

Solvable Lie algebras are not that hypo

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Abstract

We study a type of left-invariant structure on Lie groups, or equivalently on Lie algebras. We introduce obstructions to the existence of a hypo structure, namely the 5-dimensional geometry of hypersurfaces in manifolds with holonomy $SU(3)$. The choice of a splitting $\mathfrak{g}^* = V_1 \oplus V_2$, and the vanishing of certain associated cohomology groups, determine a first obstruction. We also construct necessary conditions for the existence of a hypo structure with a fixed almost-contact form. For non-unimodular Lie algebras, we derive an obstruction to the existence of a hypo structure, with no choice involved. We apply these methods to classify solvable Lie algebras that admit a hypo structure.

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Introduction

In [5], Salamon and the first author of the present work introduced *hypo structures*, namely the $SU(2)$ -structures induced naturally on orientable hypersurfaces of Calabi-Yau manifolds of (real) dimension 6. They are defined as follows. An $SU(2)$ -structure on a five-manifold is an almost-contact metric structure with additionally a reduction from the structure group $SO(4)$ to $SU(2)$; such a structure is entirely determined by the choice of differential forms $(\alpha, \omega_1, \omega_2, \omega_3)$, where α is the almost-contact 1-form and the ω_i are pointwise a distinguished orthonormal basis of $\Lambda_+^2(\ker \alpha)$, which implies that the quadruplet $(\alpha, \omega_1, \omega_2, \omega_3)$ satisfies certain relations (see Section 1, Proposition 3). Since $SU(2)$ is the stabilizer of a point under the action of $SU(3)$ on \mathbb{R}^6 , hypersurfaces in manifolds with holonomy contained in $SU(3)$ or, equivalently, with an integrable $SU(3)$ -structure, inherit a natural $SU(2)$ -structure.

In fact, if M is a Riemannian 6-manifold with holonomy contained in $SU(3)$, then M has a Hermitian structure, with Kähler form F , and a complex volume form $\Psi = \Psi_+ + i\Psi_-$, satisfying $dF = 0 = d\Psi$. Therefore, if $N \subset M$ is an orientable hypersurface, and \mathbb{U} is the unit normal vector field, the $SU(3)$ -structure induces an $SU(2)$ -structure $(\alpha, \omega_1, \omega_2, \omega_3)$ on N defined by

$$\alpha = -\mathbb{U} \lrcorner F, \quad \omega_1 = f^* F, \quad \omega_2 = \mathbb{U} \lrcorner \Psi_-, \quad \omega_3 = -\mathbb{U} \lrcorner \Psi_+,$$

where $f: N \rightarrow M$ is the inclusion.

The integrability condition on the ambient manifold immediately gives

$$d\omega_1 = 0, \quad d(\omega_2 \wedge \alpha) = 0, \quad d(\omega_3 \wedge \alpha) = 0.$$

An $SU(2)$ -structure satisfying this condition is called a *hypo* structure. Such a structure can also be characterized in terms of generalized Killing spinors, or by the condition that the intrinsic torsion is a symmetric tensor, which turns out to coincide with the second fundamental form of the hypersurface. In this sense, hypo geometry is the five-dimensional analogue of half-flat geometry in dimension six (see [3, 10]). Indeed, much as in the half-flat case, any real-analytic hypo manifold can be immersed isometrically in a Riemannian manifold with holonomy contained in $SU(3)$, so as to invert the construction outlined above, and the immersion can be determined explicitly by solving certain evolution equations ([5]).

In order to construct examples of hypo structures, a natural place to look is left-invariant structures on 5-dimensional Lie groups. In the analogous half-flat case, this was the approach of [4, 2, 6, 7], focusing on the nilpotent case, and more recently of [12], considering products of three-dimensional Lie groups. In five-dimensions, only 9 isomorphism classes of nilpotent Lie groups exist, of which exactly six admit a hypo structure [5]. If one considers solvable Lie groups, things become more complicated. By Mubarakzyanov's classification [11], there are 66 families of solvable Lie algebras of dimension 5, some of which depend on parameters; we refer to the comprehensive list of [1]. It was shown in [9] that precisely 35 out of these 66 families admit an invariant contact structure, at least generically (i.e. for generic values of the parameters). Moreover, without using Mubarakzyanov's classification, it was proved in [8] that only 5 of the 66 admit a hypo-contact structure, namely a hypo structure (α, ω_i) such that the underlying almost-contact metric structure is a contact metric structure.

In this paper we introduce some obstructions to the existence of a hypo structure on a Lie algebra, and use them to classify solvable Lie algebras with a hypo structure. The first obstruction follows a construction of [7]. One considers a splitting $\mathfrak{g}^* = V_1 \oplus V_2$, where V_1 has dimension two. This determines a doubly graded vector space $\Lambda^* \mathfrak{g}^* = \bigoplus \Lambda^{p,q}$, which is made into a double complex if

$$d(\Lambda^{p,q}) \subset \Lambda^{p+2,q-1} \oplus \Lambda^{p+1,q}. \quad (1)$$

The double complex has an associated spectral sequence that collapses at the second step. If $H^{0,3} = E_2^{0,3}$ and $H^{0,2} = E_2^{0,2}$ are zero, relative to some choice of the splitting, then no hypo structure exists (see Proposition 3). In fact, the key property is

$$Z^k \subset \Lambda^{2,k-2} \oplus \Lambda^{1,k-1}, \quad k = 2, 3,$$

where Z^k denotes the space of closed k -forms; this condition does not require (1), whose main relevance is in giving a cohomological interpretation. This obstruction applies to 27 indecomposable Lie algebras and 10 decomposable Lie

algebras, at least generically, where decomposable means isomorphic to a direct sum of ideals.

A second set of obstructions comes from the fact that if (α, ω_i) is a hypo structure on a Lie algebra \mathfrak{g} , then the forms $\omega_2 \wedge \alpha$, $\omega_3 \wedge \alpha$ lie in the space

$$V = \{\gamma \in \Lambda^3 \mathfrak{g}^* \mid \gamma \wedge \alpha = 0, d\gamma = 0\}.$$

If, for some $\beta \in \mathfrak{g}^*$, either the space $V \wedge \beta \subset \Lambda^4 \mathfrak{g}^*$ has dimension one or

$$\dim(V \wedge \beta) = 2, \quad Z^2 \wedge \alpha \wedge \beta \subset V \wedge \beta, \quad (2)$$

then necessarily α and β are linearly dependent. This is an obstruction to the existence of a hypo structure with a fixed α (see Proposition 4), but it can be combined with other arguments to prove that no hypo structure exists on a Lie algebra.

Indeed, we show that if a non-unimodular Lie algebra \mathfrak{g} has a hypo structure (α, ω_i) , then the 1-form $\beta \in \mathfrak{g}^*$ defined by $\beta(X) = \text{tr ad}(X)$ is orthogonal to α ; this gives a canonical choice for β in (2). Explicitly, in Proposition 6, we prove that there is no hypo structure if either $Z^3 \wedge \beta$ has dimension less than two, or

$$\dim(Z^3 \wedge \beta) = 2, \text{ and } \alpha \wedge \beta \wedge Z^2 \subset Z^3 \wedge \beta$$

for any $\alpha \in \mathfrak{g}^*$ such that $\alpha \wedge \beta \wedge Z^3 = 0$. This obstruction applies to 6 indecomposable families and 12 decomposable families.

On the other hand, even for unimodular Lie algebras, the structure of the space of closed 3-forms may give restrictions on α (see Lemma 10), which together with (2) enable one to show that certain Lie algebras have no structure. This obstruction accounts for 6 indecomposable families.

Finally, for 2 indecomposable families and one decomposable Lie algebra, we use the trivial fact that the space $(Z^2)^2 \wedge \alpha$ is non-zero, as it contains $(\omega_1)^2 \wedge \alpha \neq 0$.

Having obtained the classification, we can ask how often a solvable Lie algebra is hypo. We know from [5] that the answer is 6 times out of 9 for nilpotent Lie algebras. In fact, we obtain a shorter proof of this result, namely that the nilpotent Lie algebras denoted here by D_3 , $A_{5,3}$, $A_{4,1} \oplus \mathbb{R}$ have no hypo structure.

In the solvable case, the question is somewhat ambiguous, because the Lie algebras come in families. With reference to Mubarakzyanov's list, it turns out that, given a family with more than one element, the subset of Lie algebras that have a hypo structure is always a proper subset, but not always discrete. This suggests recasting the question in the following form: how many families in Mubarakzyanov's list of solvable Lie algebras contain at least one hypo Lie algebra? The answer is 21 out of 66, so the ratio is considerably less than in the nilpotent case.

If we further distinguish according to whether a family of Lie algebras is decomposable and whether it is generically contact, we obtain the following

table:

	generically contact	non-contact	all
indecomposable	7/24	9/15	16/39
decomposable	1/11	4/16	5/27
all	8/35	13/31	21/66

For instance, the top-left entry states that of the 24 families in Mubarakzyanov's list which are indecomposable and have a contact structure for generic choice of the parameters, precisely 7 have a hypo structure for some choice of the parameters.

If we count Lie algebras with a hypo structure rather than families, we obtain the following table:

	generically contact	non-contact
indecomposable	9	infinite
decomposable	1	4

Thus, there are exactly ten solvable Lie algebras that have both a hypo and a contact structure, for half of which the structures can be chosen to be compatible [8].

Finally, we point out that there are only five non-unimodular hypo Lie algebras, contained in three families, all of them indecomposable and contact.

1 A first obstruction

In this section we introduce an obstruction to the existence of a hypo structure on a 5-dimensional Lie algebra. This obstruction is given in terms of the cohomology groups of a certain double complex associated to any n -dimensional Lie algebra.

Let \mathfrak{g} be an n -dimensional Lie algebra, and denote by d the Chevalley-Eilenberg differential on the dual \mathfrak{g}^* . A *coherent splitting* of \mathfrak{g} is a splitting $\mathfrak{g}^* = V_1 \oplus V_2$, where V_1 and V_2 are vector spaces, $\dim V_1 = r \geq 2$ and

$$d(V_1) \subset \Lambda^2 V_1, \quad d(V_2) \subset \Lambda^2 V_1 + V_1 \wedge V_2.$$

Let $\Lambda^{p,q}$ be the natural image of $\Lambda^p V_1 \otimes \Lambda^q V_2$ in $\Lambda^{p+q} = \Lambda^{p+q} \mathfrak{g}^*$, with the convention that $\Lambda^{p,q} = 0$ whenever p or q is negative. A coherent splitting determines a double complex $(\Lambda^{*,*}, \delta_1, \delta_2)$, δ_1, δ_2 being the operators:

$$\delta_1 : \Lambda^{p,q} \longrightarrow \Lambda^{p+1,q}, \quad \delta_2 : \Lambda^{p,q} \longrightarrow \Lambda^{p+2,q-1}, \quad d = \delta_1 + \delta_2.$$

They satisfy

$$\delta_1^2 = 0 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1.$$

For any choice of coherent splitting on \mathfrak{g} , we can define the cohomology groups $H^{p,q}(\mathfrak{g}, V_1)$ as follows (see also [7]). For each $k \geq 0$ we define a filtration

$$\begin{aligned} \Lambda^{r,k-r} \subset \Lambda^{r,k-r} + \Lambda^{r-1,k-r+1} \subset \Lambda^{r,k-r} + \Lambda^{r-1,k-r+1} + \Lambda^{r-2,k-r+2} \subset \\ \dots \subset \Lambda^{r,k-r} + \Lambda^{r-1,k-r+1} + \dots + \Lambda^{0,k} = \Lambda^k. \end{aligned} \quad (3)$$

Notice that in (3), the space $\Lambda^{p,k-p}$ is zero if $p > k$ or $p > r$. We denote by $Z^k \subset \Lambda^k$ the space of closed invariant k -forms. Taking the intersection with Z^k , the filtration (3) determines the filtration

$$Z_r^k \subset Z_{r-1}^k \subset Z_{r-2}^k \subset \cdots \subset Z_0^k = Z^k,$$

and taking the quotient by the d -exact forms, we obtain yet another filtration

$$H_r^k \subset H_{r-1}^k \subset H_{r-2}^k \subset \cdots \subset H_0^k = H^k.$$

We can now define the cohomology groups

$$H^{p,q}(\mathfrak{g}, V_1) = \frac{H_p^{p+q}}{H_{p+1}^{p+q}}.$$

The notation is justified by the fact that whilst the spaces $\Lambda^{p,q}$ depend on both V_1 and V_2 , the filtration (3), and therefore the cohomology groups, depend only on V_1 . We define

$$h^{p,q}(\mathfrak{g}, V_1) = \dim H^{p,q}(\mathfrak{g}, V_1).$$

We can think of a coherent splitting as defined by a decomposable form which spans $\Lambda^r V_1$.

Lemma 1. *Let \mathfrak{g} be a Lie algebra of dimension n , and let ϕ be a decomposable r -form. Then ϕ defines a coherent splitting $\mathfrak{g}^* = V_1 \oplus V_2$, with $\dim V_1 = r$, if and only if*

- $d\alpha \wedge \phi = 0$ for all $\alpha \in \mathfrak{g}^*$;
- $d\phi = 0$;
- $\mathcal{L}_X \phi$ is a multiple of ϕ for all X in \mathfrak{g} , where \mathcal{L} denotes the Lie derivative.

Proof. Given a coherent splitting with $\Lambda^{r,0}$ generated by ϕ , we have

$$d\phi \in \Lambda^{r+1,0} = \{0\} \text{ and } d\alpha \wedge \phi \in \Lambda^{r+2,0} + \Lambda^{r+1,1} = \{0\}, \text{ for } \alpha \in \Lambda^1.$$

Also, since ϕ is closed, $\mathcal{L}_X \phi = d(X \lrcorner \phi) \in \Lambda^{r,0}$ (where $X \lrcorner \cdot$ denotes the contraction by X) which is spanned by ϕ .

To prove the converse, let $\phi = \alpha^1 \wedge \cdots \wedge \alpha^r$, and complete $\alpha^1, \dots, \alpha^r$ to a basis $\alpha^1, \dots, \alpha^r, \beta^1, \dots, \beta^{n-r}$. The first condition implies that the image of $d: \Lambda^1 \rightarrow \Lambda^2$ is contained in $\Lambda^{2,0} + \Lambda^{1,1}$. All we need to check in order to have a coherent splitting is that $d\alpha^j$ has type $(2,0)$. Suppose otherwise. Then

$$(d\alpha^i)^{1,1} = a_{jh}^i \beta_h \wedge \alpha_j.$$

Now since ϕ is closed, $d(X \lrcorner \phi) = \mathcal{L}_X \phi$ is a multiple of ϕ by hypothesis. So we have that

$$\begin{aligned} 0 &= (d(\alpha^1 \wedge \cdots \wedge \hat{\alpha}^i \wedge \cdots \wedge \alpha^r))^{r-1,1} = \sum_{j \neq i, h} (-1)^{j+i-1} a_{jh}^j \beta_h \wedge \alpha^1 \wedge \cdots \wedge \hat{\alpha}^j \wedge \cdots \wedge \alpha^r \\ &\quad + \sum_{j \neq i, h} a_{jh}^j \beta_h \wedge \alpha^1 \wedge \cdots \wedge \hat{\alpha}^i \wedge \cdots \wedge \alpha^r. \end{aligned}$$

Hence $a_{ih}^j = 0$ for all $i \neq j$ and h , and $\sum_{j \neq i} a_{jh}^j = 0$ for all i, h . Therefore $a_{jh}^i = 0$ for all i, j, h . \square

We introduce the following notation. Let D_j be the annihilator of the kernel of $d: \Lambda^j \mathfrak{g}^* \rightarrow \Lambda^{j+1} \mathfrak{g}^*$. In other words, if v^1, \dots, v^n is a basis of the vector space $(\Lambda^{j+1})^*$ dual to Λ^{j+1} , then D_j is spanned by the $v^j \circ d$. Likewise, for any $\phi \in \Lambda^k \mathfrak{g}^*$, let L_j^ϕ be the annihilator of the kernel of the map

$$\Lambda^j \mathfrak{g}^* \rightarrow \Lambda^{j+k} \mathfrak{g}^*, \quad \alpha \rightarrow \alpha \wedge \phi.$$

We can then give a specialized version of the lemma that accounts for the vanishing of certain cohomology groups. In the five-dimensional case we get:

Proposition 2. *Let \mathfrak{g} be a 5-dimensional Lie algebra. Then \mathfrak{g} has a coherent splitting, with $\dim V_1 = 2$ and $H^{0,2} = 0 = H^{0,3}$, if and only if there exists a nonzero 2-form ϕ such that*

- $\phi \wedge \phi = 0$;
- $d\phi = 0$;
- $\mathcal{L}_X \phi$ is a multiple of ϕ for all X in \mathfrak{g} ;
- $L_2^\phi \subset D_2, L_3^\phi \subset D_3$.

Proof. Given a coherent splitting, it is clear that exact k -forms have no component in $\Lambda^{0,k}$. Moreover, the condition $H^{0,k} = 0$ is equivalent to Z^k being contained in $\Lambda^{k,0} + \dots + \Lambda^{1,k-1}$. Thus, $L_k^\phi \subset D_k$ if and only if $H^{0,k} = 0$.

Conversely, a 2-form ϕ such that $\phi \wedge \phi = 0$ is decomposable, and therefore determines a splitting. If ϕ is as in the hypothesis, the splitting is coherent because $L_2^\phi \subset D_2$ implies that closed 2-forms, and in particular exact 2-forms, have no component in $\Lambda^{0,2}$; therefore, $d\alpha \wedge \phi = 0$ for all $\alpha \in \mathfrak{g}^*$ and Lemma 1 applies. \square

Remark. In the proof of Proposition 2, we can suppose that \mathfrak{g} has a coherent splitting, with $\dim V_1 = r \geq 2$, and conclude that $H^{0,k} = 0$ is equivalent to $L_k^\phi \subset D_k$, since this works for any dimension n of \mathfrak{g} and for all values of r . However, we need $r = 2$ to have that a 2-form ϕ is decomposable if and only if $\phi \wedge \phi = 0$.

From now on, given a 5-dimensional Lie algebra \mathfrak{g} whose dual is spanned by $\{e^1, \dots, e^5\}$, we will write $e^{ij} = e^i \wedge e^j$, $e^{ijk} = e^i \wedge e^j \wedge e^k$, and so forth.

The relevance of the above proposition comes from *hypo geometry*. First we recall some facts about $SU(2)$ -structures on a 5-manifold. (For more details, we refer to [5]). Let N be a 5-manifold and let $L(N)$ be the principal bundle of linear frames on N . An $SU(2)$ -structure on N is an $SU(2)$ -reduction of $L(N)$. We have the following (see [5, Proposition 1]):

Proposition 3. *SU(2)-structures on a 5-manifold N are in one-to-one correspondence with quadruplets $(\alpha, \omega_1, \omega_2, \omega_3)$, where α is a 1-form and ω_i are 2-forms on N satisfying at each point*

$$\omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \alpha \neq 0,$$

for some 4-form v , and

$$i_X \omega_1 = i_Y \omega_2 \Rightarrow \omega_3(X, Y) \geq 0,$$

where i_X denotes the contraction by X .

Moreover, we need recall the following definition.

Definition 4. An SU(2)-structure $(\alpha, \omega_1, \omega_2, \omega_3)$ on a 5-manifold N is said to be *hypo* if

$$d(\omega_2 \wedge \alpha) = d(\omega_3 \wedge \alpha) = d\omega_1 = 0. \quad (4)$$

Therefore, to a choice of a coframe f^1, \dots, f^5 on a Lie algebra \mathfrak{g} , we associate an SU(2)-structure given by

$$\alpha = f^5, \quad \omega_1 = f^{12} + f^{34}, \quad \omega_2 = f^{13} + f^{42}, \quad \omega_3 = f^{14} + f^{23}, \quad (5)$$

and it is called *hypo* if $\omega_1, \omega_2 \wedge \alpha, \omega_3 \wedge \alpha$ are closed.

Definition 5. Let f^1, \dots, f^5 be a coframe on a Lie algebra \mathfrak{g} such that the quadruplet $(\alpha, \omega_1, \omega_2, \omega_3)$ given by (5) defines a hypo structure on \mathfrak{g} . Then, the coframe f^1, \dots, f^5 is said to be *adapted* to the hypo structure $(\alpha, \omega_1, \omega_2, \omega_3)$.

Proposition 6. *If \mathfrak{g} has dimension 5, and there exists a coherent splitting $\mathfrak{g}^* = V_1 \oplus V_2$ with $\dim V_1 = 2$ and $h^{0,3}(\mathfrak{g}, V_1) = 0 = h^{0,2}(\mathfrak{g}, V_1)$, then there is no hypo structure.*

Proof. Let (α, ω_i) be a hypo structure, and let ϕ be a generator of $\Lambda^{2,0}$. We know that the forms $\omega_1, \alpha \wedge \omega_2, \alpha \wedge \omega_3$ are closed. Moreover, because $h^{0,3}(\mathfrak{g}, V_1) = 0 = h^{0,2}(\mathfrak{g}, V_1)$, we have

$$\phi \wedge \omega_1 = 0, \quad \phi \wedge (\alpha \wedge \omega_i) = 0.$$

If we decompose the space of two-forms on \mathbb{R}^5 according to

$$\Lambda^2 \mathbb{R}^5 = \alpha \wedge \Lambda^1 \mathbb{R}^4 \oplus \Lambda_+^2 \mathbb{R}^4 \oplus \Lambda_-^2 \mathbb{R}^4,$$

we find that ϕ must lie in $\Lambda_-^2 \mathbb{R}^4$. Since $\phi^2 = 0$, this implies $\phi = 0$. \square

Remark. Strictly speaking Proposition 6 does not use the fact that the splitting is coherent, but only the conditions $Z^3 \wedge \phi = 0, Z^2 \wedge \phi = 0$; or, in the language of Proposition 2, the inclusions $L_2^\phi \subset D_2, L_3^\phi \subset D_3$. Indeed it is sometimes the case that a splitting with this property exists without being coherent. Consider for instance the Lie algebra

$$\mathfrak{g} = (e^{13}, e^{34}, -e^{24}, 0, 0)$$

(by this notation we mean that the dual \mathfrak{g}^* has a fixed basis e^1, \dots, e^5 , such that

$$de^1 = e^{13}, \quad de^2 = e^{34}, \quad de^3 = -e^{24}, \quad de^4 = 0 = de^5).$$

Then e^{24} defines a splitting with $Z^3 \wedge \phi = Z^2 \wedge \phi = 0$, and yet this is not coherent. On the other hand, a different obstruction applies to this case (see Proposition 9 below).

2 Other obstructions

When looking at 5-dimensional solvable Lie algebras, the coherent splitting obstruction, shown in Proposition 3, is sometimes not sufficient to determine whether a hypo structure exists. In this section we describe two different obstructions that can be used in these cases.

For every 1-form γ , let $L_\gamma: \Lambda^j \rightarrow \Lambda^{j+1}$ be the map given by $L_\gamma(\eta) = \gamma \wedge \eta$.

Proposition 7. *Let α, β be linearly independent one-forms on a Lie algebra \mathfrak{g} , and set $V = \ker L_\alpha \cap Z^3$. Suppose that either*

- $\dim L_\beta(V) < 2$; or
- $\dim L_\beta(V) = 2$ and

$$L_\alpha(L_\beta(Z^2)) \subset L_\beta(V).$$

Then there is no hypo structure on \mathfrak{g} of the form (α, ω_i) (in the sense that its almost-contact form is α itself).

Proof. Suppose for a contradiction that a hypo structure (α, ω_i) exists, and let f^1, \dots, f^5 be an adapted coframe. Up to rescaling the metric and up to $SU(2)$ action, we can assume that $\beta = f^1 + af^5$, with a a constant. Then $\omega_2 \wedge \alpha, \omega_3 \wedge \alpha$ lie in V and

$$L_\beta(V) \ni L_\beta(\omega_2 \wedge \alpha) = f^{1425}, \quad L_\beta(V) \ni L_\beta(\omega_3 \wedge \alpha) = f^{1235}.$$

So $\dim L_\beta(V) \geq 2$, and if equality holds then $L_\beta(V)$ is spanned by f^{1425}, f^{1235} . But then

$$L_\alpha(L_\beta(Z^2)) \ni \alpha \wedge \beta \wedge \omega_1 = f^{5134},$$

which cannot lie in $L_\beta(V)$. □

For non-unimodular Lie algebras, it turns out that we have a canonical choice for β :

Lemma 8. *Let \mathfrak{g} be a non-unimodular Lie algebra and let $\beta \in \mathfrak{g}^*$ be the form corresponding to the linear map $\mathfrak{g} \rightarrow \mathbb{R}, X \rightarrow \text{tr ad } X$. If \mathfrak{g} has a hypo structure (α, ω_i) , then α and β are orthogonal with respect to the underlying metric.*

Proof. In an adapted coframe f^1, \dots, f^5 , with dual frame f_1, \dots, f_5 , $\alpha = f^5$, we have

$$df^{1234} = \sum_k f^k([f_k, f_5])f^{12345} = -\beta(f_5)f^{12345}.$$

However, since ω_1 is closed, the left-hand side is zero and so $\beta(f_5) = 0$. \square

Thus, in the non-unimodular case Proposition 7 gives a fairly straightforward criterion:

Proposition 9. *Let \mathfrak{g} be a non-unimodular Lie algebra, and let $\beta \in \mathfrak{g}^*$ be the form corresponding to the linear map $\mathfrak{g} \rightarrow \mathbb{R}$, $X \rightarrow \text{tr ad } X$. Suppose that either*

- $\dim L_\beta(Z^3) < 2$; or
- $\dim L_\beta(Z^3) = 2$ and for every $\alpha \in \mathfrak{g}^*$ such that $L_\alpha(L_\beta(Z^3)) = 0$,

$$L_\alpha(L_\beta(Z^2)) \subset L_\beta(Z^3).$$

Then \mathfrak{g} has no hypo structure.

Proof. Suppose \mathfrak{g} has a hypo structure (α, ω_i) . By Lemma 8, we know that α and β are linearly independent. Consider the space $V = \ker L_\alpha \cap Z^3$. If $L_\beta(Z^3)$ has dimension two, then $L_\beta(V) \subseteq L_\beta(Z^3)$ may only have dimension two if equality holds, implying $L_\alpha(L_\beta(Z^3)) = 0$. Then the statement follows from Proposition 7. \square

In order to apply Proposition 7 effectively, one needs information on what the 1-form α can be. The condition of Lemma 8 is often useful but not always sufficient, since in practice it only tells us that α and β are linearly independent; moreover, it does not apply to unimodular Lie algebras (for which β is zero). The following result gives useful restrictions on the 1-form α ; it is labeled a lemma because we view it as an aid toward the application of either Proposition 7 or Proposition 9.

Lemma 10. *Suppose \mathfrak{g} has a hypo structure (α, ω_i) . If $\dim(X \lrcorner Z^3) \wedge \gamma < 2$, where $X \in \mathfrak{g}$ and $\gamma \in \mathfrak{g}^*$, then $\alpha(X) = 0$.*

Proof. Suppose otherwise; fix an adapted coframe f^1, \dots, f^5 with dual frame f_1, \dots, f_5 . Then $X = af_1 + f_5$ up to a multiple and $\text{SU}(2)$ action. Therefore $X \lrcorner Z^3$ contains

$$\begin{aligned} X \lrcorner (f^{135} + f^{425}) &= f^{13} + f^{42} + af^{35}, \\ X \lrcorner (f^{145} + f^{235}) &= f^{14} + f^{23} + af^{45}. \end{aligned}$$

Now by hypothesis some linear combination

$$\delta = \lambda(f^{13} + f^{42} + af^{35}) + \mu(f^{14} + f^{23} + af^{45})$$

gives zero on wedging with γ . But

$$(\lambda(f^{13} + f^{42} + af^{35}) + \mu(f^{14} + f^{23} + af^{45}))^2 = 2(\lambda^2 + \mu^2)f^{1234} + \text{other terms},$$

which is nonzero. By non-degeneracy $\delta \wedge \gamma \neq 0$, which is absurd. \square

Remark. Regardless of Proposition 7, this lemma may have more immediate applications. Indeed, a hypo structure (α, ω_i) always satisfies

$$0 \neq (\omega_1)^2 \wedge \alpha \in (Z^2)^2 \wedge \alpha. \quad (6)$$

3 Diatta's algebras

Let us recall firstly that a *contact form* η on a five-dimensional Lie algebra \mathfrak{g} is a 1-form on \mathfrak{g} (that is, $\eta \in \mathfrak{g}^*$) such that

$$\eta \wedge (d\eta)^2 \neq 0.$$

The existence of a hypo structure on \mathfrak{g} is independent of the existence of a contact form. In fact, in this section we will consider indecomposable solvable Lie algebras of dimension 5 having a contact form η , and we will see that many of those Lie algebras do not admit a hypo structure. Notice that we are not requiring that the almost-contact 1-form α associated to the hypo structure coincide with the contact form η .

In [9], Diatta gives a list of 24 (families of) indecomposable five-dimensional solvable Lie algebras D_1, \dots, D_{24} that admit a left-invariant contact 1-form. They correspond to the algebras $A_{5,k}$ of [1] under

$$D_k \rightarrow \begin{cases} A_{5,k+3} & k = 1, 2, 3 \\ A_{5,k+15} & 4 \leq k \leq 24. \end{cases}$$

We shall use the notation $D_k(p_1, \dots, p_n)$ to denote special instances of a family for assigned values of the parameters. Notice that Diatta's list, as well as the one in [1] from which it was extracted, contains conditions on the parameters. We shall ignore these conditions to keep things simpler. This has two consequences: first, the same Lie algebra may appear more than once, and second, a Lie algebra $D_k(p_1, \dots, p_n)$ may not have a contact structure for some choice of the parameters p_1, \dots, p_n . However, these “degenerate” cases turn out to never have a hypo structure.

Proposition 11. *The indecomposable solvable Lie algebras that have both a*

contact structure and a hypo structure are the following:

$$\begin{aligned}
D_1 &= (e^{24} + e^{35}, 0, 0, 0, 0) \\
D_2 &= (e^{34} + e^{25}, e^{35}, 0, 0, 0) \\
D_4(-1/2, -3/2) &= \left(-\frac{1}{2}e^{15} - e^{23}, -e^{25}, \frac{1}{2}e^{35}, \frac{3}{2}e^{45}, 0\right) \\
D_4(1, -3) &= (-e^{23} - 2e^{15}, -e^{25}, -e^{35}, 3e^{45}, 0) \\
D_4(-2, 3) &= (e^{15} - e^{23}, -e^{25}, 2e^{35}, -3e^{45}, 0) \\
D_{15}(-1) &= (-e^{15} - e^{24}, -e^{34}, e^{35}, -e^{45}, 0) \\
D_{18}(-1, -1) &= (-e^{14}, -e^{25}, e^{34} + e^{35}, 0, 0) \\
D_{20}(-2, 0) &= (2e^{14}, -e^{24} - e^{35}, e^{25} - e^{34}, 0, 0) \\
D_{22} &= (e^{23} + 2e^{14}, e^{24} + e^{35}, e^{34} - e^{25}, 0, 0)
\end{aligned}$$

A Lie algebra $D_k(p_1, \dots, p_n)$ is hypo if and only if it belongs to this list.

Proof. First, we produce a hypo structure for each Lie algebra appearing in the statement.

The Lie algebras $D_4(1, -3)$, $D_{15}(-1)$, $D_{18}(-1, -1)$ and D_{22} appear in [8] and have hypo contact structures given by the coframes

$$\begin{aligned}
e^5, \frac{1}{5}(e^1 - e^4), \frac{1}{2}e^3, \frac{1}{5}e^2, -\frac{1}{5}e^1 - \frac{2}{15}e^4 & \quad D_4(1, -3). \\
\frac{1}{2}(-e^1 + e^3), e^5, \frac{\sqrt{2}}{2}e^4, \frac{\sqrt{2}}{2}e^2, -e^1 - e^3 & \quad D_{15}(-1). \\
-\frac{1}{2\sqrt{3}}(e^4 + 2e^5), \frac{1}{2\sqrt{3}}(e^2 - e^3), \frac{1}{3}e^1 - \frac{1}{6}e^2 - \frac{1}{6}e^3, \frac{1}{2}e^4, \frac{1}{3}(e^1 + e^2 + e^3) & \quad D_{18}(-1, -1). \\
e^4, e^1, \frac{\sqrt{2}}{2}e^3, \frac{\sqrt{2}}{2}e^2, \frac{1}{3}(e^5 - 3e^1) & \quad D_{22}.
\end{aligned}$$

The Lie algebra D_1 is nilpotent and has a well-known hypo-contact structure:

$$e^2, e^4, e^3, e^5, e^1 \quad D_1.$$

The Lie algebra D_2 is also nilpotent and equivalent to $(0, 0, 0, 12, 13 + 24)$, hence hypo by [5]; a hypo structure is given by the coframe

$$e^2, e^4, e^5, -e^1, -e^1 + e^3 \quad D_2.$$

Hypo structures on the three remaining Lie algebras are new. They are defined by

$$\begin{aligned}
e^1, e^3, e^2, e^5, e^4 & \quad D_4(-1/2, -3/2). \\
-e^3, 2e^5, -2e^1, 2e^2, -e^4\sqrt{2} & \quad D_4(-2, 3).
\end{aligned}$$

and, for $D_{20}(-2, 0)$, by

$$3e^2, -3\sqrt{3}e^1 - \sqrt{3}e^4 + 2\sqrt{3}e^5 - 2\sqrt{3}e^2 - \sqrt{3}e^3, 9e^1 + 3e^3 + 3e^4, \\ -2\sqrt{3}e^4 - \sqrt{3}e^2, -5e^1 - 2e^2 - 4e^3 - e^4 + 2e^5.$$

It is straightforward to check that all these Lie algebras have a contact form.

It remains to show that the remaining $D_k(p_1, \dots, p_n)$ do not have a hypo structure; to that effect, we apply the results of Sections 1, 2. Looking at the list and applying Proposition 2, we see that the algebras in the list that admit a coherent splitting with

$$H^{0,2} = H^{0,3} = 0$$

are precisely the following (ϕ denotes a generator of $\Lambda^{2,0}$ in each case):

- $D_4 = (-(1+p)e^{15} - e^{23}, -e^{25}, -pe^{35}, -qe^{45}, 0)$, $\phi = e^{25}$ if all of $p+q$, $2p+1$, $2p+1+q$ are non-zero, or $\phi = e^{35}$ if all of $p+q+2$, $p+2$, $1+q$ are non-zero;
- $D_5 = (-e^{15}(1+p) - e^{23} - e^{45}, -e^{25}, -pe^{35}, -e^{45}(1+p), 0)$, $\phi = e^{25}$ if both of $1+2p$, $2+3p$ are non-zero, or $\phi = e^{35}$ otherwise;
- $D_6 = (2e^{15} + e^{23}, e^{25}, e^{25} + e^{35}, e^{45} + e^{35}, 0)$, $\phi = e^{25}$;
- $D_7 = (e^{23}, 0, e^{25}, e^{45}, 0)$, $\phi = e^{25}$;
- $D_8 = (-2e^{15} - e^{23}, -e^{25}, -e^{25} - e^{35}, -pe^{45}, 0)$, $\phi = e^{25}$;
- $D_9 = (2e^{15} + e^{23} + e^{45}, e^{25}, e^{35} + e^{25}, 2e^{45}, 0)$, $\phi = e^{25}$;
- $D_{12} = (e^{45} + e^{15} + e^{23}, 0, e^{35}, e^{35} + e^{45}, 0)$, $\phi = e^{25}$;
- $D_{13} = (-e^{15}(1+p) - e^{23}, -pe^{25}, -e^{35} - pe^{25}, -e^{35} - e^{45}, 0)$, $\phi = e^{25}$ if both of $p+2$, $p+3$ are non-zero, or $\phi = pe^{25} + (1-p)e^{35}$ otherwise;
- $D_{14} = (e^{15} + e^{23}, e^{25}, 0, e^{45}, 0)$, $\phi = e^{25}$;
- $D_{15} = (-e^{24} - e^{15}(2+p), -e^{25}(1+p) - e^{34}, -pe^{35}, -e^{45}, 0)$, $\phi = e^{45}$ if both of $1+2p$, $1+p$ are non-zero;
- $D_{16} = (e^{24} + 3e^{15}, e^{34} + 2e^{25}, e^{35} + e^{45}, e^{45}, 0)$, $\phi = e^{45}$;
- $D_{17} = (-e^{15} - e^{24} - pe^{35}, -e^{25} - e^{34}, -e^{35}, 0, 0)$, $\phi = e^{45}$;
- $D_{18} = (-e^{14}, -e^{25}, -pe^{34} - qe^{35}, 0, 0)$, $\phi = e^{45}$ if (p, q) is not $(0, -1)$, $(-1, 0)$ or $(-1, -1)$;
- $D_{19} = (-e^{15} - pe^{14}, -e^{35} - e^{24}, -e^{34}, 0, 0)$, $\phi = e^{45}$;
- $D_{20} = (-qe^{15} - pe^{14}, -e^{24} - e^{35}, e^{25} - e^{34}, 0, 0)$, $\phi = e^{45}$ if (p, q) is not $(-2, 0)$;

- $D_{23} = (e^{14}, e^{25}, e^{45}, 0, 0), \phi = e^{45};$
- $D_{24} = (e^{14} + e^{25}, e^{24} - e^{15}, e^{45}, 0, 0), \phi = e^{45}.$

Other non-unimodular Lie algebras are ruled out by Proposition 9. They are listed below; here and throughout the paper, the 1-form β is given up to multiple.

- $D_4(1, -1) = (-e^{23} - 2e^{15}, -e^{25}, -e^{35}, e^{45}, 0), \beta = e^5$
- $D_4(0, -1) = (-e^{15} - e^{23}, -e^{25}, 0, e^{45}, 0), \beta = e^5$
- $D_{15}(-1/2) = (-e^{24} - \frac{3}{2}e^{15}, -\frac{1}{2}e^{25} - e^{34}, \frac{1}{2}e^{35}, -e^{45}, 0), \beta = e^5.$

To address the remaining Lie algebras, we apply either Proposition 7 or Equation 6.

- $D_3 = (e^{25} + e^{34}, e^{35}, e^{45}, 0, 0).$ This Lie algebra is nilpotent and isomorphic to $(0, 0, 12, 13, 23 + 14)$, therefore not hypo by [5]; however, we can prove it directly using the methods of Section 2. We compute

$$Z^2 = \text{Span}\{-e^{14} + e^{23}, e^{15} + e^{24}, e^{25}, e^{34}, e^{35}, e^{45}\}$$

$$Z^3 = \text{Span}\{-e^{125} + e^{134}, e^{135}, e^{145}, e^{234}, e^{235}, e^{245}, e^{345}\}$$

Then the spaces $(e_i \lrcorner Z^3) \wedge e^5, i = 1, 2$ are one-dimensional, so by Lemma 10 α is a linear combination of e^3, e^4, e^5 . In particular, if α is linearly independent of e^5 , then setting $\beta = e^5$ in Proposition 7 we see that

$$L_\beta(V) \subset \text{Span}\{e^{1345}, e^{2345}\}$$

contains $Z^2 \wedge \alpha \wedge \beta$. Otherwise, we may set $\beta = e^4$ and obtain the same result.

- $D_4(-2, 2) = (e^{15} - e^{23}, -e^{25}, 2e^{35}, -2e^{45}, 0).$ We compute

$$Z^2 = \text{Span}\{e^{12}, -e^{15} + e^{23}, e^{25}, e^{34}, e^{35}, e^{45}\}$$

$$Z^3 = \text{Span}\{e^{125}, e^{135}, e^{235}, e^{234} - e^{145}, e^{245}, e^{345}\}$$

Therefore $Z^3 \wedge e^5$ is one-dimensional, hence by Lemma 10 $\alpha(e_i) = 0, i = 1, 2, 3, 4$, i.e $\alpha = e^5$ up to a multiple. Now $Z^2 \wedge \alpha$ is spanned by $e^{125}, e^{235}, e^{345}$. Setting $\beta = e^3$ in Proposition 7 gives a contradiction, as $L_\beta(V)$ is spanned by e^{1235}, e^{2345} and $Z^2 \wedge \alpha \wedge \beta$ is spanned by e^{1235} .

- $D_4(-1/2, -1) = (-\frac{1}{2}e^{15} - e^{23}, -e^{25}, \frac{1}{2}e^{35}, e^{45}, 0)$ is similar to $D_4(-2, 2)$ in that

$$Z^2 = \text{Span}\left\{e^{13}, \frac{1}{2}e^{15} + e^{23}, e^{24}, e^{25}, e^{35}, e^{45}\right\},$$

$$Z^3 = \text{Span}\left\{e^{125}, e^{135}, e^{234} + \frac{1}{2}e^{145}, e^{235}, e^{245}, e^{345}\right\}.$$

The same argument applies, except that now $Z^2 \wedge \alpha \wedge \beta$ is spanned by e^{2345} .

- $D_{10} = (-2pe^{15} - e^{23}, -pe^{25} + e^{35}, -pe^{35} - e^{25}, -qe^{45}, 0)$. A basis of Z^3 is given by

$$e^{345}, e^{235}, e^{135}, e^{125}, e^{245}, (2p + q)e^{145} - e^{234},$$

plus e^{123} if $p = 0$, whereas

$$Z^2 = \text{Span} \{e^{35}, e^{45}, 2e^{15} + e^{23}, e^{25}\}.$$

Now $\beta = (4p + q)e^5$, and if $4p + q \neq 0$ then Proposition 9 applies; in general, we have that $e_1 \lrcorner Z^3$ and $e_4 \lrcorner Z^3$ wedged with e^5 are at most one-dimensional, hence by Lemma 10 α lies in the span of e^2, e^3, e^5 . But then $(Z^2)^2 \wedge \alpha = 0$, which is a contradiction.

- $D_{11} = (-2e^{15}p - e^{23} - \epsilon e^{45}, -e^{25}p + e^{35}, -e^{35}p - e^{25}, -2e^{45}p, 0)$. A basis of Z^2 is given by

$$2pe^{15} + e^{23}, e^{35}, e^{45}, e^{25},$$

whereas a basis of Z^3 is given by

$$e^{345}, e^{235}, e^{135}, e^{125}, e^{245}, -4pe^{145} + e^{234},$$

plus $e^{145} - \epsilon e^{123}$ if $p = 0$. Thus $(e_1 \lrcorner Z^3) \wedge e^5, (e_4 \lrcorner Z^3) \wedge e^5$ have dimension one and we see that α is in $\text{Span} \{e^2, e^3, e^5\}$, whence $\alpha \wedge (Z^2)^2 = 0$, a contradiction.

- $D_{18}(-1, 0) = (-e^{14}, -e^{25}, e^{34}, 0, 0)$. We compute

$$Z^2 = \text{Span} \{e^{13}, e^{14}, e^{25}, e^{34}, e^{45}\},$$

$$Z^3 = \text{Span} \{e^{125} + e^{124}, e^{134}, e^{135}, e^{145}, -e^{234} + e^{235}, e^{245}, e^{345}\},$$

and $\beta = e^5$. Moreover the spaces

$$(e_1 \lrcorner Z^3) \wedge (e^4 + e^5), \quad (e_2 \lrcorner Z^3) \wedge (e^4 + e^5), \quad (e_3 \lrcorner Z^3) \wedge (e^4 - e^5)$$

are one-dimensional, so by Lemma 10 and Lemma 8 we get $\alpha = e^4 + ae^5$, for some constant a . Then setting $\beta = e^5$ in Proposition 7, we see that $L_\beta(V)$ is at most two-dimensional, and it contains e^{1345} . Since $Z^2 \wedge \alpha \wedge \beta$ is spanned by e^{1345} , there is no hypo structure.

- $D_{18}(0, -1) = (-e^{14}, -e^{25}, e^{35}, 0, 0)$ is really isomorphic to $D_{18}(-1, 0)$, as one can check by considering the coframe $(e^2, e^1, e^3, e^5, e^4)$, so it has no hypo structure.
- $D_{21} = (e^{23} + e^{14}, e^{24} - e^{25}, e^{35}, 0, 0)$. Then $\beta = e^4$ and

$$Z^2 = \text{Span} \{e^{14} + e^{23}, e^{25} - e^{24}, e^{35}, e^{45}\},$$

$$Z^3 = \text{Span} \{e^{125} - 2e^{124}, e^{135} + e^{134}, e^{234}, e^{235} + e^{145}, e^{245}, e^{345}\}.$$

Therefore, the spaces

$$(e_1 \lrcorner Z^3) \wedge (e^4 + e^5), \quad (e_2 \lrcorner Z^3) \wedge (e^5 - 2e^4), \quad (e_3 \lrcorner Z^3) \wedge (e^4 + e^5)$$

are one-dimensional, so by Lemma 10 and Lemma 8 we get $\alpha = ae^4 + e^5$, for some constant a . Then setting $\beta = e^4$ in Proposition 7, we see that $L_\beta(V)$ is at most two-dimensional, and it contains e^{2345} . Since $Z^2 \wedge \alpha \wedge \beta$ is spanned by e^{2345} , there is no hypo structure. \square

4 Indecomposable Lie algebras without contact form

We now pass on to indecomposable solvable Lie algebras that do not have a contact structure.

Proposition 12. *The indecomposable solvable Lie algebras which have a hypo structure but not a contact structure are those given in Table 1, all of them unimodular.*

Observe that $A_{5,1}$ and $A_{5,2}$ are nilpotent, and so appear in [5].

Proof. It is straightforward to verify that the coframes given in the table define indeed hypo structures. To show that no other Lie algebras of the specified type have a hypo structure, we use the classification in [1].

- $A_{5,3} = (e^{25}, e^{45}, e^{24}, 0, 0)$ is nilpotent and known not to have a hypo structure [5]. It also has a coherent splitting $\phi = e^{45}$ with $H^{0,2} = H^{0,3} = 0$.
- $A_{5,7} = (e^{15}, pe^{25}, qe^{35}, re^{45}, 0)$ where $p, q, r \neq 0$ is not hypo unless, up to permutation of the parameters, $r = -1$ and $p + q = 0$. Indeed, suppose first that $p, q, r \neq -1$. Then if $p + q \neq 0, -1$ we find a coherent splitting $\phi = e^{45}$ with $H^{0,2} = H^{0,3} = 0$. Since we can act by an automorphism to permute p, q, r , the same happens if $p + r \neq 0, -1$ or $q + r \neq 0, -1$. Thus, still assuming $p, q, r \neq -1$, we are left with the cases $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. In the former case, $\phi = e^{15}$ defines a coherent splitting with $H^{0,2} = H^{0,3} = 0$. In the latter case, the Lie algebra is non-unimodular with $\beta = e^5$, and $L_\beta(Z^3)$ has dimension one.

Now, if $r = -1$ and $p + q \neq 0$, the Lie algebra is non-unimodular with $\beta = e^5$, and Z^3 is spanned by

$$e^{345}, e^{145}, e^{235}, e^{125}, e^{245}, e^{135}$$

plus

$$\begin{aligned} &e^{234} \text{ if } p + q - 1 = 0, \\ &e^{123} \text{ if } p + q + 1 = 0. \end{aligned}$$

Therefore, $L_\beta(Z^3)$ is at most one-dimensional.

- $A_{5,8} = (e^{25}, 0, e^{35}, pe^{45}, 0)$ has a coherent splitting given by $\phi = e^{25}$ with $H^{0,2} = H^{0,3} = 0$ if $p \neq -1$.
- $A_{5,9} = (e^{15} + e^{25}, e^{25}, pe^{35}, qe^{45}, 0)$ has a coherent splitting with $H^{0,2} = H^{0,3} = 0$. If $p + q \neq 0$ we can take $\phi = e^{25}$; if $p = -q$ but $p \neq 1, 2$ then $\phi = e^{35}$, and otherwise we can take $\phi = e^{45}$.
- $A_{5,10} = (e^{25}, e^{35}, 0, e^{45}, 0)$ has a coherent splitting given by $\phi = e^{35}$ with $H^{0,2} = H^{0,3} = 0$.
- $A_{5,11} = (e^{15} + e^{25}, e^{35} + e^{25}, e^{35}, -pe^{45}, 0)$ has a coherent splitting given by $\phi = e^{35}$ with $H^{0,2} = H^{0,3} = 0$.
- $A_{5,12} = (e^{15} + e^{25}, e^{25} + e^{35}, e^{35} + e^{45}, e^{45}, 0)$ has a coherent splitting given by $\phi = e^{45}$ with $H^{0,2} = H^{0,3} = 0$.
- $A_{5,13} = (e^{15}, pe^{25}, qe^{35} + re^{45}, qe^{45} - re^{35}, 0)$, where we assume $r \neq 0$ (as $A_{5,13}(p, q, 0)$ is isomorphic to $A_{5,7}(p, q, q)$), has a coherent splitting given by $\phi = e^{25}$ with $H^{0,2} = H^{0,3} = 0$ if $q \neq 0, -\frac{1}{2}$. If $q = -1/2$, $p \neq 1$ then the same holds of $\phi = e^{15}$. The only cases left out are $(p, q) = (1, -\frac{1}{2})$ and $q = 0$. In general, a basis of Z^3 is given by

$$e^{145}, e^{245}, e^{135}, e^{345}, e^{125}, e^{235}$$

plus e^{134} if $1 + 2q$ is zero, plus e^{234} if $p + 2q$ is zero. A basis of Z^2 is given by

$$e^{45}, e^{25}, e^{15}, e^{35}$$

plus e^{34} if q is zero, plus e^{12} if $p = -1$. Thus, if $(p, q) = (1, -\frac{1}{2})$, then $(Z^2)^2 = 0$ contradicting (6) for any α . On the other hand, if $q = 0$, then α cannot be independent of $\beta = e^5$, as $\dim L_\beta(Z^3) < 2$. But then (6) is only satisfied if $(p, q) = (-1, 0)$, in which case we already know that a hypo structure exists.

- $A_{5,14} = (e^{25}, 0, e^{45} + pe^{35}, -e^{35} + pe^{45}, 0)$ has a coherent splitting given by $\phi = e^{25}$ with $H^{0,2} = H^{0,3} = 0$ if $p \neq 0$.
- $A_{5,15} = (e^{15} + e^{25}, e^{25}, e^{45} + pe^{35}, pe^{45}, 0)$ has a coherent splitting given by $\phi = e^{45}$ with $H^{0,2} = H^{0,3} = 0$ if $p \neq -1$.
- $A_{5,16} = (e^{25} + e^{15}, e^{25}, pe^{35} + qe^{45}, pe^{45} - qe^{35}, 0)$ has a coherent splitting given by $\phi = e^{25}$ if $p \neq 0$. If $p = 0$, then $\beta = e^5$ and $L_\beta(Z^3) = 0$.
- $A_{5,17} = (pe^{15} + e^{25}, pe^{25} - e^{15}, re^{45} + qe^{35}, -re^{35} + qe^{45}, 0)$, $r \neq 0$. Then $Z^3 \wedge e^5 = 0$. So if $p + q \neq 0$, then $\beta = e^5$ and Proposition 9 applies. Otherwise, the same argument together with Proposition 7 shows that necessarily $\alpha = e^5$. Now if $p + q = 0$ but $p \neq 0$ and $r \neq \pm 1$, then $Z^2 = e^5 \wedge \Lambda^1$, so by (6) no hypo structure exists.
- $A_{5,18} = (pe^{15} + e^{35} + e^{25}, pe^{25} + e^{45} - e^{15}, pe^{35} + e^{45}, -pe^{45} - e^{35}, 0)$. Then $\beta = pe^5$, and $L_{e^5}(Z^3) = 0$. So if $p \neq 0$ we obtain an obstruction.

□

Table 1: Nondecomposable, non-contact hypo Lie algebras

Name	Structure constants	Hypo structure
$A_{5,1}$	$(e^{35}, e^{45}, 0, 0, 0)$	e^1, e^3, e^2, e^4, e^5
$A_{5,2}$	$(e^{25}, e^{35}, e^{45}, 0, 0)$	e^1, e^4, e^3, e^2, e^5
$A_{5,7}(p, -p, -1)$	$(e^{15}, pe^{25}, -pe^{35}, -e^{45}, 0)$	e^1, e^4, e^2, e^3, e^5
$A_{5,8}(-1)$	$(e^{25}, 0, e^{35}, -e^{45}, 0)$	e^1, e^2, e^3, e^4, e^5
$A_{5,13}(-1, 0, r)$	$(e^{15}, -e^{25}, re^{45}, -re^{35}, 0)$	e^1, e^2, e^3, e^4, e^5
$A_{5,14}(0)$	$(e^{25}, 0, e^{45}, -e^{35}, 0)$	e^1, e^2, e^3, e^4, e^5
$A_{5,15}(-1)$	$(e^{15} + e^{25}, e^{25}, e^{45} - e^{35}, -e^{45}, 0)$	e^1, e^4, e^3, e^2, e^5
$A_{5,17}(0, 0, r)$	$(e^{25}, -e^{15}, re^{45}, -e^{35}r, 0)$	e^1, e^2, e^3, e^4, e^5
$A_{5,17}(p, -p, 1)$	$(e^{25} + pe^{15}, -e^{15} + pe^{25}, e^{45} - pe^{35}, -e^{35} - pe^{45}, 0)$	e^1, e^3, e^2, e^4, e^5
$A_{5,17}(p, -p, -1)$	$(e^{25} + pe^{15}, -e^{15} + pe^{25}, -e^{45} - pe^{35}, e^{35} - pe^{45}, 0)$	e^1, e^3, e^4, e^2, e^5
$A_{5,18}(0)$	$(e^{35} + e^{25}, -e^{15} + e^{45}, e^{45}, -e^{35}, 0)$	e^1, e^3, e^2, e^4, e^5

5 Decomposable contact Lie algebras

By [9], there are two types of decomposable 5-dimensional Lie algebras with an invariant contact form. First, the Lie algebras $(0, e^{12}) \oplus \mathfrak{g}_3$, where $(0, e^{12})$ is the Lie algebra of affine transformations of \mathbb{R} , and \mathfrak{g}_3 is any Lie algebra of dimension three other than $(0, e^{12}, e^{13})$ or $(0, 0, 0)$. Second, the Lie algebras of the form $\mathfrak{g}_4 \oplus \mathbb{R}$, where \mathfrak{g}_4 is a four-dimensional Lie algebra carrying an exact symplectic form. In this section we show that only one of these families admits a hypo structure, and it belongs to the first type.

Proposition 13. *If \mathfrak{g}_3 is a solvable Lie algebra of dimension three, then $(0, e^{12}) \oplus \mathfrak{g}_3$ has a hypo structure if and only if $\mathfrak{g}_3 = A_{3,8} = (e^{23}, -e^{13}, 0)$.*

Proof. First, observe that $A_{3,8} \oplus (0, e^{12}) = (e^{23}, -e^{13}, 0, 0, e^{45})$ has a hypo structure given by the coframe e^1, e^2, e^4, e^3, e^5 .

To prove uniqueness, we resort once again to the list in [1]. There are nine families of solvable Lie algebras of dimension three, of which the following five have a coherent splitting with $H^{0,2} = H^{0,3} = 0$:

$$\begin{aligned}
A_{3,2} \oplus (0, e^{12}) &= (0, e^{12}, 0, 0, e^{45}), & \phi &= e^{14} \\
A_{3,4} \oplus (0, e^{12}) &= (e^{23} + e^{13}, e^{23}, 0, 0, e^{45}), & \phi &= e^{34} \\
A_{3,5} \oplus (0, e^{12}) &= (e^{13}, e^{23}, 0, 0, e^{45}), & \phi &= e^{34} \\
A_{3,7} \oplus (0, e^{12}) &= (e^{13}, qe^{23}, 0, 0, e^{45}), \quad 0 < |q| < 1, & \phi &= e^{34} \\
A_{3,9} \oplus (0, e^{12}) &= (qe^{13} + e^{23}, qe^{23} - e^{13}, 0, 0, e^{45}), \quad q > 0, & \phi &= e^{34}.
\end{aligned}$$

Therefore, by Proposition 2, there is no hypo structure on $A_{3,i}$, for $i = 2, 4, 5, 7, 9$.

For $A_{3,1} \oplus (0, e^{12}) = (0, 0, 0, 0, e^{45})$, we find that $\beta = e^4$ and $L_\beta(Z^3)$ is spanned by e^{1234} , so Proposition 9 applies.

For $A_{3,3} \oplus (0, e^{12}) = (e^{23}, 0, 0, 0, e^{45})$, we have that $\beta = -e^4$ and

$$L_\beta(Z^3) = \text{Span}\{-e^{1234}, e^{2345}\}.$$

So, by Proposition 9, if (α, ω_i) is a hypo structure then

$$\alpha \in \text{Span}\{e^2, e^3, e^4\},$$

implying that $Z^2 \wedge \alpha \wedge \beta$ is contained in $L_\beta(Z^3)$, which is absurd.

Finally, the Lie algebra $A_{3,6} \oplus (0, e^{12}) = (e^{13}, -e^{23}, 0, 0, e^{45})$ satisfies

$$\begin{aligned} Z^2 &= \text{Span}\{e^{12}, e^{13}, e^{23}, e^{34}, e^{45}\}, \\ Z^3 &= \text{Span}\{e^{123}, e^{124}, e^{134}, e^{145} - e^{135}, e^{234}, e^{245} + e^{235}, e^{345}\}. \end{aligned}$$

So α lies in the span of e^3, e^4 by Lemma 10. Moreover $\beta = e^4$, thus α has the form $e^3 + ae^4$. Defining V as in Proposition 7, we see that $e^4 \wedge V$ is contained in the span of e^{1234} and e^{2345} . Since $Z_2 \wedge e^{34} = e^{1234}$, there is no hypo structure. \square

Remark. Notice that Proposition 13 does not apply to contact Lie algebras alone, but also to the non-contact Lie algebras $A_{3,1} \oplus (0, e^{12})$ and $A_{3,5} \oplus (0, e^{12})$.

Decomposable contact Lie algebras of the type $\mathfrak{g}_4 \oplus \mathbb{R}$ are not unimodular, because the volume form is exact, and so it makes sense to apply Proposition 9. This turns out to be sufficient in order to show that no hypo structure exists on these Lie algebras.

Proposition 14. *If \mathfrak{g}_4 is a 4-dimensional solvable Lie algebra with an exact symplectic form, then $\mathfrak{g}_4 \oplus \mathbb{R}$ has no hypo structure.*

Proof. Observe that \mathfrak{g}_4 is necessarily indecomposable, because it admits an exact symplectic form. From the list in [1], \mathfrak{g}_4 must belong to one of four families, to each of which we apply Proposition 9:

- The Lie algebra $A_{4,7} \oplus \mathbb{R} = (e^{23} + 2e^{14}, e^{24} + e^{34}, e^{34}, 0, 0)$ has $\beta = e^4$, and $L_\beta(Z^3)$ has dimension one.
- The Lie algebra $A_{4,8} \oplus \mathbb{R} = (e^{23} + (1+q)e^{14}, e^{24}, qe^{34}, 0, 0)$, $-1 < q \leq 1$ has $\beta = e^4$, and $L_\beta(Z^3)$ has dimension one except if $q = -1/2$. In this case, it is spanned by e^{1345}, e^{2345} , and

$$Z^2 = \text{Span}\left\{e^{13}, e^{23} + \frac{1}{2}e^{14}, e^{24}, e^{34}, e^{45}\right\},$$

so by Proposition 9 there is no hypo structure.

- The Lie algebra $A_{4,9} \oplus \mathbb{R} = (e^{23} + 2qe^{14}, qe^{24} + e^{34}, -e^{24} + qe^{34}, 0, 0)$, with $q > 0$. Then $\beta = e^4$, and $L_\beta(Z^3)$ is one-dimensional.
- The Lie algebra $A_{4,10} \oplus \mathbb{R} = (e^{13} + e^{24}, e^{23} - e^{14}, 0, 0, 0)$ has $\beta = e^3$, and

$$L_\beta(Z^3) = \text{Span}\{e^{2345}, e^{1345}\}.$$

So L_α kills $L_\beta(Z^3)$ if and only if α lies in the span of e^3, e^4, e^5 , in which case $\alpha \wedge \beta \wedge Z^2$ is contained in $L_\beta(Z^3)$.

Thus, in neither case is there a hypo structure. \square

Remark. In the proof of Proposition 14, we have left out the cases $A_{4,8}(-1)$ and $A_{4,9}(0)$ because they do not have any exact symplectic form. They have no hypo structure either. Indeed, $A_{4,8}(-1)$ has a coherent splitting $\phi = e^{24}$ with $H^{0,2} = H^{0,3} = 0$. For $A_{4,9}(0)$, we apply Lemma 10 to show that α has no component along e^1 , contradicting (6).

6 Decomposable non-contact Lie algebras

Decomposable Lie algebras of dimension five may either be of the form $\mathfrak{g}_3 \oplus \mathfrak{h}_2$, where we are allowing the factors themselves to be decomposable, or $\mathfrak{g}_4 \oplus \mathbb{R}$. In the former case, by Proposition 13 we can assume $\mathfrak{h}_2 = \mathbb{R}^2$. Without resorting to Mubarakzyanov's classification, we can characterize which of these Lie algebras have a hypo structure.

Proposition 15. *Let \mathfrak{g}_3 be a Lie algebra of dimension 3. Then $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}^2$ admits a hypo structure if and only if \mathfrak{g}_3 is unimodular.*

Proof. Let e^1, \dots, e^5 be a coframe reflecting the splitting $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}^2$, so that e^1, e^2, e^3 is a basis of $\mathfrak{g}_3^* \subset \mathfrak{g}^*$ and e^4, e^5 a basis of $(\mathbb{R}^2)^*$.

If \mathfrak{g}_3 is unimodular, then the coframe e^1, e^2, e^4, e^5, e^3 determines a hypo structure by (5), because e^{12}, e^{13}, e^{23} are closed.

If \mathfrak{g}_3 is not unimodular, we can assume that $e^3 = \beta$ as defined in Lemma 8. Then e^3 is closed and $de^{12} \neq 0$. Moreover $de^i \wedge e^3 = 0$, $i = 1, 2$. This is because if e_1, e_2, e_3 is a basis of \mathfrak{g}_3 dual to e^1, e^2, e^3 , then

$$0 = \text{tr}(\text{ad } e_2) = e^1([e_2, e_1]) + e^3([e_2, e_3]) = e^1([e_2, e_1]).$$

Thus $Z^3 = (e^3 \wedge \Lambda^2) \oplus W$, where

$$W \subset \Lambda^3(\text{Span}\{e^1, e^2, e^4, e^5\}).$$

Since $de^{12} \neq 0$, $W \subset \text{Span}\{e^{145}, e^{245}\}$; so $L_\beta(Z_3)$ has the same dimension as $\ker d \cap \text{Span}\{e^1, e^2\}$, which is at most one since \mathfrak{g}_3 is not abelian. By Proposition 9 this concludes the proof. \square

For the other case, we must refer to Mubarakzyanov's classification.

Proposition 16. *If \mathfrak{g}_4 is an indecomposable solvable Lie algebra of dimension four, then $\mathfrak{g}_4 \oplus \mathbb{R}$ has no hypo structure.*

Proof. By [1], there are 10 families $A_{4,1}, \dots, A_{4,10}$ of solvable Lie algebras of dimension four. The families $A_{4,7}$ through $A_{4,10}$ have no hypo structure by

Proposition 14 and the subsequent remark. The following families have a coherent splitting with $H^{0,2} = H^{0,3} = 0$:

$$\begin{aligned}
A_{4,1} \oplus \mathbb{R} &= (e^{24}, e^{34}, 0, 0, 0), & \phi &= e^{34} \\
A_{4,2} \oplus \mathbb{R} &= (qe^{14}, e^{24} + e^{34}, e^{34}, 0, 0), & q \neq 0, \quad \phi &= e^{34} \\
A_{4,3} \oplus \mathbb{R} &= (e^{14}, e^{34}, 0, 0, 0), & \phi &= e^{34} \\
A_{4,4} \oplus \mathbb{R} &= (e^{14} + e^{24}, e^{24} + e^{34}, e^{34}, 0, 0), & \phi &= e^{45} \\
A_{4,5} \oplus \mathbb{R} &= (e^{14}, qe^{24}, pe^{34}, 0, 0), & p, q \neq 0, \quad \phi &= \begin{cases} e^{14}, & (p, q) = (-1, -1) \\ e^{34}, & q \neq -1 \\ e^{24}, & p \neq -1 \end{cases}
\end{aligned}$$

Notice that $A_{4,1} \oplus \mathbb{R}$ is nilpotent and isomorphic to $(0, 0, 0, e^{12}, e^{14})$.

The remaining family is

$$A_{4,6} \oplus \mathbb{R} = (qe^{14}, e^{34} + pe^{24}, -e^{24} + pe^{34}, 0, 0), \quad q \neq 0, \quad p \geq 0.$$

If $p > 0$, $\phi = e^{14}$ defines a coherent splitting with $H^{0,2} = H^{0,3} = 0$. If $p = 0$, then $\beta = e^4$, and

$$Z^3 = \text{Span} \{e^{235}, e^{145}, e^{245}, e^{134}, e^{234}, e^{124}, e^{345}\}.$$

Therefore $L_\beta(Z^3)$ has dimension one, and Proposition 9 applies. \square

Applying the classification of three-dimensional Lie algebras, we finally obtain:

Theorem 17. *A solvable Lie algebra of dimension five has a hypo structure if and only if it appears in the list of Proposition 11, it appears in Table 1, or it is one of the following:*

$$\begin{aligned}
(0, 0, 0, 0, 0), \quad (e^{23}, 0, 0, 0, 0), \quad (e^{23}, -e^{13}, 0, 0, 0), \quad (e^{13}, -e^{23}, 0, 0, 0), \\
(e^{23}, -e^{13}, 0, 0, e^{45}).
\end{aligned}$$

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