

# ON THE XIAO CONJECTURE FOR PLANE CURVES

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**ABSTRACT.** Let  $f : S \rightarrow B$  be a non-trivial fibration from a complex projective smooth surface  $S$  to a smooth curve  $B$  of genus  $b$ . Let  $c_f$  the Clifford index of the general fibre  $F$  of  $f$ . In [BGN16] it is proved that the relative irregularity of  $f$ ,  $q_f = h^{1,0}(S) - b$  is less or equal than or equal to  $g(F) - c_f$ . In particular this proves the (modified) Xiao's conjecture:  $q_f \leq \frac{g(F)}{2} + 1$  for fibrations of general Clifford index. In this short note we assume that the general fiber of  $f$  is a plane curve of degree  $d \geq 5$  and we prove that  $q_f \leq g(F) - c_f - 1$ . In particular we obtain the conjecture for families of quintic plane curves. This theorem is implied for the following result on infinitesimal deformations: let  $F$  a smooth plane curve of degree  $d \geq 5$  and let  $\xi$  be an infinitesimal deformation of  $F$  preserving the planarity of the curve. Then the rank of the cup-product map  $H^0(F, \omega_F) \xrightarrow{\cdot \xi} H^1(F, \mathcal{O}_F)$  is at least  $d - 3$ . We also show that this bound is sharp.

## 1. INTRODUCTION

Let  $f : S \rightarrow B$  be a non-isotrivial fibration from a complex projective smooth surface  $S$  to a smooth curve  $B$  of genus  $b$ . A natural question is trying to understand the relation between the invariants of the surface, the base curve  $B$  and of the general fibre  $F$ . Non-isotrivial means that the smooth fibres are not isomorphic to each other, in other words: the natural modular map  $B^0 \rightarrow \mathcal{M}_g$  in the moduli space of curves of genus  $g = g(F)$  is not constant, where  $B^0$  is the open set of  $B$  with smooth fibres. The invariants we consider are the relative irregularity  $0 \leq q_f := h^{1,0}(S) - g(B)$  and the genus  $g$  of the general fibre. Xiao proved in [X87] that for non-isotrivial fibrations the inequality

$$q_f \leq \frac{5g + 1}{6}$$

holds, and he formulated in [X88] the conjecture

$$q_f \leq \frac{g + 1}{2}.$$

After some counterexamples found by Pirola in [P92] (see also [AP16]) the conjecture is nowadays reformulated as follows

**Modified Xiao's conjecture:** For non-isotrivial fibrations it holds that

$$q_f \leq \frac{g}{2} + 1$$

Observe that it is equivalent to the initial conjecture for  $g$  odd.

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There are some evidences for this conjecture: Xiao proved that it holds if  $b = 0$  and it is known to be true when  $F$  is hyperelliptic (see [C95]). Moreover in [BGN16] a upper bound of  $q_f$  is found in terms of the Clifford index  $c_f$  of the general fibre. Remember that the Clifford index of the curve  $F$  is defined as:

$$\text{Cliff}(F) := \min\{\deg(L) - 2(h^0(F, \mathcal{O}_F(L)) - 1) \mid h^0(F, \mathcal{O}_F(L)) \geq 2, h^1(F, \mathcal{O}_F(L)) \geq 2\}.$$

Clifford's Theorem states that  $\text{Cliff}(F) \geq 0$  and it is 0 if and only if  $F$  is hyperelliptic. It is known that  $\text{Cliff}(F) = 1$  if and only if either  $F$  is trigonal or isomorphic to a smooth quintic plane curve. A smooth plane curve of degree  $d$  has Clifford index  $d - 4$  and a general curve in  $\mathcal{M}_g$  has Clifford index  $\lfloor \frac{g-1}{2} \rfloor$ .

The main theorem in [BGN16] says that for a non-isotrivial fibration it holds

$$q_f \leq g - c_f.$$

In particular the conjecture is true when the general fiber has general Clifford index. Combining the results given above one easily checks that the conjecture is true for  $g \leq 4$  and that the first open case corresponds to  $g \geq 5$  and  $c_f = 1$ . In this paper we take care of the quintic plane curve case. More generally we consider families of smooth plane curves of degree  $d \geq 5$ . Our main theorem is the following:

**Theorem 1.1.** *Let  $f : S \rightarrow B$  be a fibration such that the general fibre is a plane curve of degree  $d \geq 5$ . Then*

$$q_f \leq g - c_f - 1 = g - (d - 4) - 1 = g - d + 3$$

In the case  $d = 5$ , hence  $g = 6$ , we obtain  $q_f \leq 4$  which is the predicted bound. Hence we obtain:

**Corollary 1.2.** *The modified Xiao's conjecture holds for fibrations with general fibre a quintic plane curve.*

The idea of the proof of (1.1) is as follows: let us fix a general point of  $B$  and let  $F$  the fibre at this point, then  $f$  induces an infinitesimal deformation  $\xi \in H^1(F, T_F)$ . The kernel  $W_\xi$  of the cup-product map -

$$H^0(F, \omega_F) \xrightarrow{\cdot \xi} H^1(F, \mathcal{O}_F)$$

contains the vector space  $H^0(S, \Omega_S^1)/f^*H^0(B, \omega_B)$  (see [BGN16, Section 2] for the details). Therefore  $q_f \leq \dim W_\xi = g - \text{rank}(\cdot \xi)$ . Thus it is enough to find a lower bound of the rank of the map given by the cup-product with  $\xi$ . Then the next Theorem immediately implies Theorem (1.1):

**Theorem 1.3.** *Let  $\xi$  be an infinitesimal deformation of  $F$  as smooth plane curve, then the rank of the map  $H^0(F, \omega_F) \xrightarrow{\cdot \xi} H^1(F, \mathcal{O}_F)$  is at least  $d - 3$ , and this bound is realized for the Fermat curve.*

The rest of the paper is devoted to the proof of Theorem (1.3). In the next section we recall how to use the Jacobian ring of a plane curve in order to understand the cohomology of the curve  $F$  and the cup-product maps. We also state two theorems of Green on multiplication and restriction maps of polynomials. In section 3 we use these facts combined with the classical Macaulay's Theorem to prove Theorem (1.3).

## 2. PRELIMINARIES

Let  $F$  be a smooth curve of genus  $g \geq 3$  and let  $\xi \in H^1(F, T_F)$  be a non-trivial infinitesimal deformation of  $F$ . To simplify notations we call *rank* of  $\xi$  to the rank of the cup-product map

$$H^0(F, \omega_F) \xrightarrow{\cdot \xi} H^1(F, \mathcal{O}_F)$$

considered in the Introduction. We keep the notation  $W_\xi$  for the kernel of this map.

Let  $F$  be a smooth plane curve defined as the zero locus of the homogeneous polynomial  $f \in \mathbb{C}[x, y, z] = S$  of degree  $d$ . Its *Jacobian ring*  $R$  is the graded ring

$$R = \bigoplus_{n \geq 0} R^n = \bigoplus_{n \geq 0} (S^n / J^n).$$

Here  $J^n$  is the degree  $n$  part of the *Jacobian ideal*  $J = (f_x, f_y, f_z)$ . As  $F$  is smooth,  $\{f_x, f_y, f_z\}$  is a regular sequence and this implies that  $R$  satisfies the following properties:

**Theorem 2.1** (Macaulay). *Let  $N$  be  $3(d-2)$ . Then  $R^N$  has dimension 1 and, for every  $k$  such that  $0 \leq k \leq N$  we have that the multiplication map*

$$R^k \otimes R^{N-k} \rightarrow R^N$$

*is a perfect pairing. Moreover  $R^k = 0$  for  $k > N$  or  $k < 0$  and the dimension of  $R^k$  for  $0 \leq k \leq N$  is determined only by  $d$ .*

In addition to these, Griffiths proved that one can read canonically several pieces of the Hodge structure of  $F$  in  $R$ . More precisely

**Theorem 2.2.** *Let  $R$  be the Jacobian ring of a smooth plane curve of degree  $d$ . Then*

- $H^0(F, \omega_F) \simeq S^{d-3} = R^{d-3}$ ;
- $H^1(F, \mathcal{O}_F) \simeq R^{N-d+3} = R^{2d-3}$ ;
- the subspace of  $H^1(F, T_F)$  of all the infinitesimal deformations that preserve the planarity of  $F$  is isomorphic to  $R^d$ ;
- multiplication in  $R$  induces, using the previous identifications, cup product of the corresponding elements.

We refer to [V03] for a proof of these facts.

Next we quote two theorems of Green concerning properties of  $S = \mathbb{C}[x, y, z]$ . We stress that they are valid in any dimension although we state (and use) them only in dimension  $n = 2$ . The first theorem can be found in [G94, Lecture 7, page 74], putting  $p = 0$  :

**Theorem 2.3** (Green). *Let  $W$  be a subspace of  $S^a$  of codimension  $c$  and assume that  $|W|$  is a base point free linear system on  $\mathbb{P}^2$ . Then, for any  $m \geq c$  one has that the multiplication map*

$$W \otimes S^m \rightarrow S^{m+a}$$

*is surjective.*

In order to state the second theorem of Green we need some notation. Given a positive integer  $a$ , for any  $c \geq 0$  there is a unique expression

$$c = \binom{k_a}{a} + \binom{k_{a-1}}{a-1} + \cdots + \binom{k_1}{1} = \binom{k_a}{a} + \binom{k_{a-1}}{a-1} + \cdots + \binom{k_\delta}{\delta},$$

such that  $k_a > k_{a-1} > \cdots > k_\delta \geq \delta > 0$ . The numbers  $(k_a, k_{a-1}, \dots, k_\delta)$  are uniquely identified by this definition and are called *Macaulay's Coefficients of  $c$  with respect to  $a$* .

If  $(k_a, k_{a-1}, \dots, k_\delta)$  are the Macaulay coefficients with respect to  $a$ , we denote by  $c_{\langle a \rangle}$  the number

$$c_{\langle a \rangle} = \binom{k_a - 1}{a} + \binom{k_{a-1} - 1}{a - 1} + \dots + \binom{k_\delta - 1}{\delta},$$

where  $\binom{m}{n} = 0$  if  $m < n$ . Keeping this terminology we can state the following theorem:

**Theorem 2.4** (Green, [G89]). *Let  $W \subset S^a$  be a linear system with codimension  $c$ . Let  $H$  be a general line in  $\mathbb{P}^2$ . Then the codimension  $c_H$  of the image of the restriction map*

$$W \longrightarrow H^0(H, \mathcal{O}_H(a))$$

*satisfies  $c_H \leq c_{\langle a \rangle}$ .*

**Corollary 2.5.** *Under the assumption of Theorem 2.4, if  $c < a$  then  $c_H = 0$ , i.e. the restriction map  $W \rightarrow H^0(H, \mathcal{O}_H(a))$  is surjective.*

*Proof.* If  $c \leq a$  we can write  $c = a - r$  for some  $r$  such that  $0 \leq r \leq a$ . As

$$c = a - r = \binom{a}{a} + \binom{a-1}{a-1} + \dots + \binom{r+1}{r+1}$$

we have that  $(a, a-1, \dots, r+1)$  are the Macaulay's coefficients of  $c$  with respect to  $a$ . Therefore

$$c_{\langle a \rangle} = \binom{a-1}{a} + \binom{a-2}{a-1} + \dots + \binom{r}{r+1} = 0.$$

By definition of  $c_H$  we have that the restriction is surjective as claimed.  $\square$

### 3. PROOF OF THE THEOREM (1.3)

The Theorem (1.3) in the introduction asserts that given a smooth curve  $F$  of degree  $d$ , the rank of a non trivial infinitesimal deformation of  $F$  which preserves the planarity of  $F$  is bounded below by  $d - 3$ . By using the identifications provided by Griffiths' results in (2.2) this translates into the following statement:

**Theorem 3.1.** *Let  $F$  be a smooth plane curve of degree  $d \geq 5$  and let  $R$  be its Jacobian ring. Let  $\xi \in R^d \setminus \{0\}$ , then the rank of the map*

$$S^{d-3} = R^{d-3} \xrightarrow{\cdot \xi} R^{2d-3}$$

*is at least  $d - 3$ .*

*Proof.* Denote by  $W = W_\xi \subset S^{d-3}$  the kernel of the multiplication map

$$\cdot \xi : S^{d-3} \rightarrow R^{2d-3}.$$

Note that the codimension of  $W$  in  $S^{d-3}$  is equal to the rank of  $\xi$ . The proof is divided in two cases depending of the existence or not of a fixed loci.

#### Case 1: $|W|$ is base-point-free.

We are going to see that  $\text{rk}(\xi) \geq d - 2$ . Assume that the opposite is true, i.e. that  $\text{rk}(\xi) \leq d - 3$  holds. Then, by Green's Theorem, for every  $m \geq \text{cod}_{S^{d-3}}(W) = \text{rk}(\xi)$  we have that the multiplication map

$$\mu_m : W \otimes S^m \rightarrow S^{m+d-3}$$

is surjective. In particular, as  $\text{rk}(\xi) \leq d - 3$  we can take  $m = d - 3$ . Hence we have

$$\mu_{d-3} : W \otimes S^{d-3} \rightarrow S^{2d-6}$$

is surjective and the same holds for the map obtained by passing to the quotient, i.e. to the Jacobian ring:

$$\mu_{d-3} : W \otimes R^{d-3} \rightarrow R^{2d-6}.$$

But this is impossible since by the definition of  $W$ , the image of  $\mu_{d-3}$  is killed by  $\xi \neq 0$ , hence the pairing

$$R^{2d-6} \otimes R^d \rightarrow R^{3d-6} = R^N$$

is degenerated contradicting Macaulay's Theorem. Hence we have necessarily  $\text{rk}(\xi) \geq d-2$  as claimed.

**Case 2:  $|W|$  is not base-point-free.**

First we observe that we can assume that there are no base components. Indeed, assume that there exists a curve  $C$  of degree  $0 < d' < d-3$  in the fixed part of  $|W|$ . Then we have that  $W \subset C \cdot S^{d-d'-3}$  and therefore  $\dim W \leq \dim S^{d-d'-3} \leq \dim S^{d-4}$ , hence

$$\text{codim } W \geq \binom{d-1}{2} - \binom{d-2}{2} = d-2,$$

as wanted.

Hence now on we assume that  $|W|$  has only isolated base points. We proceed by contradiction, so assume that  $\text{rk}(\xi) \leq d-4$  holds.

Let  $Z$  be the base locus of  $|W|$  and denote by  $\mathcal{I}_Z$  the ideal sheaf of  $Z$  as subscheme of  $\mathbb{P}^2$ . Then the evaluation induces a surjection

$$W \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{I}_Z(d-3).$$

Denoting by  $M_W$  its kernel we have the short exact sequence

$$(1) \quad 0 \longrightarrow M_W \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{ev} \mathcal{I}_Z(d-3) \longrightarrow 0.$$

Let  $s$  be a general element in  $S^1 = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ . As the base points are isolated, we can assume that the line  $L = \{s = 0\}$  is disjoint with  $Z$ . By considering the multiplication by  $s$ , the short exact sequence (1) induces the following commutative diagram with exact rows and columns

$$(2) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_W & \longrightarrow & W \otimes \mathcal{O}_{\mathbb{P}^2} & \xrightarrow{ev} & \mathcal{I}_Z(d-3) \longrightarrow 0 \\ & & \downarrow \cdot s & & \downarrow \cdot s & & \downarrow \cdot s \\ 0 & \longrightarrow & M_W(1) & \longrightarrow & W \otimes \mathcal{O}_{\mathbb{P}^2}(1) & \xrightarrow{ev_1} & \mathcal{I}_Z(d-2) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_L & \longrightarrow & W \otimes \mathcal{O}_L(1) & \xrightarrow{ev_L} & \mathcal{O}_L(d-2) \longrightarrow 0. \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

The sheaf  $M_L$  in the third row is by the definition the kernel of  $ev_L$ . Notice that the rightmost sheaf of the last row is  $\mathcal{O}_L(d-2)$  because we are assuming that  $L$  is disjoint from  $Z$ .

**Claim:** The vanishing  $H^1(L, M_L) = 0$  holds.

*Proof.* (of the claim). Indeed, under our hypothesis, Corollary 2.5 implies that the restriction map

$$W \rightarrow H^0(L, \mathcal{O}_L(d-3))$$

is surjective. Tensoring with  $H^0(L, \mathcal{O}_L(1))$  we get that also the evaluation map

$$ev_l : W \otimes H^0(L, \mathcal{O}_L(1)) \longrightarrow H^0(L, \mathcal{O}_L(d-3)) \otimes H^0(L, \mathcal{O}_L(1)) \twoheadrightarrow H^0(L, \mathcal{O}_L(d-2))$$

is surjective. Then the cohomology sequence of the third row in the diagram gives the vanishing.  $\square$

By taking cohomology in the second row we have a map:

$$W \otimes S^1 \xrightarrow{\eta_1} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2)) \longrightarrow H^1(\mathbb{P}^2, M_W(1)) \longrightarrow 0$$

and denote with  $W_1$  the image of  $\eta_1$ . Our strategy is to replace  $W$  by  $W_1 \subset S^{d-2}$  and apply again the same argument in order to finally reach a subspace of  $W_2 \subset S^{d-1}$  where it is easier to finish the proof as we will see below. Hence we need to show that the codimension of  $W_1$  in  $S^{d-2}$  is lower than or equal to the codimension of  $W$  in  $S^{d-3}$ . To prove this we consider the cohomology exact sequences of the first two rows of the diagram (2):

$$\begin{array}{ccccccc} W & \hookrightarrow & H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3)) & \longrightarrow & H^1(\mathbb{P}^2, M_W) & \longrightarrow & 0 \\ \downarrow \cdot s & & \downarrow \cdot s & & \downarrow \cdot s & & \\ W \otimes S^1 & \xrightarrow{\eta_1} & H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2)) & \longrightarrow & H^1(\mathbb{P}^2, M_W(1)) & \longrightarrow & 0 \end{array}$$

The last vertical arrow is surjective due to the claim. Therefore

$$(3) \quad \text{codim}_{H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2))} W_1 = h^1(\mathbb{P}^2, M_W(1)) \leq h^1(\mathbb{P}^2, M_W) = \text{codim}_{H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3))} W.$$

Now we need to compare  $\text{codim}_{S^{d-2}} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2))$  with  $\text{cod}_{S^{d-3}} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3))$ . As we will observe in a moment, they coincide as a consequence of the vanishing of  $H^1(L, M_L) = 0$ .

**Claim:** We have the equality:

$$\text{codim}_{S^{d-2}} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2)) = \text{codim}_{S^{d-3}} H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3)).$$

*Proof.* (of the claim). We start again with the diagram (2). The cohomology exact sequence of the last two columns gives

$$\begin{array}{ccc} W & \longrightarrow & H^0(\mathbb{P}^2, \mathcal{I}_Z(d-3)) \\ \downarrow \cdot s & & \downarrow \cdot s \\ W \otimes S^1 & \xrightarrow{\eta_1} & H^0(\mathbb{P}^2, \mathcal{I}_Z(d-2)) \\ \downarrow res_1 & & \downarrow res_2 \\ W \otimes H^0(L, \mathcal{O}_L(1)) & \xrightarrow{\alpha} & H^0(L, \mathcal{O}_L(d-2)) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(\mathbb{P}^2, \mathcal{I}_Z(d-3)). \end{array}$$

Since  $H^1(L, M_L) = 0$  both  $\alpha$  and  $\alpha \circ \text{res}_1$  are surjective. Therefore  $\text{res}_2$  is also surjective. This implies the isomorphism

$$H^1(\mathbb{P}^2, \mathcal{I}_Z(d-3)) \xrightarrow{\cong} H^1(\mathbb{P}^2, \mathcal{I}_Z(d-2)).$$

Now consider the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_Z(d-3) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(d-3) & \longrightarrow & \mathcal{O}_Z(d-3) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & \mathcal{I}_Z(d-2) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(d-2) & \longrightarrow & \mathcal{O}_Z(d-2) \longrightarrow 0. \end{array}$$

Implementing the isomorphism above we obtain the claim.  $\square$

Combining the inequality (3) with the claim we have:

$$\text{cod}(W_1) \leq \text{cod}(W) \leq d-4.$$

Observe that  $|W_1|$  has, by construction, the same base locus of  $|W|$  and, as we have just proven,  $\text{cod}(W_1) \leq d-4$ . Hence, we can apply the same argument using  $W_1$  instead of  $W$  starting with the surjection

$$W_1 \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{I}_Z(d-2).$$

By doing this we obtain  $W_2 \subset S^{d-1}$  which satisfy

$$\text{cod}(W_2) \leq \text{cod}(W_1) \leq \text{cod}(W) \leq d-4.$$

To finish the proof we consider the subspace

$$\tilde{W} := W_2 + \langle f_x, f_y, f_z \rangle = W_2 + J^{d-1}.$$

Observe that in  $\tilde{W}$  there is a section which doesn't vanish on at least one point (we are assuming that  $Z$  is not empty) of the base locus, as the partial derivatives cannot all vanish in a point. Hence the dimension of  $\tilde{W}$  has increased at least by 1 and

$$\text{cod}(\tilde{W}) \leq \text{cod}(W_2) - 1 \leq d-5.$$

Moreover  $\tilde{W}$  is base point free by construction so we can apply Green's Theorem (2.3) again with  $m = d-5$  and we obtain that the multiplication map

$$\tilde{W} \otimes S^{d-5} \rightarrow S^{d-5+d-1} = S^{2d-6}$$

is surjective.

Passing to the quotient by the Jacobian ideal we have a surjective map

$$W_2 \otimes R^{d-5} \rightarrow R^{2d-6}.$$

This implies by the definition of  $W_2$  (image of  $W_1 \otimes S^1$ ) and  $W_1$  (image of  $W \otimes S^1$ ) that all the elements in  $R^{2d-6}$  are orthogonal to  $\xi$  which contradicts Macaulay's Theorem. This finishes the prove of the bound.

Finally we see that the bound is sharp. Fix  $d \geq 4$  and let  $F$  be the Fermat curve of degree  $d$  in  $\mathbb{P}^2$ , i.e. the zero locus of the polynomial  $f = x^d + y^d + z^d$ . In this case, the Jacobian ideal is simply

$$J = (x^{d-1}, y^{d-1}, z^{d-1})$$

and one can easily prove that  $R^N = \langle (xyz)^{d-2} \rangle$ . Consider the element  $[x^{d-2}y^2] \in R^d$  and denote it by  $\xi$ . Observe, moreover, that this is not zero as  $x^{d-2}y^2 \notin J$ . If  $m = x^a y^b z^c$  with

$a + b + c = d - 3$ , we have  $m \cdot \xi \neq 0$  if and only if  $a = 0$  and  $0 \leq b \leq d - 4$ . Hence the image of the multiplication by  $\xi$  is generated by the  $d - 3$  elements of

$$\{[x^{d-2}y^{2+a}z^{d-3-a}] \mid 0 \leq a \leq d - 4\}.$$

As they are independent we have  $\text{rk}(\xi) = d - 3$ . Notice that  $W$  is generated by monomials and  $x^{d-3}, y^{d-3} \in W$  but  $z^{d-3} \notin W$ , hence the base locus of  $|W|$  is  $P_2 = (0 : 0 : 1)$ .  $\square$

**Remark 3.2.** *In Theorem 3.1 we have proven that the bound  $\text{rk}(\cdot\xi) \geq d - 3$  is sharp by showing that for the Fermat curve of degree  $d$  there exists an element in  $R^d$  whose rank is exactly  $d - 3$ . There are other curves that have this property. For example, denote by  $f_\lambda \in S^5$  the polynomial  $x^5 + y^5 + z^5 + 5\lambda x^3y^2$  with  $\lambda \in \mathbb{C}$  and consider the quintic  $F_\lambda = \{f_\lambda = 0\}$ . Notice that  $F_0$  is the Fermat quintic so the general quintic of this type is indeed smooth. Moreover, as  $x^3y^2$  is not zero in the Jacobian ideal of the Fermat quintic, we know that  $F_\lambda$  is not biholomorphic to  $F_0$  for  $\lambda$  general. If we denote by  $R_\lambda$  the Jacobian ideal of  $F_\lambda$  it is easy to see that  $\xi_\lambda = [x^3y^2]$  is not zero in  $R_\lambda$ . Now we want to prove that  $\text{rk}(\xi_\lambda) = 2$  for all  $\lambda \in \mathbb{C}$ . As we have already seen,  $\xi_\lambda$  is non trivial so, by Theorem 1.3 we know that  $\text{rk}(\xi_\lambda) \geq 2$ . In order to see that the equality holds it is enough to prove that  $\dim W_{\xi_\lambda} \geq 4$  (as  $\dim R_\lambda^2 = 6$ ). With a little bit of effort one can show that*

$$x^2, xy, y^2, xz + 18\lambda^3yz \in W_{\xi_\lambda}$$

*thus proving the claim.*

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