

CONTROL MEASURES ON BOOLEAN ALGEBRAS

GIANLUCA CASSESE

ABSTRACT. In this paper we discuss the existence of a control measure for a family of measures on a Boolean algebra. We obtain a necessary and sufficient condition and several related results, including a new criterion for weak compactness for additive set functions on an algebra of sets.

This paper is dedicated to the memory of Fiamma Galgani.

1. INTRODUCTION.

In 1947, Dorothy Maharam [15] introduced and characterised the notion of measure algebra, namely a Boolean algebra \mathcal{A} endowed with a measure that is strictly positive on $\mathcal{A} \setminus \{0\}$. Obtaining a characterization of measure algebras has since then become a major topic of research in measure theory.

In this paper we investigate the somehow related question of finding necessary and sufficient conditions for a set \mathcal{M} of measures on \mathcal{A} to admit a dominating or control measure, i.e. a measure ν such that

$$(1) \quad \lim_n \nu(a_n) = 0 \quad \text{implies} \quad \lim_n \mu(a_n) = 0 \quad \mu \in \mathcal{M}.$$

Although in the measure algebra literature domination has hitherto played a minor role, it has attracted much attention in analysis, particularly in the study of vector measures in which, following Bartle, Dunford and Schwartz [2], if $F : \mathcal{A} \rightarrow X$ is an additive function with values in a vector space X , the existence of a control measure for the set $\mathcal{M} = \{x^*F : x^* \in X^*\}$ is particularly useful. See also [5].

The problem addressed in this paper has a fairly natural translation in the language of vector lattices where the domination property is reformulated into the condition that a given set belongs to some principal projection band. This general problem is fully settled in section 2. Nevertheless when it comes to additive functions on a Boolean algebra the characterization so obtained is not very explicit about the role of the underlying algebra. In order to obtain a more informative condition involving \mathcal{A} we introduce a hierarchy of different properties concerning \mathcal{M} , the (\mathbf{D}_0) , (\mathbf{D}) and (\mathbf{D}_*) properties. One may consider each of these definitions as a variant of the well known and historically important (CC) condition. In sections 4 and 5 we use the first two properties to study monotone and additive functions respectively, while in section 6 we characterize weak

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compactness via the (\mathbf{D}_*) property. Eventually, in section 7 we construct a fairly general Boolean algebra for which the (CC) condition is necessary and sufficient to be a measure algebra.

All of our results rely on two general lemmas of their own interest proved in section 3 for general Boolean algebras.

1.1. Notation. In the sequel \mathcal{A} is a Boolean algebra and, following [18], we denote binary operations on \mathcal{A} with set theoretic symbols. Thus, $a \cap b$, $a \cup b$ and a^c denote meet, join and complementation; we also write $a \cap b^c$ as $a \setminus b$, $(a \setminus b) \cup (b \setminus a)$ as $a \triangle b$ and $\mathbf{1}$ and $\mathbf{0}$ for the greatest and the least elements. By a measure on \mathcal{A} we mean a function $\mu : \mathcal{A} \rightarrow \mathbb{R}_+$ such that

$$(2) \quad \mu(\mathbf{0}) = 0 \quad \text{and} \quad \mu(a \cap b) + \mu(a \cup b) = \mu(a) + \mu(b) \quad a, b \in \mathcal{A}.$$

Of course \mathcal{A} may well be regarded as an algebra of subsets of some given set Ω , via Stone isomorphism. This remark makes available several results originally established for functions defined on an algebra of sets, at least as long as one avoids infinite operations which are generally not preserved under Boolean isomorphisms, as is well known (see e.g. [10, Example 3.1]).

The importance of a Boolean structure becomes apparent as we proceed to embed in an obvious way \mathcal{A} into the Boolean algebra $\mathfrak{S}(\mathcal{A})$ of sequences from \mathcal{A} . The n -th coordinate projection of $\sigma \in \mathfrak{S}(\mathcal{A})$ will be designated by $\sigma(n)$ and the range of σ by $[\sigma]$. The Boolean operations on $\mathfrak{S}(\mathcal{A})$ will be denoted by $\sigma \wedge \tau$, $\sigma \vee \tau$ and $\sigma \sim \tau$ defined implicitly by letting

$$(3) \quad (\sigma \wedge \tau)(n) = \sigma(n) \cap \tau(n), \quad (\sigma \vee \tau)(n) = \sigma(n) \cup \tau(n) \quad \text{and} \quad (\sigma \sim \tau)(n) = \sigma(n) \setminus \tau(n) \quad n \in \mathbb{N}.$$

$\mathfrak{S}^\downarrow(\mathcal{A})$ (resp. $\mathfrak{S}^\uparrow(\mathcal{A})$) will indicate the family of decreasing (resp. increasing) sequences on \mathcal{A} .

Denoting by $\mathfrak{S}_0(\mathcal{A})$ the ideal of sequences with finitely many non null elements, we obtain the factorial Boolean algebra $\overline{\mathfrak{S}(\mathcal{A})} = \mathfrak{S}(\mathcal{A})/\mathfrak{S}_0(\mathcal{A})$. The image of $\sigma \in \mathfrak{S}(\mathcal{A})$ under the canonical homomorphism of $\mathfrak{S}(\mathcal{A})$ onto $\overline{\mathfrak{S}(\mathcal{A})}$ will be denoted by $\bar{\sigma}$. Boolean operations on $\overline{\mathfrak{S}(\mathcal{A})}$ and on $\mathfrak{S}(\mathcal{A})$ will be indicated by the same symbols. Every function $m : \mathcal{A} \rightarrow \mathbb{R}_+$ extends to a function $\bar{m} : \overline{\mathfrak{S}(\mathcal{A})} \rightarrow \mathbb{R}_+$ via the equation

$$(4) \quad \bar{m}(\bar{\sigma}) = \limsup_n m(\sigma(n)) \quad \bar{\sigma} \in \overline{\mathfrak{S}(\mathcal{A})}, \quad \sigma \in \mathfrak{S}(\mathcal{A}).$$

2. BANACH LATTICE PRELIMINARIES

In this section we study the notion of domination in the context of a given Banach lattice X with order continuous norm. Terminology and notation are borrowed from [1]: I_A is the ideal generated by $A \subset X$, B_x is the band generated by $x \in X$ and \bar{A} denotes the norm closure of A .

Lemma 1. *Let X be a Banach lattice with order continuous norm and $I \subset X$ an ideal. If I contains no uncountable collection of non null, pairwise orthogonal elements then $I \subset B_z$ for some $z \in \bar{I}$. If X is an abstract L^p space ($1 \leq p < \infty$), the converse is also true.*

Proof. The family $\{A \subset I : x \perp y \text{ for every } x, y \in A\}$ admits, by Zorn lemma, a maximal element (relative to inclusion) which, by assumption, may be enumerated as x_1, x_2, \dots . Then

$$(5) \quad z = \sum_n 2^{-n} \frac{|x_n|}{1 + \|x_n\|} \in \bar{I}.$$

If $x \in I \setminus B_z$ then there exists $0 < y \leq |x|$ orthogonal to x_n for $n = 1, 2, \dots$, contradicting the maximality of $\{x_n : n \in \mathbb{N}\}$. Conversely, let $I \subset B_z$ for some $z \in X_+$ and let $\{y_\alpha : \alpha \in \mathfrak{A}\} \subset I_+$ be pairwise orthogonal. If X is an abstract L^p space, $\|z\|^p \geq \sum_\alpha \|z \wedge y_\alpha\|^p$ so that $y_\alpha \perp z$ – and thus $\|y_\alpha\| = 0$ – for all save countably many $\alpha \in \mathfrak{A}$. \square

Thus for a set A in a Banach lattice X with order continuous norm, $A \subset B_x$ for some $x \in X$ if and only if $A \subset B_z$ for some z of the form $z = \sum_n a_n |x_n|$, with $x_1, x_2, \dots \in A$. In the setting of countably additive set functions on a σ algebra of sets this claim was proved by Halmos and Savage [9, Lemma 7] (but see also Walsh [20, Lemma 1]) while its proof in the finitely additive case was given in [4, Theorem 2]. A similar property has also been studied recently by Lipecki [14] under the name of band domination.

Although Lemma 1 provides a clear answer to the question of a dominating element in several interesting situations, in the case of a family of additive set functions it is not particularly informative concerning the role underlying family of sets. This notwithstanding, Lemma 1 provides a first result on measure algebras, at least in a rather special case.

Corollary 1. *Let X be a Banach lattice with order continuous norm, $I \subset X$ an ideal and \mathcal{A} be the algebra of subsets of X_+ generated by $\{(0, |a|] : a \in I\}$. If \mathcal{A} is a measure algebra then $A \subset B_z$ for some $z \in \bar{I}$.*

Proof. Let the finitely additive probability μ on \mathcal{A} be strictly positive on $\mathcal{A} \setminus \{\emptyset\}$ and $\{x_\alpha : \alpha \in \mathfrak{A}\}$ a disjoint family in $I \setminus \{0\}$. Then the intervals $(0, |x_\alpha|]$ are pairwise disjoint so that $\mu(A) \geq \sum_\alpha \mu((0, |x_\alpha|])$ and \mathfrak{A} is countable. The claim follows from Lemma 1. \square

3. BOOLEAN ALGEBRA PRELIMINARIES

In this section we prove two general lemmas on Boolean algebras (the results of this section may be proved in more general structures than Boolean algebras).

Lemma 2. *Let \mathcal{A} be a Boolean algebra and let $\Gamma : \mathfrak{S}(\mathcal{A}) \rightarrow \mathcal{A}$ satisfy $\Gamma(\sigma) \geq [\sigma]$ for all $\sigma \in \mathfrak{S}(\mathcal{A})$. Assume $\emptyset \neq \mathcal{G} \subset \mathcal{F} \subset \mathcal{A}$ and (a) $\mathbf{0} \notin \mathcal{F}$, (b) $x \in \mathcal{F}$ and $y \geq x$ imply $y \in \mathcal{F}$ and (c) any family $\{x_\alpha : \alpha \in \mathfrak{A}\} \subset \mathcal{G}$ with $x_\alpha \cap x_{\alpha'} \notin \mathcal{F}$ when $\alpha \neq \alpha'$ is at most countable. There exists $\sigma_0 \in \mathfrak{S}(\mathcal{G})$ such that*

$$(6) \quad x \setminus \Gamma(\sigma_0) \notin \mathcal{G} \quad x \in \mathcal{G}.$$

Proof. If $\sigma(1) \in \mathcal{F}$ then, $\Gamma(\sigma) \geq \sigma(1)$ implies $\Gamma(\sigma) \in \mathcal{F}$; if $x \in [\sigma]$ then $x \setminus \Gamma(\sigma) \notin \mathcal{G}$. Define

$$(7) \quad \mathfrak{A} = \{(x, \sigma) \in \mathcal{G} \times \mathfrak{S}(\mathcal{G}) : x \setminus \Gamma(\sigma) \in \mathcal{G}\}.$$

If \mathfrak{A} is empty, the claim is trivial. Otherwise, write $(y, \tau) \succ (x, \sigma)$ to indicate that $\{x\} \cup [\sigma] \subset [\tau]$. Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be a maximal, linearly \succ ordered subset. If $(x, \sigma), (y, \tau) \in \mathfrak{A}_0$ and, say, $(y, \tau) \succ (x, \sigma)$ then $(y \setminus \Gamma(\tau)) \cap (x \setminus \Gamma(\sigma)) \leq x \setminus \Gamma(\tau) \notin \mathcal{F}$. Thus, the collection $\{x \setminus \Gamma(\sigma) : (x, \sigma) \in \mathfrak{A}_0\} \subset \mathcal{G}$ is such that the meet of any two elements in it does not belong to \mathcal{F} and, by property (c), \mathfrak{A}_0 must be countable. Choose $\sigma_0 \in \mathfrak{S}(\mathcal{G})$ such that $[\sigma_0] = \bigcup_{(x, \sigma) \in \mathfrak{A}_0} [\sigma]$ and set $x_0 = \Gamma(\sigma_0) \in \mathcal{F}$. Then, $x \setminus x_0 \notin \mathcal{G}$ for all x such that $(x, \sigma) \in \mathfrak{A}_0$. If $y_0 \setminus x_0 \in \mathcal{G}$ for some $y_0 \in \mathcal{G}$, this would imply $(y_0, \sigma_0) \notin \mathfrak{A}_0$ and $(y_0, \sigma_0) \succ (x_0, \sigma)$ for all $(x, \sigma) \in \mathfrak{A}_0$, contradicting the maximality of \mathfrak{A}_0 . \square

If $\phi : \mathcal{A} \rightarrow \mathbb{R}$ satisfies (2), the set $\mathcal{F} = \mathcal{G} = \{b : \sup_{a \leq b} \phi(a) > 0\}$ guarantees properties (a) and (b); $\sup_a \phi(a) < \infty$ implies (c). Lemma 2 then delivers a form of Hahn decomposition.

A trivial example of Γ is of course $\Gamma(\sigma) = \mathbf{1}$ or, if \mathcal{A} is σ complete, $\Gamma(\sigma) = \bigcup_{x \in [\sigma]} x$. A more interesting example, justifying our interest for $\overline{\mathfrak{S}(\mathcal{A})}$, is given next.

Example 1. Let \mathcal{A} be a Boolean algebra and $\{\bar{\sigma}_n : n \in \mathbb{N}\} \subset \overline{\mathfrak{S}(\mathcal{A})}$. For each $n \in \mathbb{N}$ choose $\sigma_n \in \bar{\sigma}_n$ and define $\sigma \in \mathfrak{S}(\mathcal{A})$ by letting

$$(8) \quad \sigma(n) = \bigcup_{j \leq n} \sigma_j(n) \quad n \in \mathbb{N}.$$

Then $\bar{\sigma} \geq \bar{\sigma}_n$ for each $n \in \mathbb{N}$ and one may set $\Gamma(\bar{\sigma}_1, \bar{\sigma}_2, \dots) = \bar{\sigma}$.

We shall make use of property (c) of Lemma 2 sufficiently often to justify referring to that condition by saying that \mathcal{G} is sparse in \mathcal{F} . Properties (a) and (b) imply that writing

$$(9) \quad y >_{\mathcal{F}} x \quad \text{whenever} \quad y \geq x \quad \text{and} \quad y \setminus x \in \mathcal{F}$$

implicitly defines an asymmetric partial order.

Lemma 3. Let \mathcal{A} be a Boolean algebra and let $\mathcal{F} \subset \mathcal{A}$ be sparse in itself and satisfy properties (a) and (b) of Lemma 2. Any $\mathcal{G} \subset \mathcal{A}$ linearly $>_{\mathcal{F}}$ ordered either admits an $>_{\mathcal{F}}$ maximum (resp. minimum) or a countable subset \mathcal{G}_0 having the same upper (res. lower) \geq bounds as \mathcal{G} .

Proof. Put $\mathfrak{A} = \{(x, x') \in \mathcal{G} \times \mathcal{G} : x' >_{\mathcal{F}} x\}$. For $(x, x'), (y, y') \in \mathfrak{A}$ write $(y, y') \succ (x, x')$ when $y \geq x'$ and let $\mathfrak{A}_0 \subset \mathfrak{A}$ be a maximal, linearly \succ ordered subset. If $(x, x'), (y, y') \in \mathfrak{A}_0$ then

$$(10) \quad (x' \setminus x) \cap (y' \setminus y) \leq x' \setminus y \leq y \setminus y \notin \mathcal{F}.$$

Given that \mathcal{F} is sparse, \mathfrak{A}_0 must be countable as well as $\mathcal{G}_0 = \{x, x' : (x, x') \in \mathfrak{A}_0\}$. If $z_0 \in \mathcal{G}$ is an upper bound for \mathcal{G}_0 but not for \mathcal{G} , then there exists $z \in \mathcal{G}$ such that $z >_{\mathcal{F}} x$ for all $x \in \mathcal{G}_0$. If z is not an $>_{\mathcal{F}}$ maximum for \mathcal{G} , then there exists $z' \in \mathcal{G}$ such that $z' >_{\mathcal{F}} z$ and therefore such that $(z, z') \in \mathfrak{A}$ and $(z, z') \succ (x, x')$ for all $(x, x') \in \mathfrak{A}_0$, a contradiction. \square

4. MONOTONIC SET FUNCTIONS

In this section we fix $\mathcal{B} \subset \mathcal{A}$ closed under \cup , \cap and \setminus . If $\psi : \mathcal{B} \rightarrow \mathbb{R}_+$ we define $\bar{\psi} : \overline{\mathfrak{S}(\mathcal{B})} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ as in (4).

Definition 1. A function $\psi : \mathcal{B} \rightarrow \mathbb{R}_+$ possesses property (\mathbf{D}_0) if any collection $\{\sigma_i : i \in I\} \subset \mathfrak{S}(\mathcal{B})$ satisfying

$$(11) \quad \inf_{i \in I} \bar{\psi}(\bar{\sigma}_i) > 0 \quad \text{and} \quad \bar{\psi}(\bar{\sigma}_i \cap \bar{\sigma}_j) = 0 \quad i, j \in I, i \neq j,$$

is at most countable.

Theorem 1. Let $\psi : \mathcal{B} \rightarrow \mathbb{R}_+$ be monotonic with $\psi(\mathbf{0}) = 0$. If ψ satisfies property (\mathbf{D}_0) then there exists $\sigma_0 \in \mathfrak{S}^\uparrow(\mathcal{B})$ such that

$$(12) \quad \lim_n \psi(\sigma(n) \setminus \sigma_0(n)) = 0 \quad \sigma \in \mathfrak{S}(\mathcal{B}).$$

Proof. Let $\mathcal{F} = \{\bar{\sigma} \in \overline{\mathfrak{S}(\mathcal{B})} : \bar{\psi}(\bar{\sigma}) > 0\}$. If a family $\{\bar{\sigma}_i : i \in I\} \subset \mathcal{F}$ is such that $\bar{\sigma}_i \wedge \bar{\sigma}_j \notin \mathcal{F}$ and $\bar{\psi}(\bar{\sigma}_i) > 0$ with I uncountable, there must then be $\eta > 0$ and $I_0 \subset I$ uncountable such that $\inf_{i \in I_0} \bar{\psi}(\bar{\sigma}_i) > \eta$, contradicting (11). Thus \mathcal{F} is sparse in itself and, in view of Example 1, the conditions of Lemma 2 are satisfied with $\mathcal{F} = \mathcal{G}$. We deduce the existence of $\bar{\sigma}_0 \in \mathcal{F}$ such that

$$(13) \quad \bar{\psi}(\bar{\sigma} \sim \bar{\sigma}_0) = 0 \quad \bar{\sigma} \in \mathcal{F}$$

which holds trivially even when $\bar{\sigma} \notin \mathcal{F}$. Of course, since ψ is monotonic, the above conclusion still holds if we replace each set $\sigma_0(n)$ with $\bigcup_{j \leq n} \sigma_0(j) \in \mathcal{B}$, so as to make the sequence increasing. \square

Theorem 1 applies e.g. to a family $\{\varphi_t : t \in T\}$ of functions $\varphi_t : \mathcal{B} \rightarrow \mathbb{R}$ satisfying

$$(14) \quad \varphi_t(\mathbf{0}) = 0 \quad \text{and} \quad \sup_t \varphi_t(x) < \infty \quad t \in T, x \in \mathcal{B}$$

upon letting $\psi(x) = \sup_{t \in T} \sup_{\{y \in \mathcal{B} : y \leq x\}} \varphi_t(y)$.

Loosely speaking, one may interpret Theorem 1 as asserting that the sequence σ_0 summarizes most of the relevant information conveyed by ψ . Notice that if \mathcal{B} is σ complete then (12) implies

$$(15) \quad \lim_k \psi\left(\bigcup_{n > k} \sigma_0(n) \setminus \sigma_0(k)\right) = 0.$$

The following are examples in which condition (11) may fail or take a rather special form.

Example 2. Let the range of ψ in Theorem 1 be a finite set (e.g. when ψ is the supremum of a set of $0 - 1$ valued additive functions). Then for each pair $i \neq j$, $\bar{\psi}(\bar{\sigma}_i \wedge \bar{\sigma}_j) = 0$ if and only if $\psi(\sigma_i(n) \cap \sigma_j(n)) = 0$ for n sufficiently large. Fix $\eta > 0$ and let $\{\bar{\sigma}_i : i \in I\}$ be a maximal (with respect to inclusion) set in $\overline{\mathfrak{S}(\mathcal{B})}$ satisfying

$$(16) \quad \inf_{i \in I} \bar{\psi}(\bar{\sigma}_i) > \eta \quad \text{and} \quad \bar{\psi}(\bar{\sigma}_i \cap \bar{\sigma}_j) = 0 \quad i, j \in I, i \neq j.$$

Under the assumptions of Theorem 1, I is at most countable. However, if I is countably infinite, we may choose iteratively n_k such that

$$(17) \quad n_k > n_{k-1} \vee k, \quad \inf_{n > n_k} \psi(\sigma_k(n)) > \eta \quad \text{and} \quad \sup_{n \geq n_k} \sup_{i < k} \psi(\sigma_i(n) \cap \sigma_k(n)) = 0$$

and define $\sigma_0(k) = \sum_k \sigma_k(n) \mathbf{1}_{\{n_k \leq n < n_{k+1}\}}$. Then (16) extends to $\{\bar{\sigma}_i : i \in I\} \cup \{\bar{\sigma}_0\}$, contradicting the maximality of I . In other words, in the special case under consideration a collection $\{\bar{\sigma}_i : i \in I\}$ as in Theorem 1 is at most countable if and only if it is finite.

The following example is related to weak compactness, as will be clear after Theorem 3. Two sequences $\sigma, \tau \in \mathfrak{S}(\mathcal{B})$ are said to be quasi disjoint if $\sigma \wedge \tau \in \mathfrak{S}_0(\mathcal{B})$, i.e. if $\bar{\sigma} \wedge \bar{\tau} = 0$.

Example 3. Let $\mathcal{B} = \mathcal{P}(\mathbb{N})$ in Theorem 1. By a diagonal argument the maximal family $\{B^i : i \in I\}$ of infinite subsets of \mathbb{N} with finite pairwise intersection is uncountable. A formal proof may be found in [12, Lemma 9.21]. If we write $\sigma_i(n) = B^i \cap \{n, n+1, \dots\}$ and denote by σ_i the corresponding sequence, we obtain an uncountable, pairwise quasi disjoint family $\{\sigma_i : i \in I\} \subset \mathfrak{S}^\perp(\mathbb{N})$. By quasi disjointness, $\bar{\psi}(\bar{\sigma}_i \wedge \bar{\sigma}_j) = 0$ for all $i, j \in I$ with $i \neq j$. Thus in order for ψ to be of class (\mathbf{D}_0) , we need to have $\bar{\psi}(\sigma_i) = 0$ for all save countably many $i \in I$. In fact this conclusion holds under a weaker condition than property (\mathbf{D}_0) that will be introduced in the next section as property (\mathbf{D}) .

5. ADDITIVE SET FUNCTIONS.

In this section we fix a given family \mathcal{M} of measures on \mathcal{A} . Our purpose is to obtain a characterization of dominated sets of measures that may be given entirely in terms of the underlying algebra \mathcal{A} . The following property is the one considered in Example 3.

Definition 2. \mathcal{M} possesses property (\mathbf{D}) if every pairwise quasi disjoint collection $\{\sigma_\alpha : \alpha \in \mathfrak{A}\} \subset \mathfrak{S}^\perp(\mathcal{A})$ satisfying

$$(18) \quad \sup_{\mu \in \mathcal{M}} \bar{\mu}(\bar{\sigma}_\alpha) > 0 \quad \alpha \in \mathfrak{A}$$

is at most countable. If the same conclusion holds with (18) replaced by the weaker condition

$$(19) \quad \lim_n \sup_{\mu \in \mathcal{M}} \mu(\sigma_\alpha(n)) > 0 \quad \alpha \in \mathfrak{A}$$

then \mathcal{M} is said to be of class (\mathbf{D}_*) .

In case the elements of \mathcal{M} are countably additive and \mathcal{A} a σ algebra of subsets of some set Ω , each sequence σ_α in Definition 2 may be replaced with the element $b_\alpha = \bigcap_n \sigma_\alpha(n)$. Property (\mathbf{D}) takes then a somewhat easier form: each pairwise disjoint collection $\{b_\alpha : \alpha \in \mathfrak{A}\}$ in \mathcal{A} with $\sup_{\mu \in \mathcal{M}} \mu(b_\alpha) > 0$ is at most countable. This weaker version of property (\mathbf{D}) was introduced long ago in the literature under the name of “countable chain” (CC) condition by Maharam [15, p. 160] in her study of measure algebras and plays an important role in the papers by Musiał [16] and Drewnowski [6] (who credits Dubrovskii [7] for its first formulation). Maharam, differing from the other authors cited, considers this condition in the case in which \mathcal{M} is the set of all measures on \mathcal{A} . Drewnowski, [6, Theorem 2.3] and Musiał [16, Theorem 2], prove that (CC) is necessary and sufficient for a countably additive measure with values in a locally convex vector space to admit a control measure. Their claim may be easily adapted to show that such condition is necessary and sufficient for a dominated set of countably additive set functions on a σ algebra to

be dominated, a result rediscovered in [4, Theorem 3] and whose proof is an immediate corollary of the following Theorem 2 of the present paper. I am grateful to professor Lipecki who, in a private communication, called my attention on these references giving me the opportunity to acknowledge the results obtained by a group of outstanding mathematicians whose work is perhaps too little known.

It should be mentioned that the need for an extension from *sets* to *families of sets*, exemplified in the shift from property (CC) to property (D), was already clear to Maharam who formulated “postulate II” (p. 159) as a reinforcement of property (CC). In another paper on measure algebras, Kelley [13] considered families of sets with positive intersection number. More comments on the relationship with the measure algebra literature will appear in the closing section of the paper.

Before moving to the general implications of these definitions, three elementary facts may be easily established.

(1). A set \mathcal{M} consisting of a single element ν possesses property (D). In fact for each sequence σ_α as in Definition 2 one may let

$$(20) \quad \nu_\alpha(b) = \lim_n \nu(a \cap \sigma_\alpha(n)) \quad b \in \mathcal{A}$$

obtaining a family $\{\nu_\alpha : \alpha \in \mathfrak{A}\}$ of pairwise orthogonal, non null elements contained in the ideal generated by ν , so that \mathfrak{A} must be countable, by Lemma 1.

(2). Thus every dominated set possesses property (D). This same conclusion is no longer valid if ν dominates \mathcal{M} weakly (i.e. $\nu(a) = 0$ implies $\mu(a) = 0$ for all $\mu \in \mathcal{M}$) as this latter condition is not sufficient to infer from (18) that $\lim_n \nu(\sigma_\alpha(n)) > 0$.

In this section we have a special interest for those subfamilies of $\overline{\mathfrak{S}(\mathcal{A})}$ in restriction to which $\bar{\mu}$ is additive. An important such class is

$$(21) \quad \Sigma^\downarrow(\mathcal{A}) = \mathfrak{S}^\downarrow(\mathcal{A})/\mathfrak{S}_0(\mathcal{A}).$$

Another one, given a measure ν on \mathcal{A} , is the subclass $\Sigma^\downarrow(\mathcal{A}, \nu) \subset \Sigma^\downarrow(\mathcal{A})$ obtained upon replacing $\mathfrak{S}^\downarrow(\mathcal{A})$ in (21) with $\mathfrak{S}^\downarrow(\mathcal{A}, \nu) = \mathfrak{S}^\downarrow(\mathcal{A}) \cap \mathfrak{S}(\mathcal{A}, \nu)$ where

$$(22) \quad \mathfrak{S}(\mathcal{A}, \nu) = \left\{ \sigma \in \mathfrak{S}(\mathcal{A}) : \lim_n 2^n \sup_{k > n} \nu(\sigma(n) \triangle \sigma(k)) = 0 \right\}$$

is the Boolean algebra of sequences with exponential rate of ν -convergence.

Both $\Sigma^\downarrow(\mathcal{A})$ and $\Sigma^\downarrow(\mathcal{A}, \nu)$ contain the zero and the unit of $\overline{\mathfrak{S}(\mathcal{A})}$ and are closed with respect to join and meet. Since $\bar{\mu}$ is additive on $\Sigma^\downarrow(\mathcal{A})$ it is then so also on the algebra $\Sigma(\mathcal{A})$ generated by $\Sigma^\downarrow(\mathcal{A})$ [11, p. 478] and, *a fortiori*, on $\Sigma(\mathcal{A}, \nu)$, the algebra generated by $\Sigma^\downarrow(\mathcal{A}, \nu)$. Moreover, since $\mathfrak{S}(\mathcal{A}, \nu)$ is a Boolean algebra, then $\Sigma(\mathcal{A}, \nu) \subset \mathfrak{S}(\mathcal{A}, \nu)/\mathfrak{S}_0(\mathcal{A})$.

(3). A final, simple conclusion is obtained in the following:

Lemma 4. *Let ν be a measure on \mathcal{A} , $\varepsilon > 0$ and $\bar{\sigma} \in \Sigma(\mathcal{A}, \nu)$. There exist $\bar{\tau} \in \mathfrak{S}^\downarrow(\mathcal{A})$ and $\bar{v} \in \mathfrak{S}^\uparrow(\mathcal{A})$ such that*

$$(23) \quad \bar{\tau} \leq \bar{\sigma} \leq \bar{v} \quad \text{and} \quad \bar{\nu}(\bar{\tau}) + \varepsilon \geq \bar{\nu}(\bar{\sigma}) \geq \bar{\nu}(\bar{v}) - \varepsilon.$$

Proof. Pick $\sigma \in \bar{\sigma}$ and fix N large enough so that

$$(24) \quad 2^{-N} < \varepsilon/2 \quad \text{and} \quad \sup_{k \geq n \geq N} \nu(\sigma(n) \triangle \sigma(k)) < 2^{-n}.$$

Define $\tau(n) = \mathbf{1}$ and $v(n) = \mathbf{0}$, if $n < N$, or else $\tau(n) = \bigcap_{N \leq j \leq n} \sigma(j)$ and $v(n) = \bigcup_{N \leq j \leq n} \sigma(j)$. Clearly, $\bar{\tau} \leq \bar{\sigma} \leq \bar{v}$, $\tau \in \mathfrak{S}^\downarrow(\mathcal{A})$ and $v \in \mathfrak{S}^\uparrow(\mathcal{A})$. If $k \geq N$ then

$$\tau(k) \triangle \sigma(k) \subset \bigcup_{j=N}^{k-1} \sigma(j) \triangle \sigma(k), \quad v(k) \triangle \sigma(k) \leq \bigcup_{j=N}^{k-1} \sigma(j) \triangle \sigma(k)$$

$$\text{and } \nu\left(\bigcup_{j=N}^{k-1} \sigma(j) \triangle \sigma(k)\right) \leq 2^{-(N-1)} < \varepsilon. \quad \square$$

We shall use the notation $\nu \perp \mu$ and $\nu \ll \mu$ in exactly the same sense as for set functions.

Proposition 1. *Let \mathcal{M} possess property **(D)**. Choose a measure ν on \mathcal{A} such that $\nu \perp \mu$ for every $\mu \in \mathcal{M}$ and fix $0 < t < 1$. Then there exists $\tau_* \in \mathfrak{S}^\downarrow(\mathcal{A})$ such that*

$$(25) \quad \lim_n \nu(\tau_*(n)) \geq (1-t)\|\nu\| \quad \text{while} \quad \sup_{\mu \in \mathcal{M}} \lim_n \mu(\tau_*(n)) = 0.$$

Proof. If $\nu = 0$ choose $\tau_* = 0$. If $\|\nu\| > 0$ consider the sets

$$(26) \quad \Delta = \{\bar{\sigma} \in \Sigma(\mathcal{A}, \nu) : \bar{\nu}(\bar{\sigma}) \geq (1-t/2)\|\nu\|\} \quad \text{and} \quad \mathcal{F} = \left\{ \bar{\sigma} \in \Sigma(\mathcal{A}, \nu) : \sup_{\mu \in \mathcal{M}} \bar{\mu}(\bar{\sigma}) > 0 \right\}.$$

Since \mathcal{F} satisfies properties (a) and (b) of Lemma 2, the order $>_{\mathcal{F}}$ may be defined. We claim that Δ admits $\bar{\tau}$ such that $\bar{\tau} \not>_{\mathcal{F}} \bar{\tau}'$ for any $\bar{\tau}' \in \Delta$. Let to this end Δ_0 be a maximal, linearly $>_{\mathcal{F}}$ ordered subset of Δ . If Δ_0 admits a $>_{\mathcal{F}}$ minimum, the claim is proved. If not, then by Lemma 3 we may assume that Δ_0 admits a countable subset having the same bounds as Δ_0 . Given that Δ_0 is linearly $>_{\mathcal{F}}$ ordered we can extract an $>_{\mathcal{F}}$ decreasing sequence $\langle \bar{\sigma}_n \rangle_{n \in \mathbb{N}}$ from Δ_0 such that Δ_0 has the same lower bounds as $\{\bar{\sigma}_n : n \in \mathbb{N}\}$. Upon passing to a subsequence, if necessary, we can further assume

$$(27) \quad \sup_{k > n} \bar{\nu}(\bar{\sigma}_n \sim \bar{\sigma}_k) < 2^{-2n}.$$

For each $n \in \mathbb{N}$, choose $\sigma_n \in \bar{\sigma}_n$ so that $\sigma_n \geq \sigma_{n+1}$. Given that $\sigma_n \in \Sigma(\mathcal{A}, \nu)$ the quantity $\lim_k \nu(\sigma_n(k))$ exists for each $n \in \mathbb{N}$. But then, exploiting a diagonal argument, we can construct a sequence $\langle i_k \rangle_{k \in \mathbb{N}}$ of integers such that $i_k > i_{k-1} \vee k$ and that

$$(28) \quad \sup_{\{n,p,q: n \leq k \leq p \wedge q\}} \nu(\sigma_n(i_p) \triangle \sigma_n(i_q)) \leq 2^{-2k} \quad k \in \mathbb{N}.$$

Letting $\sigma'_n(k) = \sigma_n(i_k)$ we conclude that $\sigma'_n \geq \sigma'_{n+1}$, $\sigma'_n \in \Sigma(\mathcal{A}, \nu)$ and that $\bar{m}(\bar{\sigma}'_n) \leq \bar{m}(\bar{\sigma}_n)$ for all $n \in \mathbb{N}$ and all additive set function m on \mathcal{A} while $\bar{\nu}(\bar{\sigma}'_n) = \bar{\nu}(\bar{\sigma}_n)$.

Define now $\tau \in \mathfrak{S}(\mathcal{A})$ by letting $\tau(n) = \sigma'_n(n)$ for all $n \in \mathbb{N}$. Then $\tau(k) \subset \sigma'_n(k)$ for each $k \geq n$ so that $\bar{\tau} \leq \bar{\sigma}'_n$. To show that τ is the desired lower bound we need to show that $\tau \in \Delta$. If $k > n$

$$\begin{aligned} \nu(\tau(n) \triangle \tau(k)) &= \nu(\sigma'_n(n) \triangle \sigma'_k(k)) \\ &\leq \sup_{j>k} \nu(\sigma'_n(n) \triangle \sigma'_n(j)) + \sup_{j>k} \nu(\sigma'_k(k) \triangle \sigma_k(j)) + \lim_j \nu(\sigma'_n(j) \setminus \sigma'_k(j)) \\ &\leq 2^{-2(n-1)} \end{aligned}$$

by (27) and (28) so that $\tau \in \mathfrak{S}(\mathcal{A}, \nu)$. In addition, the inequality

$$\bar{\nu}(\bar{\tau}) \geq \lim_n \lim_j \nu(\sigma'_n(j)) - \lim_n \sup_{j>n} \nu(\sigma'_n(n) \triangle \sigma'_n(j)) = \lim_n \bar{\nu}(\bar{\sigma}'_n) \geq (1 - t/2)\|\nu\|$$

which follows from (28) implies that $\bar{\tau} \in \Delta$. Thus τ is a lower bound for Δ_0 and, since Δ_0 is maximal, it admits no $\bar{v} \in \Delta$ with $\bar{\tau} >_{\mathcal{F}} \bar{v}$. This conclusion translates into the statement

$$(29) \quad \sup_{\mu \in \mathcal{M}} \bar{\mu}(\bar{\tau} \sim \bar{v}) = 0 \quad \bar{v} \in \Delta, \bar{v} \leq \bar{\tau}.$$

Choose, e.g., $\tau_0 \in \mathfrak{S}^\downarrow(\mathcal{A})$ such that $\nu(\tau_0(n)) < 2^{-2n}$ and let $v = \tau \sim \tau_0$. Then $v \in \mathfrak{S}(\mathcal{A}, \nu)$, $v \leq \tau$ and $\bar{\nu}(v) = \bar{\nu}(\tau \sim \tau_0) = \bar{\nu}(\tau) \geq (1 - t/2)\|\nu\|$ i.e. $\bar{v} \in \Delta$. But then by (29)

$$0 = \sup_{\mu \in \mathcal{M}} \bar{\mu}(\bar{\tau} \sim \bar{v}) = \sup_{\mu \in \mathcal{M}} \bar{\mu}(\bar{\tau} \wedge \bar{\tau}_0) = \sup_{\mu \in \mathcal{M}} \lim_j \lim_k \mu(\tau(k) \cap \tau_0(j)).$$

The same conclusion holds *a fortiori* if we replace $\bar{\tau}$ with $\bar{\tau}_* \in \Sigma^\downarrow(\mathcal{A})$ chosen, in accordance with Lemma 4, such that $\bar{\tau}_* \leq \bar{\tau}$ and $\bar{\nu}(\bar{\tau}_*) \geq (1 - t)\|\nu\|$. This leads to

$$0 = \sup_{\mu \in \mathcal{M}} \lim_j \lim_k \mu(\tau_*(k) \cap \tau_0(j)) = \sup_{\mu \in \mathcal{M}} \lim_j \lambda_\mu(\tau_0(j))$$

where we have implicitly defined $\lambda_\mu \in ba(\mathcal{A})$ via

$$(30) \quad \lambda_\mu(H) = \lim_k \mu(\tau_*(k) \cap H) \quad H \in \mathcal{A}.$$

According to Orlicz [17, Theorem 3, p. 124] this is enough to conclude that $\lambda_\mu \ll \nu$. However, by construction, $\lambda_\mu \ll \mu$. Then necessarily, $0 = \sup_{\mu \in \mathcal{M}} \lambda_\mu(1) = \sup_{\mu \in \mathcal{M}} \bar{\mu}(\bar{\tau}_*)$ for all $\mu \in \mathcal{M}$. \square

It will be clear after the next result that the condition stated in Proposition 1 is not only necessary for property **(D)** but sufficient as well.

Theorem 2. \mathcal{M} possesses property **(D)** if and only if it is dominated.

Proof. Necessity has already been proved. To prove sufficiency, consider the collection D of all pairs (M, M') of subsets of \mathcal{M} with $M \subset M'$. For given $(M, M'), (N, N') \in D$, write $(M, M') \leq (N, N')$ whenever $M' \subset N$. Since this defines a partial order, consider the maximal linearly ordered family $\{(M_\alpha, M'_\alpha) : \alpha \in \mathfrak{A}\} \subset D$ such that for each $\alpha \in \mathfrak{A}$ (a) M_α, M'_α are countable and (b) there exists $0 < \nu_\alpha \perp M_\alpha$ and $\mu'_\alpha \in M'_\alpha$ such that $\nu_\alpha \leq \mu'_\alpha$. If \mathcal{M} possesses property **(D)**, then according to Proposition 1 for each $\alpha \in \mathfrak{A}$ there exists $\tau_\alpha \in \mathfrak{S}^\downarrow(\mathcal{A})$ such that

$$(31) \quad \bar{\nu}_\alpha(\bar{\tau}_\alpha^c) < \varepsilon \|\nu_\alpha\| \quad \text{while} \quad \sup_{\{\mu \in \mathcal{M} : \mu \perp \nu_\alpha\}} \bar{\mu}(\bar{\tau}_\alpha) = 0.$$

By construction, each $\alpha \in \mathfrak{A}$ admits countably many predecessors, $\alpha_1, \alpha_2, \dots$ and for each of these it is possible to construct $\tau_{\alpha_j} \in \mathfrak{S}^\downarrow(\mathcal{A})$ as in (31). We define then $\sigma_\alpha \in \mathfrak{S}^\downarrow(\mathcal{A})$ by letting

$$(32) \quad \sigma_\alpha(n) = \tau_\alpha(n) \setminus \bigcup_{j < n} \tau_{\alpha_j}(k_j^\alpha \vee n) \quad n \in \mathbb{N}$$

where, exploiting $\nu_\alpha \perp \nu_{\alpha_j}$, k_j^α is chosen so that

$$\sup_{k \geq k_j^\alpha} \nu_\alpha(\tau_{\alpha_j}(k)) < 2^{-j-1} [\varepsilon \|\nu_\alpha\| - \bar{\nu}_\alpha(\bar{\tau}_\alpha^c)] \quad j \in \mathbb{N}.$$

Notice that $\sigma_\alpha \leq \tau_\alpha$ and that

$$\sigma_\alpha(n) \cap \tau_{\alpha_j}(n) \subset \tau_{\alpha_j}(n) \setminus \tau_{\alpha_j}(k_j \vee n) = 0 \quad n \geq k_j^\alpha$$

so that σ_α and τ_{α_j} are quasi disjoint and, *a fortiori*, so are σ_α and σ_{α_j} . Moreover,

$$\nu_\alpha(\sigma_\alpha(n)^c) \leq \nu_\alpha(\tau_\alpha(n)^c) + \sum_{j \leq n} \nu_\alpha(\tau_{\alpha_j}(k_j^\alpha \vee n)) \leq \nu_\alpha(\tau_\alpha(n)^c) + \frac{1}{2} [\varepsilon \|\nu_\alpha\| - \bar{\nu}_\alpha(\bar{\tau}_\alpha^c)] < \varepsilon \|\nu_\alpha\|.$$

The collection $\{\sigma_\alpha : \alpha \in \mathfrak{A}\} \subset \mathfrak{S}^\downarrow(\mathcal{A})$ is thus pairwise quasi disjoint and satisfies (31). By property (D), \mathfrak{A} must be countable. Let $M = \bigcup_{\alpha \in \mathfrak{A}} M'_\alpha \in D$. If one could find $\mu \in \mathcal{M}$ and $0 < \nu \leq \mu$ such that $\nu \perp M$, then the pair $(M, M \cup \{\mu\})$ would contradict the maximality of $\{(M_\alpha, M'_\alpha) : \alpha \in \mathfrak{A}\}$. Thus each $\mu \in \mathcal{M}$ is dominated by some $m \in M$ and, *a fortiori*, by the σ -convex combination of its elements. \square

If \mathcal{M} is dominated it is then clear by Lemma 1 that a dominating measure is of the form

$$(33) \quad \mu_0 = \sum_n a_n \mu_n \quad \text{for some} \quad \mu_1, \mu_2, \dots \in \mathcal{M}, \quad a_1, a_2, \dots \in \mathbb{R}_+.$$

Therefore $\nu \perp \mathcal{M}$ if and only if $\nu \perp \mu_0$. But then for every $0 < t \leq 1$ there exists $\sigma \in \mathfrak{S}^\downarrow(\mathcal{A})$ such that

$$(34) \quad \bar{\nu}(\bar{\sigma}) \geq (1-t)\|\nu\| \quad \text{while} \quad \bar{\mu}_0(\bar{\sigma}) = \sup_{\mu \in \mathcal{M}} \bar{\mu}(\bar{\sigma}) = 0.$$

In other words after Theorem 2 the condition of Proposition 1 is sufficient for property (D).

6. WEAK COMPACTNESS IN THE SPACE OF ADDITIVE SET FUNCTIONS.

Let us now consider the case in which \mathcal{A} is an algebra of subsets of some non empty set Ω and $\mathcal{M} \subset ba(\mathcal{A})$. In the special case in which \mathcal{M} is norm bounded, uniform strong additivity is equivalent to relative weak compactness (see [3]) and implies that \mathcal{M} must be dominated. This implication is true even without norm boundedness.

Corollary 2. *A uniformly strongly additive set \mathcal{M} is dominated.*

Proof. Suppose that \mathcal{M} fails to possess property (D). Then it is possible to find $\eta > 0$ and a pairwise quasi disjoint sequence $\langle \sigma_k \rangle_{k \in \mathbb{N}}$ in $\mathfrak{S}^\downarrow(\mathcal{A})$ such that $\inf_k \sup_{\mu \in \mathcal{M}} |\bar{\mu}|(\bar{\sigma}_k) > \eta$. By picking B_k from the sequence σ_k for each $k \in \mathbb{N}$ accurately we can then form a pairwise disjoint sequence $\langle B_k \rangle_{k \in \mathbb{N}}$ such that $\inf_k \sup_{\mu \in \mathcal{M}} |\mu|(B_k) > \eta$ so that uniform strong additivity fails. \square

The connection between property (\mathbf{D}) and weak compactness is made precise in the following:

Theorem 3. \mathcal{M} is relatively weakly compact if and only if it is norm bounded and of class (\mathbf{D}_*) .

Proof. The set $\{\nu\}$ is trivially of class (\mathbf{D}_*) . If ν dominates \mathcal{M} uniformly, then \mathcal{M} is of class (\mathbf{D}_*) . Thus relative weak compactness implies property (\mathbf{D}_*) .

To prove the converse, denote by $\sigma\mathcal{A}$ the σ algebra generated by \mathcal{A} and, if $m \in ba(\mathcal{A})_+$, by m_* the set function on $\sigma\mathcal{A}$ defined by

$$(35) \quad m_*(B) = \sup_{\{A \in \mathcal{A} : A \subset B\}} m(A) \quad B \in \sigma\mathcal{A}.$$

We first show that the collection $\{|\mu|_* : \mu \in \mathcal{M}\}$ itself possesses property (\mathbf{D}_*) . In fact, if $\{\sigma_\alpha : \alpha \in \mathfrak{A}\}$ is a pairwise quasi disjoint family in $\mathfrak{S}^\downarrow(\sigma\mathcal{A})$ such that

$$\limsup_n \sup_{\mu \in \mathcal{M}} |\mu|_*(\sigma_\alpha(n)) > 0 \quad \alpha \in \mathfrak{A}$$

for each $\varepsilon > 0$, $\alpha \in \mathfrak{A}$ and $n \in \mathbb{N}$ we can find $\tau_\alpha(n) \in \mathcal{A}$ such that $\tau_\alpha(n) \subset \sigma_\alpha(n)$ and $\sup_{\mu \in \mathcal{M}} |\mu|_*(\sigma_\alpha(n) \setminus \tau_\alpha(n)) \leq \varepsilon 2^{-n}$. Let $\tau'_\alpha(n) = \bigcap_{j=1}^n \tau_\alpha(j)$. Then $\tau'_\alpha \in \mathfrak{S}^\downarrow(\mathcal{A})$, $\tau'_\alpha \leq \sigma_\alpha$ and

$$\begin{aligned} \sup_{\mu \in \mathcal{M}} |\mu|(\tau'_\alpha(n)) &\geq \sup_{\mu \in \mathcal{M}} |\mu| \left(\bigcap_{j=2}^n \tau_\alpha(j) \right) - \sup_{\mu \in \mathcal{M}} |\mu| \left(\bigcap_{j=2}^n \tau_\alpha(j) \setminus \tau_\alpha(1) \right) \\ &\geq \sup_{\mu \in \mathcal{M}} |\mu| \left(\bigcap_{j=2}^n \tau_\alpha(j) \right) - \sup_{\mu \in \mathcal{M}} |\mu|_*(\sigma_\alpha(1) \setminus \tau_\alpha(1)) \\ &\geq \sup_{\mu \in \mathcal{M}} |\mu| \left(\bigcap_{j=2}^n \tau_\alpha(j) \right) - \varepsilon 2^{-1} \\ &\geq \sup_{\mu \in \mathcal{M}} |\mu|_*(\sigma_\alpha(n)) - \varepsilon \sum_{j=1}^n 2^{-j}. \end{aligned}$$

This shows that $\{\tau'_\alpha : \alpha \in \mathfrak{A}\}$ forms a pairwise quasi disjoint family of decreasing sequences that satisfies the condition $\lim_n \sup_{\mu \in \mathcal{M}} |\mu|(\tau'_\alpha(n)) > 0$ for each $\alpha \in \mathfrak{A}$. By property (\mathbf{D}_*) , \mathfrak{A} must then be countable, thus proving the preceding claim.

Take a disjoint sequence $v \in \mathfrak{S}(\mathcal{A})$ and define

$$(36) \quad v(E) = \bigcup_{k \in E} v(k) \quad \text{and} \quad \psi(E) = \sup_{\mu \in \mathcal{M}} |\mu|_*(v(E)) \quad E \subset \mathbb{N}.$$

Given that \mathcal{M} is norm bounded, that the sequence is disjoint and that $v(E) \in \sigma\mathcal{A}$ we conclude that $\psi : \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}_+$ is monotone, $\psi(\emptyset) = 0$ and that ψ is of class (\mathbf{D}) . As shown in Example 3, there exists an infinite set $E_0 \subset \mathbb{N}$ such that, letting $E_n = E_0 \cap \{n, n+1, \dots\}$,

$$(37) \quad \lim_n \psi(E_n) = \lim_n \sup_{\mu \in \mathcal{M}} |\mu|_*(v(E_n)) = 0.$$

Upon passing to a subsequence if necessary, we can assume the existence of $i_n \in E_n \setminus E_{n+1}$. Then,

$$\lim_n \sup_{\mu \in \mathcal{M}} |\mu|(v(i_n)) \leq \lim_n \psi(E_n) = 0.$$

This rules out the possibility that $\limsup_k \sup_{\mu \in \mathcal{M}} |\mu|(v(n)) > 0$ and proves that \mathcal{M} is uniformly strongly additive and thus relatively weakly compact. \square

We deduce easily the following special version of a result of Zhang [21, Theorem 1.3]. In this claim it is essential to take \mathcal{M} to consist of positive set functions.

Corollary 3. *A weakly* compact set $\mathcal{M} \subset ba(\mathcal{A})_+$ is weakly compact if and only if of class (\mathbf{D}) .*

Proof. If \mathcal{M} is weakly* compact it is then weakly closed and bounded. By Theorem 3 it remains to prove that is of class (\mathbf{D}_*) . But for a weakly* compact set of positive, additive set functions this is equivalent to property (\mathbf{D}) , by virtue of Dini's Theorem. \square

7. RELATION WITH THE LITERATURE

When \mathcal{M} is the set of all measures on \mathcal{A} , properties (\mathbf{D}) and (CC) are rightfully interpreted as properties of the algebra \mathcal{A} . Given that each element of \mathcal{A} other than 0 is assigned positive mass by some measure, property (\mathbf{D}) is sufficient to imply the existence of a set function that vanishes only on 0, i.e. that \mathcal{A} is a measure algebra. Maharam conjectured that the (CC) condition may possibly be sufficient for a Boolean algebra to be measure algebra (see also [11, Theorem 2.4]). Gaifman [8] later constructed an example of a Boolean algebra satisfying the (CC) condition but failing to be a measure algebra. Quite recently, Talagrand [19] provided an example of a Boolean σ algebra satisfying the (CC) property and the so-called weak distributive law but which is not a measure algebra. A necessary and sufficient condition has been given by Kelley [13].

We can show a special case of a fairly general Boolean algebra in which the (CC) property is necessary and sufficient to be a measure algebra.

Theorem 4. *Let \mathcal{A} be a Boolean algebra. $\Sigma(\mathcal{A})$ is a measure algebra if and only if it possesses property (CC) .*

Proof. Measure algebras possess the (CC) property. Conversely, if $\{\sigma_\alpha : \alpha \in \mathfrak{A}\}$ is a pairwise quasi disjoint family then $\{\bar{\sigma} : \alpha \in \mathfrak{A}\}$ is disjoint in $\Sigma(\mathcal{A})$; moreover $\sup \bar{m}(\bar{\sigma}_\alpha) > 0$, the supremum being over all measures on \mathcal{A} , is equivalent to $\bar{\sigma}_\alpha \neq 0$ in $\Sigma(\mathcal{A})$. If $\Sigma(\mathcal{A})$ satisfies the (CC) property, \mathfrak{A} must be countable so that \mathcal{A} has property (\mathbf{D}) and the family of all measures on \mathcal{A} is dominated by some μ_0 , by Theorem 2. Then, $\bar{\mu}_0(\bar{\sigma}) = 0$ if and only if $\bar{m}(\bar{\sigma}) = 0$ for all measures m on \mathcal{A} , i.e. if $\bar{\sigma} = 0$. \square

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UNIVERSITÀ MILANO BICOCCA

Email address: gianluca.cassese@unimib.it

Current address: Department of Economics, Statistics and Management Building U7, Room 2097, via Bicocca degli Arcimboldi 8, 20126 Milano - Italy