v_y w_y - \widehat{V})(v_x w_x + v_y w_y)^2 + \frac{(1 - \beta)v_x w_x - \beta v_y w_y}{v_x w_x + v_y w_y}}. \quad (18)$$

In particular, a flip bifurcation occurs at $a = a^*$ if $\Delta = \Delta^*$.

Proof: The conditions for the local stability of $a = 0$ and $a = 1$ immediately follow by those found Proposition 2.6, when making Δ explicit.

The expression for Δ^* in (18) follows by that of μ^* in (14), rewriting the factor $(\alpha v_y w_y - (1 - \alpha)v_x w_x)$ on the denominator of (14) as $\Delta(v_x w_x + v_y w_y) + \beta v_y w_y - (1 - \beta)v_x w_x$ and making Δ explicit in (14). ■

We stress that, like happened in Proposition 2.6, the stability conditions found in Corollary 2.7 for $a = 0$ and $a = 1$ do not depend on the expression of $\mathcal{A}_i$.

We also notice that, when investigating the local stability of the nontrivial equilibria with respect to σ , it is not possible to derive general conditions similar to those found in Proposition 2.6 for μ (cf. (14) and (15)). Namely, even if V_i , $i \in \{\alpha, \beta\}$, do not depend on σ , such parameter is usually present in the expressions of $(\partial \mathcal{A}_\alpha(a)/\partial(\sigma d_\alpha^2))|_{a=a^*}$ and of $(\partial \mathcal{A}_\alpha(a)/\partial(\sigma d_\alpha^2))|_{a=a_j^*}$, as it is easy to check, for instance, with the functional form for attractiveness in (19). In fact, with the Gaussian map in (19), for suitable parameter configurations, there may not exist threshold stability values analogous to those in (14) and (15) with respect to σ , since e.g. a^* may be stable for every $\sigma > 0$.¹

Since by (8) and (9) the value of the nontrivial equilibria is not constant with respect to $\Delta = \alpha - \beta$ (cf. also Corollary 2.5), when moving such parameter $a = a^*$, $a = a_1^*$ and $a = a_2^*$ may become negative or exceed 1, and transcritical bifurcations occur when a trivial and a nontrivial equilibria merge, leading to an exchange in stability, so that the previously stable equilibrium loses its stability in favour of the unstable one, which becomes stable.

All the possibilities for the transcritical bifurcations occurring at $a = 0$ and at $a = 1$ when considering Δ as bifurcation parameter are summarized in the next result.

Proposition 2.8: *When varying Δ , in correspondence to the only possible bifurcations occurring at $a = 0$, it holds that $a = 0$ gains stability, while a_1^* or a_2^* become unstable. When varying Δ , in correspondence to the bifurcations occurring at $a = 1$, the latter may gain or lose stability, both when it collides with a^* and with a_1^* or a_2^* .*

Proof: By the proof of Proposition 2.6 and by Corollary 2.7, we have that $a = 0$ is locally asymptotically stable for map g if $d^2(V_\beta(0), \widehat{V}) = (v_x w_x + v_y w_y - \widehat{V})^2 < (v_x w_x + v_y w_y + \Delta(v_x w_x/\beta - v_y w_y/(1 - \beta)) - \widehat{V})^2 = d^2(V_\alpha(0), \widehat{V})$. Hence, $a = 0$ loses stability when $V_\beta(0) = V_\alpha(0)$ or when $(V_\alpha(0) + V_\beta(0))/2 = \widehat{V}$. In the former case, we have that $a^* = 0$, and thus a transcritical bifurcation occurs for g at $a = 0$ when a^* collides with it, while, in the latter case, we have that $a_1^* = 0$ or $a_2^* = 0$, and thus a transcritical bifurcation occurs for g at $a = 0$ when a_1^* or a_2^* collide with it. We stress that, by increasing Δ , the transcritical bifurcation at $a = 0$ cannot involve a^* as, recalling (13), a^* does not change sign on varying Δ . Since $V_\beta(0)$ does not depend on Δ and $V_\alpha(0)$ is increasing (decreasing) with Δ when $a^* > 0$ ($a^* < 0$), and since $V_\alpha(0) > V_\beta(0)$ if and only if a^* is positive, then two different scenarios may occur at $a = 0$. In the first one $V_\alpha(0) < \widehat{V} < V_\beta(0)$, so that $a^* < 0$ and $V_\alpha(0)$ is decreasing with Δ . Hence, the transcritical bifurcation occurs when $(V_\alpha(0) + V_\beta(0))/2 = \widehat{V}$ and a_1^* or a_2^* collide with $a = 0$, which steals the stability of a_1^* or a_2^* . In the second (symmetric) scenario we have $V_\beta(0) < \widehat{V} < V_\alpha(0)$, so that $a^* > 0$ and $V_\alpha(0)$ is increasing with Δ . Thus, the transcritical bifurcation at $a = 0$ occurs again when

$(V_\alpha(0) + V_\beta(0))/2 = \widehat{V}$ and a_1^* or a_2^* collide with $a = 0$, which also in this case steals the stability of a_1^* or a_2^* . This concludes the proof for $a = 0$.

By the proof of Proposition 2.6 and by Corollary 2.7, we have that $a = 1$ is locally asymptotically stable for map g if $d^2(V_\alpha(1), \widehat{V}) = (v_x w_x + v_y w_y - \widehat{V})^2 < (v_x w_x + v_y w_y + \Delta(v_y w_y / (1 - \beta - \Delta) - v_x w_x / (\beta + \Delta)) - \widehat{V})^2 = d^2(V_\beta(1), \widehat{V})$. Hence, $a = 1$ loses stability when $V_\beta(1) = V_\alpha(1)$ or when $(V_\alpha(1) + V_\beta(1))/2 = \widehat{V}$. In the former case, we have that $a^* = 1$, while, in the latter case, we have that $a_1^* = 1$ or $a_2^* = 1$. Since a^* is decreasing with Δ , $V_\alpha(1)$ does not depend on Δ , $V_\beta(1)$ is always increasing with Δ , and $V_\alpha(1) > V_\beta(1)$ if and only if $a^* > 1$, then four different scenarios may occur at $a = 1$. In the first one $V_\beta(1) < V_\alpha(1) < \widehat{V}$, so that the transcritical bifurcation occurs when $V_\beta(1)$ reaches $V_\alpha(1)$ and a^* collides with $a = 1$, which loses stability in favour of a^* . In the second (symmetric) scenario $\widehat{V} < V_\beta(1) < V_\alpha(1)$, so that the transcritical bifurcation occurs when $V_\beta(1)$ reaches $V_\alpha(1)$ and a^* collides with $a = 1$, which steals the stability of a^* . In the third scenario $V_\alpha(1) < \widehat{V} < V_\beta(1)$, so that $a^* < 1$. The transcritical bifurcation occurs when $(V_\alpha(1) + V_\beta(1))/2 = \widehat{V}$ and a_1^* or a_2^* collide with $a = 1$, which steals the stability of a_1^* or a_2^* . In the fourth (symmetric) scenario we have $V_\beta(1) < \widehat{V} < V_\alpha(1)$, so that $a^* > 1$. The transcritical bifurcation occurs again when $(V_\alpha(1) + V_\beta(1))/2 = \widehat{V}$ and a_1^* or a_2^* collide with $a = 1$, which loses stability in favour of a_1^* or a_2^* . Also the proof for $a = 1$ is complete. ■

In regard to $a = 0$, we stress that the case $V_\alpha(0) < \widehat{V} < V_\beta(0)$, implying that $a^* < 0$, is prevented by the assumption $(v_x w_x)/(v_y w_y) > \beta/(1 - \beta)$, which ensures that $a^* > 0$.

3. Bifurcation analysis, possible scenarios and their economic interpretation

3.1. Bifurcation analysis and multistability phenomena

In the present subsection, in view of investigating the model asymptotic heterogeneity, i.e. of discussing the possible dynamics arising when choosing initial conditions characterized by the coexistence between heterogeneous agents in order to understand whether the initial heterogeneity eventually disappears and, in case it persists, which new equilibria or more complex attractors emerge, we perform a qualitative bifurcation analysis on varying the parameter Δ measuring the heterogeneity degree in the structure of preferences between groups.

More precisely, we deal with groups' attractiveness $\mathcal{A}_{i,t}$, for $i \in \{\alpha, \beta\}$, described either by the Gaussian map

$$\mathcal{A}'_{i,t} = f_1(\sigma d^2(V_{i,t}, \widehat{V})) = \exp(-\sigma(V_{i,t} - \widehat{V})^2), \quad (19)$$

or by the parabolic function

$$\mathcal{A}''_{i,t} = f_2(\sigma d^2(V_{i,t}, \widehat{V})) = 1 - \sigma(V_{i,t} - \widehat{V})^2, \quad (20)$$

where σ , as explained after Definition 2.3, is a positive parameter describing the sensitivity of the attractiveness to the distance between the visibility level $V_{i,t}$ induced by the preference structure $i \in \{\alpha, \beta\}$ and the threshold visibility level $\widehat{V}$, which allows maximizing the attractiveness degree. We notice that $\mathcal{A}'_{i,t}$ in (19) and $\mathcal{A}''_{i,t}$ in (20) satisfy all the assumptions in Definition 2.3.

Figure 2. The graph of map g for f_1 in (19) with $\Delta = 0.2$ in (A), $\Delta = 0.35$ in (B) and $\Delta = 0.66$ in (C).

In regard to the parameter values, we let Δ free to vary, while we fix the remaining parameters as follows:² $v_x = 0.9$, $w_x = 0.2$, $v_y = 0.15$, $w_y = 2$, $\beta = 0.1$, $\mu = 6.5$, $\sigma = 8$, $\widehat{V} = 0.8$. Such parameter values coincide with those considered in [40], where the groups attractiveness is described by the map in (3). We recall that in [40, Subsection III.A] also the scenarios with $\widehat{V} = 0.2$ and $\widehat{V} = 0.5$ were investigated. However, since in those frameworks only transcritical bifurcations of $a=0$ and $a=1$ were detected, at which the two trivial equilibria switch from stability to instability or vice-versa, and since the stability conditions found in Corollary 2.7 with respect to Δ for $a=0$ and $a=1$ do not depend on the expression of $\mathcal{A}_i$, such scenarios remain exactly as in [40] for the Gaussian and parabolic formulations of attractiveness in (19) and (20). Thus we omit the corresponding analysis, focussing just on the third scenario considered in [40, Subsection III.A], in which $\widehat{V} = 0.8$.

Figure 3. A magnification of the bifurcation diagram in Figure 1(A) for $\Delta \in (0.55, 0.68)$. The initial conditions are $a_0 = 0.001$ for the orange dots, $a_0 = 0.20$ for the red dots, $a_0 = 0.25$ for the blue dots and $a_0 = 0.999$ for the green dots. We denote in green also $a = 1$. In agreement with Figure 1(A), we draw in red $a = a^*$ and in blue $a = a_1^*$ and $a = a_2^*$, too. Solid (dashed) curves refer to stable (unstable) equilibria and cycles.

We start dealing with the Gaussian formulation of attractiveness, showing in Figure 1(A) the bifurcation diagram of g , with f_1 as in (19), for $\Delta \in (0, 0.9)$. We notice that for small values of Δ only $a = 1$ is stable, while for intermediate values of Δ just $a = a^*$ is stable. When Δ increases, we observe first that also $a = 0$ becomes locally stable and then that $a = a^*$ loses stability, leading to a one-piece chaotic attractor, which coexists with the locally stable equilibria $a = 0$ and $a = 1$, and that disappears for still larger values of Δ , when the unique attractors are given by the trivial equilibria.

In view of understanding the various transitions, in Figure 2 we draw the graph of map g for $\Delta = 0.2$ in (A), $\Delta = 0.35$ in (B) and $\Delta = 0.66$ in (C). In Figure 2(A) the only equilibria are $a = 0$ and $a = 1$, denoted by black squares, of which the former is unstable and the latter is stable, leading to the eventual extinction of the agents of group β . In (B) the nontrivial equilibrium $a = a^*$, denoted by a black dot, belongs to $(0, 1)$ and it is stable, while $a = 0$ and $a = 1$ are unstable. In (C) also $a = a_1^*$ and $a = a_2^*$, denoted by empty squares, have entered $(0, 1)$, but they are unstable, as well as $a = a^*$, and separate the basins of attraction of $a = 0, a = 1$ and of the chaotic attractor surrounding $a = a^*$. We stress that, due to the closeness of $a = a_2^*$ to $a = 1$, in (C) we omit denoting the latter by a black square. The same remark applies to Figure 4.

We notice that, when $a = a^*$ enters the interval $(0, 1)$ by increasing the value of Δ from (A) to (B), a transcritical bifurcation occurs for the map g at $a = 1$ for $\Delta = 0.275$, at which $a = 1$ loses stability in favour of $a = a^*$, as it happened in [40]. Namely, like recalled above, the stability conditions found for $a = 0$ and $a = 1$ in Proposition 2.6 and in Corollary 2.7 do not depend on the expression of $\mathcal{A}_i$. This means that, as we shall confirm below, also the values of Δ at which the other transcritical bifurcations occur coincide with those detected in [40]. For such reason, for brevity's sake, we omit the proofs of the occurrence of the various transcritical bifurcations, recalling that they would require to check the conditions in [47, page 507], like done in [40, Proposition III.1]. In particular, we highlight that the value of Δ at which $a = a^*$ undergoes the transcritical bifurcation directly follows by (12).

Raising further the value of Δ from (B) to (C) in Figure 2, we observe the emergence of $a = a_1^*$ and $a = a_2^*$ in $(0, 1)$. More precisely, when $a = a_1^*$ enters the interval $(0, 1)$, a transcritical bifurcation of the map g occurs at $a = 0$ for $\Delta = 0.436$, at which $a = a_1^*$ loses stability in favour of $a = 0$, which from that moment on is locally stable; similarly, when $a = a_2^*$ enters the interval $(0, 1)$, a transcritical bifurcation of the map g occurs at $a = 1$ for

Figure 4. The graph of map g for f_1 in (19) with $\Delta = 0.660, a_0 = 0.790$ in (A), $\Delta = 0.662, a_0 = 0.786$ in (B) and $\Delta = 0.664, a_0 = 0.784$ in (C).

$\Delta = 0.654$, at which $a = a_2^*$ loses stability in favour of $a = 1$, which becomes locally stable. As expected, also such bifurcation values coincide with those found in [40].

We stress that, for slightly different parameter configurations, it may happen that no transcritical bifurcations occur at $a = 0$. Namely, taking e.g. $\hat{V} = 0.2$ and fixing the remaining parameters as in the present section, the only transcritical bifurcation is that occurring at $a = 1$, at which a^* loses stability when it enters the interval $(0,1)$. See [40, Figure 1].

Increasing the value of Δ in Figure 2 from (B) to (C), in addition to the two transcritical bifurcations of g described above, it also happens that $a = a^*$ loses stability via a subcritical flip bifurcation occurring for $\Delta = \Delta^* = 0.5171$, as shown in the next result.

Proposition 3.1: *For the map $g = g(a; \Delta)$ in (7) with f_1 as in (19) a subcritical flip bifurcation occurs at $a = a^* = 0.532$ for $\Delta = \Delta^* = 0.517$.*

Proof: According to Wiggins [47, page 516], for the occurrence of a flip bifurcation for the map g at a point $a = \hat{a}$ for a certain $\Delta = \hat{\Delta}$ we have to check the following conditions:

$$\begin{aligned} g(\hat{a}; \hat{\Delta}) &= \hat{a}, & \frac{\partial g}{\partial a}(\hat{a}; \hat{\Delta}) &= -1, & \frac{\partial g^2}{\partial \Delta}(\hat{a}; \hat{\Delta}) &= 0, \\ \frac{\partial^2 g^2}{\partial a \partial \Delta}(\hat{a}; \hat{\Delta}) &\neq 0, & \frac{\partial^2 g^2}{\partial a^2}(\hat{a}; \hat{\Delta}) &= 0, & \frac{\partial^3 g^2}{\partial a^3}(\hat{a}; \hat{\Delta}) &\neq 0. \end{aligned}$$

Moreover, according to [44, Theorem 8.2], if $(\partial^3 g^2 / \partial a^3)(\hat{a}; \hat{\Delta}) > 0$ and $(\partial^2 g / \partial a \partial \Delta)(\hat{a}; \hat{\Delta}) < 0$ the flip bifurcation is subcritical. Direct (software-assisted) computations show that the needed conditions are satisfied at $\hat{a} = 0.532$ for $\hat{\Delta} = 0.517$. In particular, it holds that $(\partial^2 g^2 / \partial a \partial \Delta)(\hat{a}; \hat{\Delta}) = 16.523$, $(\partial^3 g^2 / \partial a^3)(\hat{a}; \hat{\Delta}) = 4.722$ and $(\partial^2 g / \partial a \partial \Delta)(\hat{a}; \hat{\Delta}) = -6.136$. This completes the proof. ■

We remark that, although looking at Figure 1(A) the flip bifurcation occurring for the map g at $a^* = 0.532$ seems to be supercritical, with $a = a^*$ losing stability for $\Delta = 0.517$ in favour of a stable period-two cycle, followed by a sequence of flip bifurcations leading to a chaotic attractor, as proven in Proposition 3.1 that flip bifurcation is indeed subcritical, like the one detected in [40] for the attractiveness formulation recalled in (3). Namely, the bifurcation diagram in Figure 1(B) shows that a stable period-two cycle (in green) coexists with $a = a^*$ (in magenta) when the latter is still locally stable. The stable period-two cycle emerges, together with an unstable period-two cycle which collides with $a = a^*$ when the flip bifurcation occurs, via a fold bifurcation of g for $\Delta = 0.5168$ with periodic points $\{a_1, a_2\} = \{0.429, 0.646\}$. The stable period-two cycle undergoes a sequence of flip bifurcations leading to a two-piece chaotic attractor, followed by a one-piece chaotic attractor, which suddenly disappears for $\Delta = 0.662$, so that for larger values of Δ the only attractors are given by the trivial equilibria $a = 0$ and $a = 1$. Before explaining the latter transition, we show in Figure 3 a magnification of the bifurcation diagram in Figure 1(A), focussing now on $\Delta \in (0.55, 0.68)$, where we notice that for $\Delta \in (0.599, 0.608)$ the stable period-two cycle preceding the flip bifurcation of $a = a^*$, then followed by a chaotic attractor, coexists with a period-two cycle which emerges via a fold bifurcation of g close to its minimum point and which gives rise to a two-piece chaotic attractor via a cascade of flip bifurcations, that disappears when the iterates of the minimum point of g enter the basin of attraction of

the main attractor. Moreover, for $\Delta = 0.671$ we observe the emergence of a period-three cycle, which gives rise to a three-piece chaotic attractor that soon disappears.

In order to explain the change occurring for $\Delta = 0.662$, we draw in Figure 4 the graph of map g for $\Delta = 0.660$ in (A), $\Delta = 0.662$ in (B) and $\Delta = 0.664$ in (C), together with the forward iterates of the minimum point a_m . We find that for $\Delta < 0.662$ such iterates do not overcome the unstable equilibrium $a = a_1^*$, whose stable set (which includes also its rank-1 preimage) separates the basins of attraction of $a = 0$ and of the chaotic attractor surrounding $a = a^*$. For $\Delta = 0.662$ it holds that $g(a_m) = g(0.786) = a_1^* = 0.043$ and the first homoclinic bifurcation of $a = a_1^*$ occurs. For $\Delta > 0.662$ the forward iterates of a_m belong to $(0, a_1^*)$ and converge to $a = 0$. Hence, for $\Delta > 0.662$ the chaotic attractor surrounding $a = a^*$ does not exist anymore and, according to the initial condition, the generic orbit tends toward $a = 0$ or toward $a = 1$. We remark however that, after the contact bifurcation with $a = a_1^*$, the latter admits homoclinic orbits and a chaotic repeller survives.

In greater detail, in regard to the detected multistability phenomena, in Figure 1(A) $a = 0$ initially coexists with $a = a^*$, then with the period-two cycle surrounding it, leading first to a two-piece chaotic attractor, and then to a one-piece chaotic attractor. Together with $a = 0$ and the chaotic attractor, multistability involves also $a = 1$, which is locally stable for sufficiently low and for sufficiently high values of $\Delta \in (0, 0.9)$. For small intervals of values of Δ in Figures 1(B) and 3 we also find multistability phenomena involving further periodic and chaotic attractors.

We stress that, for the considered parameter configuration, the only nontrivial equilibrium which may be locally stable is given by $a = a^*$, but this is not true in general. Taking, for instance, $\beta = 0.1, \mu = 6.5, \sigma = 8, v_x = 0.8, w_x = 0.1, v_y = 0.5, w_y = 2, \Delta = 0.8$ and $\widehat{V} = 0.85$, the only nontrivial equilibria are $a = a_1^*$ and $a = a_2^*$, of which the former is locally stable, as well as $a = 1$. Dealing instead with $\beta = 0.1, \mu = 6.5, \sigma = 8, v_x = 0.2, w_x = 0.3, v_y = 0.25, w_y = 1.4, \Delta = 0.8$ and $\widehat{V} = 0.36$, we find that also $a = a^*$ belongs to $(0, 1)$, and the locally stable equilibria are given by $a = 0, a = a_1^*$ and $a = 1$. Hence, both with $a = a^*$ belonging to $(0, 1)$ or not, we may have multistability phenomena involving $a = a_1^*$ and $a = a_2^*$, too. Indeed, as shown in Proposition 2.6, $a = a_{1,2}^*$ lose stability via a flip bifurcation at $\mu = \mu_{1,2}^*$ on increasing μ .

When looking at the graph of map g , we notice that in Figure 2(A,B) no multistability phenomena arise, while in Figure 2(C), in addition to the stable trivial equilibria, we observe a chaotic attractor surrounding $a = a^*$. As explained above, the unstable equilibria in Figure 2(C) play the role of separating the basins of attraction: trajectories will be attracted by one or the other of the three attractors according to the chosen initial conditions. In particular, when the initial conditions, which represent a summary of the past history, are excessively close to the extreme values $a = 0$ and $a = 1$, the trajectories tend towards one of the two trivial equilibria, and a preference structure totally prevails over the other. However, when the initial conditions are not too close to $a = 0$ and $a = 1$, for intermediate values of the heterogeneity degree parameter, our system generates oscillatory behaviours (see, for instance, Figures 1 and 2(C)), thanks to the combined effect of the evolutionary mechanism and of the price formation mechanism. Indeed, the group coexistence may be dynamic in nature, both regular, just after the flip bifurcation, and irregular, due to the presence of the chaotic attractor. The dynamic coexistence between groups then displays the recurrent behaviour typical of the fashion cycle, characterized by booms and busts. We display, for the same parameter configurations considered in Figure 2(C) and

for the time periods $t \in [200, 300]$, the time series for all the relevant variables in Figure 6. Namely, the fashion cycle for us emerges at the aggregate level as a continual oscillation of the attractiveness and of the shares of the two groups, while on the individual level it is characterized by oscillatory consumption choices over the two goods, presenting booms and busts, for the agents of the two groups.

We will present in Subsection 3.2 the interpretation of the frameworks in Figure 2. Namely, we wish to highlight how the sociological and economic interpretation of the main scenarios provided in [40] cannot be precisely repeated in correspondence to the Gaussian formulation for attractiveness in (19), despite the resemblance in the dynamic outcomes which is apparent comparing Figure 5 in [40] with Figure 1(A).

Before performing such interpretative task, let us now briefly investigate how the bifurcation analysis conducted dealing with the Gaussian map changes when considering the parabolic formulation for attractiveness in (20). Showing in Figure 5 the bifurcation diagram of g with f_2 as in (20), for $\Delta \in (0, 0.9)$, at first we find a confirmation of the theoretical results in Section 2, which ensure that the same transcritical bifurcations as in Figure 1(A) and in Figure 5 in [40] occur at $a = 0$ and at $a = 1$, and that the number and value of the nontrivial equilibria coincide for any Δ when different attractiveness functional forms are taken into account. Moreover, for the considered parameter configuration, $a = a_1^*$ and $a = a_2^*$ are still unstable when they lie in the interval $(0, 1)$. In spite of the many similarities among the three frameworks, we notice two differences about the flip bifurcation occurring at $a = a^*$. The first one concerns the value of Δ at which it occurs. Namely, in [40], for the attractiveness recalled in (3), $a = a^*$ undergoes a flip bifurcation for $\Delta = 0.628$; as shown in Proposition 3.1, for f_1 as in (19) the flip bifurcation occurs for $\Delta = 0.517$; similarly to what in that result, it is possible to prove that for f_2 in (20) $a = a^*$ undergoes a flip bifurcation for $\Delta = 0.382$. Since the unique transcritical bifurcation of the map g at $a = 0$, at which $a = a_1^*$ loses stability in favour of $a = 0$ when entering the interval $(0, 1)$, occurs for all attractiveness formulations for $\Delta = 0.436$, then the flip bifurcation at $a = a^*$ follows the transcritical bifurcation at $a = 0$ in the first two frameworks, while the flip bifurcation precedes the transcritical bifurcation when considering the parabolic functional form for attractiveness. This means that, while in [40] and with the Gaussian map in (19) there is an interval of values of Δ for which the stable equilibria $a = 0$ and $a = a^*$ coexist, when considering f_2 as in (20) such coexistence phenomenon cannot occur anymore, since $a = a^*$ becomes unstable before $a = 0$ gains stability. The second, more crucial, difference among the three considered frameworks concerns the nature of the flip bifurcation occurring at $a = a^*$. Indeed, the flip bifurcation has been proven to be subcritical both in Proposition III.2 in [40] for the attractiveness recalled in (3) and in Proposition 3.1 above for f_1 as in (19). On the other hand, setting $\Delta^* = 0.382$ g with f_2 as in (20) it holds that $(\partial^3 g^2 / \partial a^3)(a^*; \Delta^*) = -127.739 < 0$ and $(\partial^2 g / \partial a \partial \Delta)(a^*; \Delta^*) = -14.117 < 0$, and thus, according to [44, Theorem 8.2], the flip bifurcation is supercritical. Namely, as shown in Figure 5 it is followed by a cascade of period-doubling bifurcations. Like with the other two attractiveness formulations, those period-doubling bifurcations lead first to a two-piece chaotic attractor, and then to a one-piece chaotic attractor, which disappears due to a contact bifurcation with $a = a_1^*$ for $\Delta = 0.551$. After the contact bifurcation, only (one of or both) the trivial equilibria are stable, and the same holds true for low values of the degree of heterogeneity. Hence, also with the parabolic functional form for attractiveness, for intermediate values of the heterogeneity degree parameter, our system generates

Figure 5. The bifurcation diagram of g for f_2 in (20) with $\Delta \in (0, 0.9)$. We denote in orange $a = 0$, in cyan $a = 1$, in red $a = a^*$ and the attractor which follows its flip bifurcation, in blue $a = a_1^*$ and $a = a_2^*$. Solid (dashed) curves refer to stable (unstable) equilibria and cycles.

Figure 6. The time series corresponding to the periods $t \in [200, 300]$ for $\mathcal{S}'_{\alpha,t}$ (in blue) and $\mathcal{S}'_{\beta,t}$ (in green) in (A), for a_t (in blue) and $b_t = 1 - a_t$ (in green) in (B), for $x_{\alpha,t}^*$ (in blue) and $x_{\beta,t}^*$ (in green) in (C), for $y_{\alpha,t}^*$ (in blue) and $y_{\beta,t}^*$ (in green) in (D), for the same parameter configuration considered in Figure 2(C).

oscillatory dynamics and the coexistence between groups displays the recurrent dynamic behaviour typical of the fashion cycle, characterized by booms and busts. Like done in Figure 6 for the Gaussian map, it would be possible to show the time series for all the relevant variables, which would illustrate the oscillatory behaviour, not only of the shares of the two groups, but also of the attractiveness and of the consumption choices over the two goods. We omit to present the corresponding plot for brevity's sake.

Summarizing, both for the attractiveness formulations in (19) and (20), the above bifurcation analysis has highlighted that in our model interesting dynamics are generated by an intermediate heterogeneity degree between groups. Indeed, when the level of agents' heterogeneity is too small, the only equilibria are the trivial ones, in which just one of the two kinds of agents is present, and this excludes the possibility of any sort of non-convergent dynamics. When increasing the heterogeneity degree parameter, three nontrivial equilibria emerge via transcritical bifurcations and this allows for more complex behaviours. Namely, such nontrivial equilibria may lose stability, giving rise to periodic and chaotic dynamics. However, when the distance between the two preference structures becomes excessive, no

oscillatory behaviours occur anymore, as the main chaotic attractor disappears due to a contact bifurcation with one of the nontrivial equilibria, and thus the system, according to the chosen initial condition, asymptotically converges toward one or the other of the trivial steady states.

For the considered parameter configuration, the only nontrivial equilibrium which may be locally stable is given by $a = a^*$. In particular, we showed that when $a = a^*$ enters the interval $(0, 1)$, a transcritical bifurcation occurs at $a = 1$, which loses stability in favour of $a = a^*$. We proved that $a = a^*$ loses stability via a flip bifurcation, which is subcritical for the Gaussian attractiveness formulation in (19), while it is supercritical for the parabolic map in (19). In the former case, like happened in [40] for the attractiveness functional form recalled in (3), we found period-two cycles emerging via fold bifurcations of the map governing the dynamics far from the nontrivial equilibrium, also when the latter is locally stable. On the contrary, for f_2 as in (19), the supercritical flip bifurcation is followed by a cascade of period-doubling bifurcations leading to chaos.

Despite the dissimilarity in the global dynamics, we found interesting multistability phenomena, involving equilibria, as well as periodic or chaotic attractors for both the attractiveness formulations in (19) and (20) (see Figures 1 and 5). The main difference between the scenarios detected in the two frameworks lies in the fact that subcritical flip bifurcations suddenly lead to strong oscillations in the agents' consumption choices and in the population shares, while supercritical flip bifurcations are initially followed by small oscillations, which then grow for increasing values of the heterogeneity parameter, until we observe large oscillations, similar to those which immediately follow a subcritical flip bifurcation.

We end the present subsection by stressing that the proof of chaos performed in [40] using the method of the turbulent maps in [4], working with homoclinic orbits, can be directly transposed to the just considered frameworks. On the other hand, as mentioned above, the sociological and economic interpretation of the main scenarios provided in [40] cannot be precisely repeated here, as we do not find mixed visibility frameworks in relation to any of our main scenarios (see the beginning of Subsection 3.2 for the corresponding definition). Moreover, the interpretation of the convergence toward a nontrivial steady state was missing in [40]. For such reasons, due also to its importance in view of understanding the functioning of the model dynamics, we describe the interpretation of the main scenarios in the next subsection. In particular, therein we will deal with the Gaussian rather than with the parabolic formulation for attractiveness in order to show how, despite the similarity between the dynamic outcomes obtained in [40] for the attractiveness recalled in (3) and here for f_1 in (19), some features related to the model interpretation differ in the two frameworks.

3.2. Economic and social scenarios

We now give an economic interpretation of the main scenarios we found with the Gaussian functional form for attractiveness, i.e. those in Figure 2, remarking that the framework symmetric with respect to Figure 2(A), in which agents of group α eventually disappear, admits a symmetric explanation with respect to the one we shall provide below for Figure 2(A), as in that case we are in a *strong visibility framework*, in which the visibility levels for both groups of agents lie above the threshold visibility level $\hat{V}$. We also notice that

the other possible scenarios locally reproduce one of those four frameworks, according to the chosen initial condition.

In particular, differently from [40], where, in relation to the scenario in Figure 2(C), we encountered a *mixed visibility framework* characterized by the lack of persistence of a given ordering among V_β , V_α and $\widehat{V}$, we find that in the three scenarios in Figure 2 we are in *weak visibility frameworks*, in which the visibility levels for the two groups of agents lie below the threshold visibility level $\widehat{V}$. In all cases, in agreement with (5), we observe a reduction of the share of the group with the lower attractiveness degree, caused by a social visibility level more distant from $\widehat{V}$. However, while for the scenario in Figure 2(A) there is a fixed ordering among V_β , V_α and $\widehat{V}$, which leads to the extinction of the group with the lower attractiveness degree, in regard to Figure 2(C), still remaining in a weak visibility framework, we observe a continual alternation in the relative position of V_α and V_β , which allows for oscillatory non-convergent dynamics. In Figure 2(B) we observe an initial alternation in the relative position of V_α and V_β , which soon stabilizes in a fixed ordering among V_β , V_α and $\widehat{V}$, this time leading to a stationary framework characterized by the coexistence between the two groups of agents.

More precisely, as concerns Figure 2(A), for instance taking $a_0 = 0.5$, we have that $V_\beta < V_\alpha < \widehat{V}$. Thus, since V_α is closer to $\widehat{V}$ than V_β , we expect an increase in the share of the agents of group α and, consequently, a raise in the aggregate demand of commodity x . Indeed, we do observe a decrease in the relative price p . The optimal consumption quantities for the two goods are determined and, since the value of x_α^* decreases more than x_β^* and v_x is much larger than v_y , the value of V_α decreases, while the value of V_β increases, so that the distance between V_β and V_α is reduced, but the ordering among V_β , V_α and $\widehat{V}$ is maintained. The repetition of such process eventually leads to the extinction of the agents of group β .

In regard to Figure 2(C), we change the value of Δ from 0.66 to 0.55, in order to observe the simplest oscillatory behaviour, i.e. a period-two cycle. Let us start from an initial condition close to the smallest between the two elements of the period-two cycle, taking for instance $a_0 = 0.2$. In such framework, the agents of type α , with their stronger preference for commodity x , are few and thus we expect that the aggregate demand for that good is low. Indeed, we do observe a high value for the relative price p . The optimal consumption quantities for the two goods are determined and, since p is high, the value of x_α^* is high, too. Due to the fact that v_x is much larger than v_y , we find that $V_\beta < V_\alpha < \widehat{V}$. Since the distance between V_α and $\widehat{V}$ is smaller than the distance between V_β and $\widehat{V}$, the share of agents of type α raises and approaches $a = 0.91$. We then expect that the aggregate demand for commodity x is higher than before. Indeed, we observe a lower value for the relative price p . The optimal consumption quantities for the two goods are determined and, since p is lower now, the value of x_α^* decreases. As v_x is still much larger than v_y , we find that $V_\alpha < V_\beta < \widehat{V}$. Since V_β is closer to $\widehat{V}$ than V_α , the share of agents of type α decreases and is again near $a = 0.2$, giving rise to a period-two cycle.

As explained in Subsection 3.1, in Figure 2(C) we witness the recurrent dynamic behaviour typical of the fashion cycle, which for us emerges at the aggregate level as a continual oscillation of the attractiveness and of the shares of the two groups, while on the individual level it is characterized by oscillatory consumption choices over the two goods, presenting booms and busts, for the agents of the two groups. See Figure 6 for the corresponding time series of the main variables for the time periods $t \in [200, 300]$.

Finally, the interpretation of Figure 2(B) follows by a combination of the arguments presented for Figure 2(A,C). Namely, starting for instance from $a_0 = 0.4$, we have that the agents of type α are few and thus we expect that the aggregate demand for commodity x is low. Indeed, we do observe a high value for the relative price p . The optimal consumption quantities for the two goods are determined and, since p is high, the value of x_α^* is high, too. Due to the fact that v_x is much larger than v_y , we find that $V_\beta < V_\alpha < \widehat{V}$. Since the distance between V_α and $\widehat{V}$ is smaller than the distance between V_β and $\widehat{V}$, the share of agents of type α raises and exceeds $a = 0.8$. We then expect that the aggregate demand for commodity x is higher than before. Indeed, we observe a lower value for the relative price p . The optimal consumption quantities for the two goods are determined and, since p is lower now, the value of x_α^* decreases. As v_x is still much larger than v_y , we find that $V_\alpha < V_\beta < \widehat{V}$. Since V_β is slightly closer to $\widehat{V}$ than V_α , the share of agents of type α decreases a little and approaches $a = 0.795$. Then, as the share of agents of type α is still large, the same process repeats again, so that the ordering $V_\alpha < V_\beta < \widehat{V}$ does not change anymore, but V_α progressively increases, while V_β progressively decreases, until they meet at $V_\alpha = V_\beta = 0.48$. Accordingly, the share a slowly but monotonically decreases from $a = 0.795$ as long as it reaches the stationary value $a^* = 0.786$.

4. Conclusions

In the present work we extended the exchange economy evolutionary discrete-time model with heterogeneous agents introduced in [40] by considering a generic bell-shaped attractiveness function for the two groups of agents. In particular, in the social framework we dealt with along the paper, the reproduction level of a group is related to its attractiveness degree, which depends on the social visibility level, determined by the consumption choices of the agents in that group.

In such context we investigated the asymptotic heterogeneity among agents, i.e. we discussed the possible dynamics arising when choosing initial conditions characterized by the coexistence between heterogeneous agents in view of understanding whether the initial heterogeneity eventually disappears from the system and, in case it persists, which new equilibria or more complex attractors emerge. We found that, although the same results on existence and results on local stability of the equilibria very similar to those obtained in [40] still hold true when considering a generic bell-shaped attractiveness function for the two groups of agents, the global dynamics may differ according to the chosen functional form for the attractiveness. Nonetheless, like in [40], also in this work we observed interesting multistability phenomena, involving equilibria, as well as periodic or chaotic attractors, for intermediate values of the heterogeneity degree parameter, which still plays an ambiguous role on the complexity of the dynamics. In more detail, thanks to the combined action of the price formation mechanism and of the share updating rule which produces the alternation between the bandwagon and snob regimes, we found that, for values of the heterogeneity degree parameter neither too small, nor too large, the dynamic coexistence between groups and the oscillatory nature of the consumption activities over the two goods for the agents belonging to the two groups display the recurrent dynamic behaviour typical of the fashion cycle (see [45]), characterized by booms and busts.

As concerns future study directions, we could generalize the setting analyzed in this work by allowing the threshold value $\widehat{V}$, at which the social attractiveness of a preference

structure becomes a decreasing function of the visibility level, not to coincide across the different preference structures or by letting the utility functions vary within a suitable class of maps, containing Cobb-Douglas utility functions, in order to understand how the number, the position and the local stability of the equilibria are affected by such changes. Moreover, our framework could be employed to represent the fashion cycle in settings with capital accumulation, such as the OLG model by Diamond [17]. A further extension of our setting would consist in assuming two different time scales for consumption choices and for the evolutionary mechanism. Indeed, in order to model the greater speed at which agents perform consumption choices with respect to the time required to update the population shares on the basis of the relative attractiveness between groups, it would be suitable to assume a continuous-time formulation for the consumption choices and a discrete-time description for the share updating rule. This would lead to the study of a hybrid dynamical system, like those analyzed e.g. in [9,12]. Furthermore, we could assume that the utility of some agents depends also on attractiveness, in order to investigate whether the more rational agents, which are aware of the functioning of the share updating rule, perform better from an evolutionary viewpoint. We recall that a partial negative answer to such question has been given in [41] in a framework in which the attractiveness of each group of agents depends in a monotone increasing manner on visibility. We finally stress that, modifying the argument of agents' utility functions so as to enter a setting with strategic interaction, like those considered in [29,30], and displaying strategic complementarity and/or substitutability, we could investigate the possible dynamical effects produced by heterogeneity between groups in those frameworks, too.

Notes

1. This occurs for instance with the parameter configuration we will deal with in Section 3 when considering $\mu = 2$ rather than $\mu = 6.5$, i.e. for $v_x = 0.9, w_x = 0.2, v_y = 0.15, w_y = 2, \beta = 0.1, \alpha = 0.9, \mu = 2, \bar{V} = 0.8$ and letting $\sigma > 0$ free to vary. On the other hand, for the attractiveness formulation in (20) it holds that $\partial \mathcal{A}_\alpha(a) / \partial (\sigma d_\alpha^2) \equiv -1$ and thus, in the case of a parabolic map, $(\partial \mathcal{A}_\alpha(a) / \partial (\sigma d_\alpha^2))|_{a=a^*}$ and $(\partial \mathcal{A}_\alpha(a) / \partial (\sigma d_\alpha^2))|_{a=a_j^*}$ do not depend on σ , and no issues arise.
2. We here consider polarized values for v_x and v_y , as both of them are positive but one is much larger than the other. Such case approximates those frameworks in which visibility and attractiveness are produced by the consumption of a single good. However, our results hold true also in the case of more balanced values for v_x and v_y .

Acknowledgments

The authors thank the anonymous Referees for the helpful and valuable comments.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

- [1] M. Bianchi, *Novelty, preferences, and fashion: When goods are unsettling*, J. Econ. Behav. Organ. 47 (2002), pp. 1–18.

- [2] G.I. Bischi, U. Merlone, and E. Pruscini, *Evolutionary dynamics in club goods binary games*, J. Econ. Dyn. Control 91 (2018), pp. 104–119.
- [3] A. Bisin and T. Verdier, *The economics of cultural transmission and the dynamics of preferences*, J. Econ. Theory 97 (2001), pp. 298–319.
- [4] L.S. Block and W.A. Coppel, *Dynamics in One Dimension*, Springer Lecture Notes 1513, Springer-Verlag, Berlin, 1992.
- [5] L.E. Blume and D. Easley, *Optimality and natural selection in markets*, J. Econ. Theory 107 (2002), pp. 95–135.
- [6] W. Brian Arthur, *Increasing Returns and Path Dependence in the Economy*, University of Michigan Press, Ann Arbor, MI, 1994.
- [7] A. Cabrales and J. Sobel, *On the limit points of discrete selection dynamics*, J. Econ. Theory 57 (1992), pp. 407–419.
- [8] J.P. Caulkins, R.F. Hartl, P.M. Kort, and G. Feichtinger, *Explaining fashion cycles: Imitators chasing innovators in product space*, J. Econ. Dyn. Control 31 (2007), pp. 1535–1556.
- [9] F. Cavalli and A. Naimzada, *A multiscale time model with piecewise constant argument for a boundedly rational monopolist*, J. Differ. Equ. Appl. 22 (2016), pp. 1480–1489.
- [10] F. Cavalli, A. Naimzada, and M. Pireddu, *Emergence of complex social behaviors from the canonical consumption model*, Mind Soc. (2015), DOI:10.1007/s11299-015-0169-9.
- [11] F. Cavalli, A. Naimzada, and M. Pireddu, *A family of models for Schelling binary choices*, Physica A 444 (2016), pp. 276–296.
- [12] F. Cavalli, A. Naimzada, and M. Sodini, *Oligopoly models with different learning and production time scales*, submitted.
- [13] J. Chang and R. Stauber, *Evolution of preferences in an exchange economy*, Econ. Lett. 103 (2009), pp. 131–134.
- [14] P.R.P. Coelho and J.E. McClure, *Toward an economic theory of fashion*, Econ. Inq. 31 (1993), pp. 595–608.
- [15] G. Corneo and O. Jeanne, *Segmented communication and fashionable behavior*, J. Econ. Behav. Organ. 39 (1999), pp. 371–385.
- [16] V. Di Giovinazzo and A. Naimzada, *A model of fashion: Endogenous preferences in social interaction*, Econ. Model. 47 (2015), pp. 12–17.
- [17] P.A. Diamond, *National debt in a neoclassical growth model*, Am. Econ. Rev. 55 (1965), pp. 1126–1150.
- [18] P. Dindo, *A tractable evolutionary model for the minority game with asymmetric payoffs*, Physica A 355 (2005), pp. 110–118.
- [19] E. Ekaterinchuk, J. Jungeilges, T. Ryazanova, and I. Sushko, *Dynamics of a minimal consumer network with uni-directional influence*, J. Evol. Econ. 27 (2017), pp. 831–857.
- [20] E. Ekaterinchuk, J. Jungeilges, T. Ryazanova, and I. Sushko, *Dynamics of a minimal consumer network with bi-directional influence*, Commun. Nonlinear Sci. Numer. Simul. 58 (2018), pp. 107–118.
- [21] P. Frijters, *A model of fashions and status*, Econ. Model. 15 (1998), pp. 501–517.
- [22] W. Gaertner, *A dynamic model of interdependent consumer behavior*, J. Econ. 34 (1974), pp. 327–344.
- [23] W. Gaertner, *Periodic and aperiodic consumer behavior*, Appl. Math. Comput. 22 (1987), pp. 233–254.
- [24] W. Gaertner and J. Jungeilges, *A non-linear model of interdependent consumer behaviour*, Econ. Lett. 27 (1988), pp. 145–150.
- [25] W. Gaertner and J. Jungeilges, *“Spindles” and coexisting attractors in a dynamic model of interdependent consumer behavior: A note*, J. Econ. Behav. Organ. 21 (1993), pp. 223–231.
- [26] W. Gaertner and J. Jungeilges, *A model of interdependent consumer behavior: Nonlinear dynamics in $\mathbb{R}^2$* , in *Models and Measurement of Welfare and Inequality*, W. Eichhorn, ed., Springer-Verlag, Berlin, 1994, pp. 940–962.
- [27] L. Gardini, I. Sushko, and K. Matsuyama, *2D discontinuous piecewise linear map: Emergence of fashion cycles*, Chaos 28 (2018), p. 055917, DOI:10.1063/1.5018588.

- [28] R.E. Haymond and E.S. Thomas, *Busy Dogs: A class of iteration problems*, J. Diff. Equ. Appl. (1996), DOI:10.1080/10236199608808059.
- [29] A. Heifetz, C. Shannon, and Y. Spiegel, *The dynamic evolution of preferences*, Econ. Theory 32 (2007), pp. 251–286.
- [30] A. Heifetz, C. Shannon, and Y. Spiegel, *What to maximize if you must*, J. Econ. Theory 133 (2007), pp. 31–57.
- [31] J. Hofbauer and J. Weibull, *Evolutionary selection against dominated strategies*, J. Econ. Theory 71 (1996), pp. 558–573.
- [32] J. Jungeilges and T. Ryazanova, *Transitions in consumption behaviors in a peer-driven stochastic consumer network*, submitted.
- [33] J. Jungeilges, T. Ryazanova, A. Mitrofanova, and I. Popova, *Sensitivity analysis of consumption cycles*, Chaos 28 (2018), p. 055905, DOI:10.1063/1.5024033.
- [34] E. Karni and D. Schmeidler, *Fixed preferences and changing tastes*, Am. Econ. Rev. 80 (1990), pp. 262–267.
- [35] K. Matsuyama, *Custom versus fashion: Path-dependence and limit cycles in a random matching game*, Working Paper in Economics, E-92-11, Hoover Institution; Northwestern University CMS-EMS DP # 1030, (1991).
- [36] A. Naimzada and M. Pireddu, *Chaotic social interaction via endogenous reactivity*, J. Diff. Equ. Appl. (2014), DOI:10.1080/10236198.2014.903938.
- [37] A. Naimzada and M. Pireddu, *Endogenous evolution of heterogeneous consumers preferences: Multistability and coexistence between groups*, Econ. Lett. 142 (2016), pp. 22–26.
- [38] A. Naimzada and M. Pireddu, *An evolutive discrete exchange economy model with heterogeneous preferences*, Chaos Solit. Fract. 111 (2018), pp. 35–43.
- [39] A. Naimzada and M. Pireddu, *Complex dynamics in an evolutionary general equilibrium model*, Discrete Dyn. Nat. Soc. 2018 (2018), Article ID 8471624, 13 pages.
- [40] A. Naimzada and M. Pireddu, *Fashion cycle dynamics in a model with endogenous discrete evolution of heterogeneous preferences*, Chaos 28 (2018), p. 055907, DOI:10.1063/1.5024931.
- [41] A. Naimzada and M. Pireddu, *Strong coexistence for a model with endogenous evolution of heterogeneous agents*, Commun. Nonlinear Sci. Numer. Simul. 65 (2018), pp. 35–53.
- [42] W. Pesendorfer, *Design innovation and fashion cycles*, Am. Econ. Rev. 85 (1995), pp. 771–792.
- [43] W.H. Sandholm, *Population Games and Evolutionary Dynamics*, MIT Press, Cambridge, 2010.
- [44] A.N. Sharkovsky, S.F. Kolyada, A.G. Sivak, and V.V. Fedorenko, *Dynamics of One-Dimensional Maps*, Kluwer Academic, Boston, 1997.
- [45] G. Simmel, *Fashion*, Am. J. Sociol. 62 (1957), pp. 541–558.
- [46] F. Vigneron and L. Johnson, *A review and a conceptual framework of prestige-seeking consumer behavior*, Acad. Mark. Sci. Rev. 3 (1999), pp. 1–15.
- [47] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Texts in Applied Mathematics, 2nd ed., Springer, New York, 2003.
- [48] W.-B. Zhang, *Fashion with snobs and bandwagoners in a three-type households and three-sector neoclassical growth model*, Rev. Mex. Econ. Finan. 11 (2016), pp. 1–19.
- [49] W.-B. Zhang, *Fashion and business cycles with snobs and bandwagoners in a multi-sector growth model*, J. Bus. 2 (2017), pp. 1–13.