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LINEAR SOURCE LATTICES AND THEIR RELEVANCE IN THE REPRESENTATION THEORY OF FINITE GROUPS

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For a finite group G and a prime number p the local-global study of the representation theory of G looks for properties and invariants of G that can be detected on its local subgroups, i.e., the normalizer $N_G(D)$ of a p -subgroup D of G , and vice versa. For simplicity we fix a splitting p -modular system $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ of G . In this context the *category of (left) $\mathcal{O}G$ -lattices* plays a key role for studying the local and global structure of a group. In fact, one of the main tool - the *Green correspondence* - establishes a one-to-one correspondence between the isomorphism types of indecomposable $\mathcal{O}G$ -lattices with vertex set ${}^G D = \{gD = gDg^{-1} \mid g \in G\}$ and the isomorphism types of indecomposable $\mathcal{O}N_G(D)$ -lattices with vertex set $\{D\}$. However, in general it is extremely difficult to give a description of the category of left $\mathcal{O}G$ -lattices as there are usually an infinite number of (isomorphism classes of) indecomposable $\mathcal{O}G$ -lattices. Indeed, describing all indecomposable $\mathcal{O}G$ -lattices is in general a “wild” problem.

In order to avoid this difficulty, but to maintain the main tool - the Green correspondence - we restrict our consideration to the category of $\mathcal{O}G$ -lattices with linear and trivial source, respectively. In fact, the Grothendieck rings $\mathbf{L}_{\mathcal{O}}(G)$ (resp. $\mathbf{T}_{\mathcal{O}}(G)$), i.e., the free $\mathbb{Z}$ -module spanned by the isomorphism types of indecomposable linear (resp. trivial) source $\mathcal{O}G$ -lattices, are in particular finitely generated free abelian groups and their species are explicitly known (cf. [Bol98b]). By definition, the category of left $\mathcal{O}G$ -lattices functions as a kind of “bridge” between the category of finitely generated (left) $\mathbb{K}G$ and $\mathbb{F}G$ -modules. To move from lattices to modules it suffices to apply the tensor product functors $- \otimes_{\mathcal{O}} \mathbb{K}$ and $- \otimes_{\mathcal{O}} \mathbb{F}$, respectively. Let $\mathbf{R}_{\mathbb{K}}(G)$ (resp. $\mathbf{R}_{\mathbb{F}}(G)$) be the Grothendieck ring of $\mathbb{K}G$ -representations (resp. $\mathbb{F}G$ -representations). It is well known that the map $-_{\mathbb{K}} = - \otimes_{\mathcal{O}} \mathbb{K}: \mathbf{T}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G)$ is not surjective, while $-_{\mathbb{K}}: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G)$ is surjective.

The first question arising naturally in this context is what kind of sections of $-_{\mathbb{K}}$ (resp. $-_{\mathbb{F}} = - \otimes_{\mathcal{O}} \mathbf{R}_{\mathbb{F}}(G)$) exists. An answer to this question is given

in Chapter 2, where two canonical sections

$$\tau_G: \mathbf{R}_{\mathbb{F}}(G) \longrightarrow \mathbf{T}_{\mathcal{O}}(G) \text{ and } \lambda_G: \mathbf{R}_{\mathbb{K}}(G) \longrightarrow \mathbf{L}_{\mathcal{O}}(G) \quad (0.0.1)$$

of the surjective maps

$$-\mathbb{F} = - \otimes_{\mathcal{O}} \mathbb{F}: \mathbf{T}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{F}}(G) \text{ and } -\mathbb{K} = - \otimes_{\mathcal{O}} \mathbb{K}: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G)$$

have been constructed. Here we follow two different strategies. The first one uses the construction of *dual* maps between the set of species of the Grothendieck rings involved. For this approach one needs an explicit description of the species of the linear source ring. Although the sections defined in this way just yield maps

$$\mathbf{t}_G: \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}} \longrightarrow \mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}} \text{ and } \mathbf{l}_G: \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}} \longrightarrow \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}},$$

defined on the complexification of the Grothendieck rings involved, it shows how these maps are linked to the representation tables defined by D. Benson (cf. [Ben06]). Secondly, we followed an approach suggested by Robert Boltje in [Bol98a], where he introduces a canonical induction formula for the Grothendieck rings. Following his fundamental observation one may define maps τ_G and λ_G . The final result of this chapter is the proof that these two approaches lead to the same canonical sections, i.e.,

$$\mathbf{t}_G \mid_{\mathbf{T}_{\mathcal{O}}(G)} = \tau_G \text{ and } \mathbf{l}_G \mid_{\mathbf{L}_{\mathcal{O}}(G)} = \lambda_G.$$

It is a somehow remarkable fact that the ring $\mathbf{L}_{\mathcal{O}}(G)$ of linear source lattices is equipped with two canonical symmetric bilinear forms. The first one arises from the canonical map $-\mathbb{K}: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G)$,

$$\langle \cdot, \cdot \rangle: \mathbf{L}_{\mathcal{O}}(G) \times \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbb{Z}$$

while the second one

$$\langle\langle \cdot, \cdot \rangle\rangle: \mathbf{L}_{\mathcal{O}}(G) \times \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbb{Z},$$

is induced by the canonical projection $\pi: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \simeq \mathbf{L}_{\mathcal{O}}(G)/\mathbf{L}_{\mathcal{O}}^{\prec}(G)$, where $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(G)$ is the free $\mathbb{Z}$ -module spanned by the isomorphism classes of indecomposable linear source $\mathcal{O}G$ -lattices with maximal vertex and $\mathbf{L}_{\mathcal{O}}^{\prec}(G)$ is the free $\mathbb{Z}$ -module spanned by the isomorphism classes of indecomposable linear source $\mathcal{O}G$ -lattices with smaller vertex. Even more important seems to be their difference (see Chapters 3 and 4)

$$(\cdot, \cdot) = \langle \cdot, \cdot \rangle - \langle\langle \cdot, \cdot \rangle\rangle.$$

The values of these bilinear forms can be computed explicitly in terms of the species of the linear source ring, exactly in the same way as one may

express the inner product in $\mathbf{R}_{\mathbb{K}}(G)$ in terms of the values of characters. In this context the ring

$$\mathbf{E}_{\mathcal{O}}(G) = \mathbf{L}_{\mathcal{O}}(G) / (\text{rad}(\langle\langle \cdot, \cdot \rangle\rangle) \cap \text{rad}(\langle \cdot, \cdot \rangle)).$$

of *essential linear source* $\mathcal{O}G$ -lattices (cf. § 3.2) arises naturally.

The next step will be to transfer the previous discussion in a block-wise version. Given a finite group G , the group ring $\mathcal{O}G$ decomposes into a direct sum of p -blocks B (see § 1.2). Each block B has a unique (up to conjugation) defect group D . The *Brauer correspondence* permits to associate to each block B with a defect group D a unique block b of $N_G(D)$ with D as defect group. The $\mathcal{O}N_G(D)$ -block b is called the *Brauer correspondent* of B . The irreducible characters and the isomorphism types of indecomposable linear and trivial source lattices are divided into blocks. Then, given a block B of G , it is possible to define the Grothendieck groups $\mathbf{R}_{\mathbb{K}}(B)$, $\mathbf{L}_{\mathcal{O}}(B)$ and $\mathbf{T}_{\mathcal{O}}(B)$ of $\mathbb{K}B$ -representations, linear source and trivial source $\mathcal{O}G$ -lattices belonging to B , respectively. In particular, an $\mathcal{O}G$ -lattice belonging to a block B is said to be of maximal vertex if it has a defect group of B as a vertex; $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(B)$ will denote the $\mathbb{Z}$ -span of the isomorphism types of indecomposable linear source lattices with maximal vertex in B . Considering a block B with a defect group D , particular interest is given to the set $\text{Irr}_0(B)$ of irreducible characters of *height zero*, i.e., the irreducible characters χ such that $\chi(1)_p \cdot |D| = |P|$, where $\chi(1)_p$ is the p -part of the degree of χ and P is a Sylow p -subgroup of G . These characters play a central role in some local-global conjectures deeply studied and investigated. Let $\mathbf{R}_{\mathbb{K}}^0(B)$ denote the free abelian group of irreducible height zero representations belonging to the block B . Considering blocks with normal defect group, in Chapter 4 the following somehow astonishing result will be proved (see Theorem A).

Theorem. Let b be a p -block with normal defect. Then for every indecomposable linear source b -lattice L with maximal vertex the $\mathbb{K}G$ -module $L_{\mathbb{K}}$ is irreducible and has height zero. Moreover, the canonical map

$$-_{\mathbb{K}}: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(b) \longrightarrow \mathbf{R}_{\mathbb{K}}^0(b)$$

is an isomorphism.

This result shows that thanks to Green correspondence there is a bijection between the groups $\mathbf{R}_{\mathbb{K}}^0(b)$ and $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(B)$, where the $\mathcal{O}N_G(D)$ -block b is the Brauer correspondent of the $\mathcal{O}G$ -block B .

At this point we will investigate the relationship between $\mathbf{R}_{\mathbb{K}}^0(B)$ and $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(B)$. Unfortunately, in this case the tensor product $-_{\mathbb{K}}: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{R}_{\mathbb{K}}^0(B)$ is not in general a bijection (e.g. A_5 with $p = 2$, see § 2.9). So, one can ask whether it is possible to construct a bijection and, in case of positive answer, how. Thanks to the previous mentioned considerations on linear source lattices, this is equivalent to the *Alperin-McKay conjecture*. In fact, a

first positive answer is given in the case of the existence of a *McKay bijection* between the sets $\text{Irr}_0(B)$ and $\text{Irr}_0(b)$ established by restriction (see Theorem D). In this case the canonical section λ_G (see (0.0.1)) is the answer. This case can be applied for example to p -solvable groups with self-normalizing Sylow p -subgroups or to the symmetric groups S_{2^n} for the prime $p = 2$, in both cases considering blocks with the set of Sylow p -subgroups as set of defect groups. But the bijection constructed with the canonical section λ_G turns out to be much more interesting than this. To understand why, we need to proceed by steps and to summarize what we have already stated in this context. When there exists a McKay bijection, we have the following chain of isomorphisms,

$$\mathbf{R}_{\mathbb{K}}^0(b) \leftrightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(b) \leftrightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \leftrightarrow \mathbf{R}_{\mathbb{K}}^0(B),$$

where the first map is the bijection induced by the tensor product with $\mathbb{K}$, the second one is the Green correspondence and the third map

$$\tilde{\sigma}_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{R}_{\mathbb{K}}^0(B)$$

is constructed considering the canonical section λ_G restricted to the block B . In particular, we will see that the bijection $\tilde{\sigma}_B$ is the composition of a (good) section $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ of the canonical projection $\pi_B: \mathbf{L}_{\mathcal{O}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \simeq \mathbf{L}_{\mathcal{O}}(B)/\mathbf{L}_{\mathcal{O}}^{\vee}(B)$ and the tensor product $-\mathbb{K}: \mathbf{L}_{\mathcal{O}}(B) \rightarrow \mathbf{R}_{\mathbb{K}}^0(B)$. The section σ_B has the strong property that its image is totally isotropic in $\mathbf{L}_{\mathcal{O}}(B)$ with respect to the bilinear form $(\cdot, \cdot)$ and this has three very important consequences. The first one is that $\tilde{\sigma}_B([L]) = \sigma_B([L])_{\mathbb{K}}$ is an irreducible height zero representation for every indecomposable linear source B -lattice L with maximal vertex, the second is that it is injective and the last one is that the p' -part of $\tilde{\sigma}_B([L])(1)$ and the p' part of plus or minus the degree of the character of the Green correspondent (cf. 1.4.1) are congruent modulo p . But then, we can conclude that if there exists a map $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ such that its image is totally isotropic with respect to the bilinear form $(\cdot, \cdot)$, then the *Alperin-McKay* conjecture implies its refinement due to M. I. Isaacs and G. Navarro known as Conjecture B and presented in [IN02] (see Theorem B). Moreover if such a section $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ exists and $\tilde{\sigma}_B$ is surjective, then the Alperin-McKay conjecture and Conjecture B are positively verified (see Theorem B). The strength of this approach is reflected in the following theorem proved in § 4.4 (see Theorem C).

Theorem. Let B be an $\mathcal{O}G$ -block and let b be its Brauer correspondent with respect to a defect group D . If B and b are splendid derived equivalent, then there exists a section $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ whose image is totally isotropic in $\mathbf{L}_{\mathcal{O}}(B)$ with respect to the bilinear form $(\cdot, \cdot)$. Thus Conjecture B in [IN02] holds for B .

This theorem, that can be applied to the case of blocks with cyclic defect groups, establishes a link between *Broué (splendid) conjecture* and Conjecture B. A positive answer to Broué splendid conjecture will allow us to apply Theorem C to all blocks with abelian defect and thus Conjecture B would hold for blocks with abelian defect.

After all this considerations it is clear that linear source lattices have a key role in the representation theory of finite groups and in particular in the study of local-global conjectures since they establish new non-trivial connections between them. Of course, one can ask which role they can have in other interesting refinements of the Alperin-McKay conjecture. The last section of this thesis is devoted to convince the reader that linear source lattices could also contribute for analysing Galois actions on the set $\mathbf{R}_{\mathbb{K}}^0(B)$ and $\mathbf{R}_{\mathbb{K}}^0(b)$. This might be particularly interesting for approaching Conjecture D of G. Navarro and I. M. Isaacs (see [IN02]) and Conjecture B of G. Navarro (see [Nav04]). Unfortunately, the restriction of time of a Ph.D. thesis has not permitted to exploit this direction further.

Outline of the thesis

The first part of Chapter 1 has been written after the participation to the inspiring course *Basic local representation theory* given by B. Külshammer during the *Introductory workshop on the representation theory of finite groups* in the semester on representation theory that took place at the École polytechnique fédérale de Lausanne in July 2016. This chapter is dedicated to the main definitions and results of the representation theory used through all the thesis. Special emphasis is laid on trivial and linear source lattices, and their detection.

The main result of Chapter 2 is the construction of the canonical sections $\tau_G: \mathbf{R}_{\mathbb{F}}(G) \rightarrow \mathbf{T}_{\mathcal{O}}(G)$ and $\lambda_G: \mathbf{R}_{\mathbb{K}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}(G)$ of the surjective maps $- \otimes_{\mathcal{O}} \mathbb{F}: \mathbf{T}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{F}}(G)$ and $- \otimes_{\mathcal{O}} \mathbb{K}: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G)$, respectively.

Chapter 3 is divided in two different parts. In the first part the ring $\mathbf{E}_{\mathcal{O}}(G)$ of essential linear source $\mathcal{O}G$ -lattices is formally introduced and, thanks to an explicit description of the set of its species, its rank is calculated. In the last part of the chapter the link between trivial source lattices with maximal vertex and irreducible characters is studied in two particular cases: groups with normal subgroups of index p and groups with cyclic Sylow p -subgroup of prime order.

The last chapter is part of a joint work with Shigeo Koshitani and Thomas Weigel dedicated to the connection between the Alperin-McKay conjecture, its refinements and the Alperin conjecture and the Grothendieck group $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(B)$ of linear source $\mathcal{O}G$ -lattices with maximal vertex in an $\mathcal{O}G$ -block B .

CHAPTER 1

Preliminaries

The aim of this chapter is to present most of the definitions and results of the local representation theory of finite groups which will be useful in the following chapters. For convenience of the reader, they will be collected here without the proofs if already known. For a more complete and precise reference see [Alp86], [Ben98], [Ben98], [CR90], [Nav98] and others. From now on, p will denote a prime number and G a finite group.

1.1 Modules

Let $\mathbb{F}$ be a finite field of positive characteristic $p > 0$ and A a finite dimensional algebra over $\mathbb{F}$, then $A - \mathbf{mod}$ will denote the category of finitely generated (left) A -modules. Given a finite group G , then $\mathbb{F}G$ will denote the group algebra over the field $\mathbb{F}$.

Definition 1.1.1. A module $0 \neq L \in A - \mathbf{mod}$ is *simple* if 0 and L are the only submodules of L .

The cardinality of the set of isomorphism classes of simple A -modules will be denoted by $l(A)$.

Let $\mathcal{C}l(G)$ be the set of conjugacy classes of G , then $C \in \mathcal{C}l(G)$ is called p -regular if, and only if, p does not divide the order of g , for all $g \in C$. In 1935, R. Brauer proved that, if $\mathbb{F}$ is a splitting field for G , the cardinality of the set of isomorphism classes of simple $\mathbb{F}G$ -modules is equal to the number of p -regular conjugacy classes of G (cf. [Ben06, § 2.11]), i.e.,

$$l(\mathbb{F}G) = |\{p\text{-regular conjugacy classes of } G\}| \quad (1.1.1)$$

Let $ZA = \{z \in A | z \cdot a = a \cdot z \ \forall a \in A\}$ denote the center of the algebra A , and $\kappa(A) = \dim(ZA)$. Moreover, for all $X \subseteq G$, let $X^+ = \sum_{x \in X} x \in \mathbb{F}G$.

Definition 1.1.2. A composition series of $M \in A - \mathbf{mod}$ is a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M \quad (1.1.2)$$

such that M_i/M_{i-1} is simple $\forall i \in \{1, \dots, r\}$.

Then, by the *Jordan-Hölder theorem* (cf. [CR90, Theorem 1.17]), if

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M \text{ and } 0 = N_0 \subset N_1 \subset \cdots \subset N_s = M \quad (1.1.3)$$

are two composition series of $M \in A - \mathbf{mod}$, then $r = s$ and there is a permutation σ of $1, \dots, r$ such that $M_i/M_{i-1} \simeq N_{\sigma(i)}/N_{\sigma(i)-1}$. The terms M_i/M_{i-1} are called *composition factors* of M and r *length* of M . Moreover, $[M : L]$ will denote the multiplicity of the simple module L as a composition factor of M .

Definition 1.1.3. Let $0 \neq M \in A - \mathbf{mod}$, M is called *indecomposable* if there is not decomposition $M = M' \oplus M''$, where M' and M'' are proper submodules of M .

In this context an important role is played by the *Krull-Schmidt theorem* (cf. [CR90, Theorem 6.12]): let $M \in A - \mathbf{mod}$ such that $M = M_1 \oplus \cdots \oplus M_r = N_1 \oplus \cdots \oplus N_s$, where $M_1, \dots, M_r, N_1, \dots, N_s$ are indecomposable submodules. Thus $s = r$ and there is a permutation π of $1, \dots, r$ such that $M_i \simeq N_{\pi(i)}$, $\forall i \in \{1, \dots, r\}$. In particular, $M_1, \dots, M_r$ will be called *components* of M .

For $N \in A - \mathbf{mod}$, let $N|M$ denote that $N \in A - \mathbf{mod}$ is a direct summand of M , i.e., there exists $N' \in A - \mathbf{mod}$ such that $M \simeq N \oplus N'$.

Definition 1.1.4. An algebra A has *finite representation type* if the number of isomorphism classes of indecomposable A -modules is finite.

For example, the group algebra $\mathbb{F}G$ has finite representation type if, and only if, $\text{char}(\mathbb{F}) = 0$, or if $\text{char}(\mathbb{F}) = p$ and the Sylow p -subgroups of G are cyclic (cf [Ben06, Corollary 2.12.9]).

Let $\text{char}(\mathbb{F}) = p > 0$ and $G = \langle g \rangle$, $|G| = p^n$. Then the group algebra $\mathbb{F}G$ has precisely p^n isomorphism classes of indecomposable modules $M_1, \dots, M_{p^n}$.

Definition 1.1.5. An ideal I of A is *nilpotent* if $I^n = 0$ for some $n \in \mathbb{N}$. In particular A contains a unique largest nilpotent ideal, the *Jacobson radical* JA .

Thanks to the *Wedderburn-Artin theorem* (see [Ben06, § 1.2]), there is an isomorphism of algebras

$$A/JA \simeq \mathbb{F}^{d_1 \times d_1} \times \cdots \times \mathbb{F}^{d_l \times d_l} \quad (1.1.4)$$

where $l = l(A)$, $d_i \in \mathbb{N}$ and $\mathbb{F}^{d_i \times d_i}$ is the $\mathbb{F}$ -algebra of $(d_i \times d_i)$ -matrices.

Remark 1.1.6. Simple A -modules L_i , for $i \in \{1, \dots, l(A)\}$ are obtained by letting A act on $\mathbb{F}^{d_i}$ via $A \twoheadrightarrow A/JA \rightarrow \mathbb{F}^{d_i \times d_i}$

One of the most important results in “Representation Theory” is *Maschke’s theorem*:

Theorem 1.1.7. *Let G be a finite group. Then, $J\mathbb{F}G = 0$ if, and only if, the characteristic of $\mathbb{F}$ does not divide the order of G .*

Proof. See [Isa06, Theorem 1.9 and Problems page 11]. $\square$

Definition 1.1.8. For $M \in A - \mathbf{mod}$, the chain

$$M \supseteq (JA)M \supseteq (JA)^2M \supseteq \dots \subseteq 0 \quad (1.1.5)$$

is the *Loewy series* of M . The *Loewy length* of M is the minimal $t \in \mathbb{N}_0$ such that $(JA)^t M = 0$. The A -modules $(JA)^{i-1}M/(JA)^i M$ for $i \in \{1, \dots, t\}$ are the *Loewy layers* of M .

If $(JA)M = 0$, then M is called *semisimple*. In this case

$$M = L_1 \oplus \dots \oplus L_s, \quad (1.1.6)$$

with simple $L_1, \dots, L_s \in A - \mathbf{mod}$.

Definition 1.1.9. Let $M \in A - \mathbf{mod}$, if $M \simeq A^n$ (regular module) for some $n \in \mathbb{N}_0$, then M is called *free*. A module $P \in A - \mathbf{mod}$ is called *projective* if $P|M$ for some free $M \in A - \mathbf{mod}$.

Remark 1.1.10. If $P \in A - \mathbf{mod}$ is indecomposable and projective, then $(JA)P$ is the only maximal submodule of P . Moreover,

$$P \mapsto P/(JA)P \quad (1.1.7)$$

gives a bijection between the isomorphism classes of indecomposable projective A -modules and the isomorphism classes of simple A -modules.

For example, if $P \in \mathbb{F}G - \mathbf{mod}$ is indecomposable and projective, then P has a unique simple submodule L such that $L \simeq P/(J\mathbb{F}G)P$.

Let $P_1, \dots, P_{l(A)}$ represent the isomorphism classes of indecomposable and projective A -modules. Let $L_i \simeq P_i/(JA)P_i$, for $i \in \{1, \dots, l(A)\}$ and set $c_{ij} = [P_i : L_j] \in \mathbb{N}_0$ for $i, j \in \{1, \dots, l(A)\}$. The integers c_{ij} are called *Cartan invariants* of A and the matrix $C = (c_{ij}) \in \mathbb{N}_0^{l(A) \times l(A)}$ is called *Cartan matrix* of A . In particular, if $\mathbb{F}$ is a splitting field for G the Cartan matrix C of $\mathbb{F}G$ is symmetric (cf. [Ben06, Remark page 16]) and $\det(C) = p^\alpha$ (cf. [Ben06, Theorem 2.16.5]). The elementary divisors of C are the order of the Sylow p -subgroup of $C_G(g_i)$, where $g_1, \dots, g_{l(A)}$ represent the p -regular conjugacy classes of G .

1.2 Blocks

1.2.1 Idempotents

Definition 1.2.1. If $e^2 = e \in A$, then e is an *idempotent* of A . Two idempotents e and f are *orthogonal* if $ef = 0 = fe$. An idempotent $0 \neq e \in A$ is *primitive* if one can not write $e = e' \oplus e''$, with nonzero orthogonal idempotents $e', e'' \in A$.

Remark 1.2.2. If $e = e^2 \in A$, then $e(1 - e) = 0$ and $(1 - e)^2 = 1 - e$.

Then $A = Ae \oplus A(1 - e)$, where $Ae \in A - \mathbf{mod}$ is projective. For idempotents $e, f \in A$, $Ae \simeq Af$ if, and only if, $f = ueu^{-1}$ for some invertible element u in A . This induces a bijection between the conjugacy classes of primitive idempotents in A and the isomorphism classes of indecomposable projective modules, i.e., $e \mapsto Ae$ (cf. [Ben06, pages 11-12]).

Theorem 1.2.3 (Lifting theorem). *Let I be an ideal of A and let $\varepsilon = \varepsilon^2$ be an idempotent of A/I . Then $\varepsilon = e + I$ for some idempotent $e \in A$. If ε is primitive in A/I , then e can be chosen to be primitive in A .*

Proof. See [Ben06, Corollary 1.5.2]. □

Definition 1.2.4. An idempotent $e \in ZA$ which is primitive in ZA is a *block idempotent* of A

1.2.2 Blocks

It is important to remind that an algebra A contains finitely many block idempotents $e_1, \dots, e_r$. Moreover, $\forall i \neq j, e_i e_j = 0$, and $e_1 + \dots + e_r = 1$. For $i \in \{1, \dots, r\}$, $B_i = Ae_i = e_i A$ is a block ideal of A . Each B_i is an $\mathbb{F}$ -algebra with identity element e_i , called a *block algebra*. Then

$$A = B_1 \oplus \dots \oplus B_r. \quad (1.2.1)$$

In this context, $\text{Bl}(A)$ will denote the set of block ideals of A , and $\text{Bl}(G)$ will denote the set of blocks of the group algebra $\mathbb{F}G$. In particular a block B of $\mathbb{F}G$ is an indecomposable two-sided ideal of the group algebra $\mathbb{F}G$. A block B of $\mathbb{F}G$ will equivalently be called *block of G* , an *$\mathbb{F}G$ -block* or a *p -block*.

Let $M \in A - \mathbf{mod}$, then

$$M = B_1 M \oplus \dots \oplus B_r M \quad (1.2.2)$$

with submodules $B_i M$. If M is indecomposable, then $M = B_i M$ for a unique $i \in \{1, \dots, r\}$ and $B_j M = 0$ for all $j \neq i$. Thus M becomes a B_i -module and it is said that M belongs to the block B_i . This gives a partition of the isomorphism classes of indecomposable A -modules into a disjoint union

of isomorphism classes of indecomposable B_i -modules, for $i \in \{1, \dots, r\}$. Similarly, it is possible to obtain a partition of the isomorphism classes of simple A -modules into a disjoint union of isomorphism classes of simple B_i -modules, for $i \in \{1, \dots, r\}$. Hence,

$$l(A) = l(B_1) + \dots + l(B_r). \quad (1.2.3)$$

Since $ZA = ZB_1 \oplus \dots \oplus ZB_r$,

$$\kappa(A) = \kappa(B_1) + \dots + \kappa(B_r). \quad (1.2.4)$$

This brings to a decomposition of the Cartan matrix C of A :

$$\begin{bmatrix} C_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_r \end{bmatrix} \quad (1.2.5)$$

where C_i is the Cartan matrix of the block B_i .

1.2.3 The Brauer homomorphism and theorems

Given a field $\mathbb{F}$ of positive characteristic, for any subgroup Q of G

$$\begin{aligned} \text{Br}_Q : Z\mathbb{F}G &\rightarrow Z\mathbb{F}C_G(Q) \\ \sum_{g \in G} \alpha_g \cdot g &\mapsto \sum_{g \in C_G(Q)} \alpha_g \cdot g \end{aligned} \quad (1.2.6)$$

is a homomorphism of algebras, the *Brauer homomorphism* with respect to the subgroup Q .

Definition 1.2.5. Let $B \in \text{Bl}(G)$ be a p -block with unity element e_B . Let D be a maximal p -subgroup of G , with respect to the inclusion, such that

$$\text{Br}_D(e_B) \neq 0. \quad (1.2.7)$$

Then D is called a *defect group* of B (cf. [Sam14, Definition 1.4]).

Let us observe that there are different equivalent ways to define a defect group of a block, see for example [Nav98, Chapter 4, in particular Theorem 4.11] and [Ben06, §2.7, §2.8 and in particular Theorem 2.8.2].

Remark 1.2.6. If D is a defect group of a block B , then D is unique up to conjugation (cf. [Isa06, Corollary 15.36]). So, $\text{df}(B) = {}^G D = \{gD \mid g \in G\}$ will denote the set of defect groups of the block B . In the following a block B will be said to have defect groups $\text{df}(B)$.

If $|D| = p^d$, $d \in \mathbb{N}_0$, then $d = d(B)$ is the *defect* of the block B . The defect group of a block gives some measure of how complicated the representation theory of the block is. In fact a block has defect 0 if, and only if, there is only one indecomposable module in the block (cf 1.2.15), and it has cyclic defect group if, and only if, there are only finitely many indecomposable modules in the block (cf [Ben06, Corollary 2.12.9]).

Definition 1.2.7. Let $\mathbb{F}$ be a splitting field of positive characteristic. Let H be a subgroup of G and $b \in \text{Bl}(H)$. The natural projection

$$\begin{aligned} Pr_H^G: \mathbb{F}G &\rightarrow \mathbb{F}H \\ \sum_{g \in G} \alpha_g \cdot g &\mapsto \sum_{g \in H} \alpha_g \cdot g \end{aligned} \quad (1.2.8)$$

is linear and $Pr_H^G(Z\mathbb{F}G) \subseteq Z\mathbb{F}H$. If ω_b is the central homomorphism corresponding to b (cf. [Ben06, § 1.6]) and

$$\omega_b \circ Pr_H^G: Z\mathbb{F}G \rightarrow \mathbb{F} \quad (1.2.9)$$

is a homomorphism of algebras, then $\omega_b \circ Pr_H^G = \omega_B$ for a unique block $B \in \text{Bl}(G)$ - the *induced block* $B = b^G$ (cf. [Nav98, page 87]).

If $C_G(d) \subseteq H$ for a defect group d of b , then b is called *admissible* in G . In this case b^G is always defined. Whenever b^G is defined, then there are defect groups d of b and D of $B = b^G$ such that $d \subseteq D$.

Theorem 1.2.8 (Brauer's first main theorem). *If G is a finite group and $D \subseteq G$ is a p -subgroup, then $b \mapsto b^G$ induces a bijection between blocks of $\mathbb{F}N_G(D)$ with defect group D and blocks of $\mathbb{F}G$ with defect group D .*

Proof. See [Ben06, Theorem 2.8.6] □

Then b and $B = b^G$ are said to be in *Brauer correspondence*.

If e_B and e_b denote the blocks idempotents of B and b respectively, then $\text{Br}_D(e_B) = e_b$.

Also the following properties of defect groups hold.

- If D is a defect group of $B \in \text{Bl}(G)$, then D is a Sylow p subgroup of $C_G(g)$ for some p -regular element $g \in G$, i.e., D is a defect group of the conjugacy class of $g \in G$ (cf. [Ben06, Page 47]).
 - If Q is an arbitrary p -subgroup of G , then the number of blocks of $\mathbb{F}G$ with defect group Q is less or equal to the number of p -regular conjugacy classes of G with defect group Q . If Q is a Sylow p -subgroup, then the equality holds.
-

Theorem 1.2.9 (Green). *Let D be a defect group of $B \in \text{Bl}(G)$ and let S be a Sylow p -subgroup of G containing D . Then $D = S \cap gSg^{-1}$ for some $g \in C_G(D)$. In particular, $O_p(G) \subseteq D$. Thus $D = O_p(N_G(D))$, i.e., D is a radical p -subgroup of G .*

Proof. See [Alp86, Theorem 6, §IV.13] □

Let K be a normal subgroup of G , then G acts on $\mathbb{F}K$ permuting its the blocks.

Definition 1.2.10. If B is a block of $\mathbb{F}G$ and b is a block of $\mathbb{F}K$ such that $Bb \neq 0$, then B covers b .

For a block $B \in \text{Bl}(G)$, the blocks of $\mathbb{F}K$ covered by B form a single G -orbit. Their number is given by the index $|G : I_G(b)|$, where $b \in \text{Bl}(H)$ is covered by B and $I_G(b) = \{g \in G | gbg^{-1} = b\}$ is the *inertia group* of b in G . Let B , b and β denote a block of $\mathbb{F}G$, $\mathbb{F}K$ and $\mathbb{F}I$ respectively, then the following result holds.

Theorem 1.2.11 (Fong-Reynolds). *Let $K \trianglelefteq G$, $b \in \text{Bl}(K)$ and $I = I_G(b)$.*

- (i) *Then $\beta \mapsto \beta^G$ induces a bijection between blocks of $\mathbb{F}I$ covering b and blocks of $\mathbb{F}G$ covering b .*
- (ii) *The number of irreducible characters in the block β is equal to the number of irreducible characters in the block β^G .*
- (iii) *If D is a defect group of β , then D is also a defect group of β^G and $D \cap K$ is a defect group of b .*

Proof. See [Nav98, Theorem 9.14] □

Example 1.2.1. Let $G = S_4$, $K = V_4$ and $p = 3$. Then $\mathbb{F}K = b_1 \oplus b_2 \oplus b_3 \oplus b_4$, where $b_i \simeq \mathbb{F}$ for $i \in \{1, \dots, 4\}$. In particular, G acts on $\mathbb{F}K$ with orbits $\{b_1\}$ and $\{b_2, b_3, b_4\}$. Let $b = b_2$; then $I = I_G(P)$ is a Sylow p -subgroup of G and there are two blocks β_1 and β_2 of $\mathbb{F}I$ covering b . Since $\beta_i \simeq \mathbb{F}$, $B_i \simeq \beta_i^G \simeq \mathbb{F}^{3 \times 3}$. Also, β_i , β_i^G and B have defect 0.

Remark 1.2.12. Let $N \trianglelefteq G$ and let $\nu_N: \mathbb{F}G \rightarrow \mathbb{F}[G/N]$ be the canonical projection. If $B \in \text{Bl}(G)$, then $\nu_N(B) = \bar{B}_1 \oplus \dots \oplus \bar{B}_t$, where $\bar{B}_1, \dots, \bar{B}_t$ are blocks of G/N . The blocks $\bar{B}_1, \dots, \bar{B}_t$ are said to be *dominated* by B .

Let D be a defect group of B , then each $\bar{B}_i$ has a defect group contained in DN/N and at least one $\bar{B}_i$ has defect group DN/N . So, in general the blocks of $\mathbb{F}N_G(D)$ with defect group D are not in one-one correspondence with blocks of $\mathbb{F}N_G(D)/D$ of defect zero.

To consider the defect zero case, there is the following extended version of Theorem 1.2.8 (cf. [Ben06, §2.8.6a]).

Theorem 1.2.13 (Brauer's Extended First Main theorem). *Let G be a finite group, then there is a one-to-one correspondence between the following.*

- (i) *Blocks of G with defect group D .*
- (ii) *Blocks of $N_G(D)$ with defect group D .*
- (iii) *$N_G(D)$ -conjugacy classes of blocks of $C_G(D)$ with D as defect group in $N_G(D)$.*
- (iv) *(Assuming $\mathbb{F}$ is a splitting field for $C_G(D)$) $N_G(D)$ -conjugacy classes of blocks b of $C_G(D)$ with D as defect group in $DC_G(D)$ and index $|N_G(b) : DC_G(D)|$ coprime to p .*
- (v) *(Assuming $\mathbb{F}$ is a splitting field for $C_G(D)$) $N_G(D)$ -conjugacy classes of blocks b of defect zero of $DC_G(D)/D$ with $|N_G(b) : DC_G(D)|$ coprime to p .*

The trivial $\mathbb{F}G$ module is the field $\mathbb{F}$ where G acts via $g.\alpha = \alpha$ for $g \in G$ and $\alpha \in \mathbb{F}$. The trivial module $\mathbb{F}$ is an irreducible $\mathbb{F}G$ -module, then $\mathbb{F}$ belongs to a unique block $B_0 = B_0(\mathbb{F}G)$, which is called the *principal block* of $\mathbb{F}G$.

Theorem 1.2.14 (Brauer's third main theorem). *If b is a block of the subgroup H of G , D is a defect group of b and $C_G(D) \subseteq H$, then $b^G = b_0(G)$, if, and only if, $b = b_0(H)$.*

Proof. See [Alp86, Theorem 1, §IV.16] □

Another important result about blocks is given by the following proposition.

Proposition 1.2.15 (Brauer). *Let F be a splitting field for G and let B be a block of $\mathbb{F}G$. Then the following are equivalent:*

- (i) $d(B) = 0$
- (ii) $B \simeq \mathbb{F}^{n \times n}$ for some n
- (iii) $\kappa(B) = 1$
- (iv) *There is a simple projective $\mathbb{F}G$ -module belonging to B*

Proof. See [Ben06, Corollary 2.7.5] □

Then there is a bijection between blocks of defect 0 in $\mathbb{F}G$ and isomorphism classes of simple projective $\mathbb{F}G$ -modules. This result can be seen as a generalization of the Maschke's theorem.

1.3 Modular systems

In the representation theory of finite groups the field (or ring) considered to construct the group ring plays a key role. That is why the following definition is so important.

Definition 1.3.1. Let p be a prime number. A p -modular system is a triple $(\mathbb{K}, \mathcal{O}, \mathbb{F})$, where:

- $\mathcal{O}$ is a complete discrete valuation domain of characteristic 0;
- $\mathbb{K} = \text{quot}(\mathcal{O})$;
- $\mathbb{F} = \text{res}(\mathcal{O})$ is of characteristic p .

If $\mathbb{K}$ and $\mathbb{F}$ are splitting for G and all its subgroup, then $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ is called *splitting p -modular system*.

A finitely generated $\mathcal{O}G$ -module M is an $\mathcal{O}G$ -lattice if it is - considered as $\mathcal{O}$ -module - free. Let $\mathcal{O}G\text{-lat}$ be the category of $\mathcal{O}G$ -lattices. If $M \in \mathcal{O}G\text{-lat}$, then $M_{\mathbb{K}} = \mathbb{K} \otimes_{\mathcal{O}} M$ is the associated $\mathbb{K}G$ -module and $M_{\mathbb{F}} = \mathbb{F} \otimes_{\mathcal{O}} M$ is the associated $\mathbb{F}G$ -module

$$\begin{array}{ccc} & \mathcal{O}G\text{-lat} & \\ \swarrow -_{\mathbb{K}} & & \searrow -_{\mathbb{F}} \\ \mathbb{K}G\text{-mod} & & \mathbb{F}G\text{-mod} \end{array} \quad (1.3.1)$$

The importance of the category $\mathcal{O}G\text{-lat}$ will be analysed in the following chapters.

As seen before, a decomposition of $\mathbb{F}G$ into blocks $\mathbb{F}G = B_1 \oplus \cdots \oplus B_s$ corresponds to a decomposition of the identity element $1 = e_1 + \cdots + e_s$ as a sum of orthogonal primitive central idempotents. The correspondence is given by $B_i = e_i \cdot \mathbb{F}G$. Since both $Z(\mathbb{F}G)$ and $Z(\mathcal{O}G)$ have a basis consisting of the conjugacy class sums in G , it follows that the *reduction modulo p* is a surjective map $Z(\mathcal{O}G) \rightarrow Z(\mathbb{F}G)$, and so by [Ben98, Theorem 1.9.4(iii)] the idempotents $e_i \in \mathbb{F}G$ may be lifted to orthogonal primitive central idempotents $f_i \in \mathcal{O}G$. Then

$$\mathcal{O}G = \hat{B}_1 \oplus \cdots \oplus \hat{B}_s, \quad (1.3.2)$$

where $\hat{B}_i = f_i \cdot \mathcal{O}G$ and $B_i = \mathbb{F} \otimes_{\mathcal{O}} \hat{B}_i$.

Remark 1.3.2. All the consideration of § 1.2 can be “lifted” to the $\mathcal{O}G$ -blocks. As before $\text{Bl}(G)$ will denote the set of $\mathcal{O}G$ -blocks. It will be clear from the context if the blocks considered are $\mathbb{F}G$ or $\mathcal{O}G$ -blocks.

Let $M_1, \dots, M_k$ be all irreducible $\mathbb{K}G$ -modules and let $\chi_1, \dots, \chi_k$ be the correspondent (irreducible) characters. For $K \in \mathcal{C}l(G)$, let $\hat{K} = \sum_{x \in K} x$. Then the set $\{K^+ \mid K \in \mathcal{C}l(G)\}$ is an $\mathcal{O}$ -basis of $Z\mathcal{O}G$. Moreover for every

K^+ , $\frac{|K^+|\chi_i(g)}{\chi_i(1)}$ is an algebraic integer (see [Isa06, Theorem 3.7]). Then for $i = 1, \dots, k$, it is possible to define a map (cf. [Nav98, Chapter 3])

$$\begin{aligned} \omega_i : Z\mathbb{F}G &\rightarrow \mathbb{F} \\ K^+ &\mapsto \frac{K^+\chi_i(x_k)}{\chi_i(1)} + \mathfrak{p} \end{aligned} \quad (1.3.3)$$

where $x_K \in \mathbb{K}$ and $\mathfrak{p}$ is the unique maximal ideal of $\mathcal{O}$. Then ω_i is a homomorphism of algebras and there is a unique block B of G such that $\omega_i = \omega_B$. In this case the irreducible module M_i belongs to the block B . It follows that there is a partition of irreducible $\mathbb{K}G$ -modules (and then characters) in the $\mathbb{F}G$ -blocks, and thus in the $\mathcal{O}G$ -blocks.

1.4 Vertices and sources

In this section let $\Gamma \in \{\mathbb{F}, \mathcal{O}\}$.

Let $M \in \Gamma G - \mathbf{mod}$ and $H \leq G$. If $M|_{\text{ind}_H^G(\text{res}_H^G(M))}$, then M is H -projective or projective relative to H .

For example, if S is a Sylow p -subgroup of G , then every $M \in \Gamma G - \mathbf{mod}$ is relatively S -projective. If M belongs to a block with defect group D , then M is relatively D -projective (cf. [Ben06, Proposition 2.7.4]). Moreover, $M \in \Gamma G - \mathbf{mod}$ is relatively 1-projective if, and only if, M is projective.

Definition 1.4.1. Let $M \in \Gamma G - \mathbf{mod}$ be indecomposable. A subgroup V of G is called a *vertex* of M if M is relatively V -projective, but non relatively W -projective for any proper subgroup W of V .

In particular, the vertices of M form a conjugacy class of p -subgroup of G ; $\mathbf{v}(M) = {}^G V$ will denote the G -conjugacy class of vertices of M (cf. [Ben98, Proposition 3.10.2]). It follows from the definition, that the indecomposable projective ΓG -modules have vertex 1. Let M be an indecomposable ΓG -module belonging to a block B and let V be a vertex of M , then $V \subseteq D$, for a suitable defect group D of B . If $B \in \text{Bl}(G)$ with defect group D , then D is a vertex of some simple ΓG -module belonging to B (cf. [Ben06, Remark page 48 and § 2.12]).

A module M is said to have *maximal vertex* if $\mathbf{v}(M) = \text{Syl}_p(G)$ is the set of Sylow p -subgroups of G ; for example the trivial ΓG -module has maximal vertex. More generally, if $M \in \Gamma G - \mathbf{mod}$ is indecomposable with vertex $V \leq S \in \text{Syl}_p(G)$, then $|S : V| \mid \text{rk}(M)$.

Theorem 1.4.2 (Green's indecomposability theorem). *Suppose $K \trianglelefteq G$ such that G/K is a p -group and let $N \in \Gamma K - \mathbf{mod}$ be indecomposable. Then $\text{ind}_K^G(N)$ is indecomposable.*

Proof. See [Ben98, Theorem 3.13.3]. □

Definition 1.4.3. Let $M \in \Gamma G - \mathbf{mod}$ be indecomposable and let V be a vertex of M . Then there is an indecomposable module $S \in \Gamma V - \mathbf{mod}$ such that $M|_{\text{ind}_V^G(S)}$. The module S is a V -source of M (or a source of M). Moreover, S is unique up to isomorphism and $N_G(V)$ -conjugation. Let $\mathbf{s}_V(M) = {}^{N_G(V)}S$ denote the set of isomorphism classes of left ΓV -modules which are sources of M .

1.4.1 Green correspondence

One of the principal tools in local representation theory is given by *Green correspondence* (see [Ben06, §2.12]). Let us consider the following situation. Let $P \leq G$ be a p -subgroup and $H \leq G$ such that $N_G(P) \leq H \leq G$. Let

$$\begin{aligned}\mathfrak{X} &= \{Q \mid Q \leq P \cap gPg^{-1} \text{ for some } g \in G \setminus H\} \\ \mathfrak{Y} &= \{Q \mid Q \leq H \cap gPg^{-1} \text{ for some } g \in G \setminus H\} \\ \mathfrak{Z} &= \{Q \leq P \mid Q \notin \mathfrak{X}\}\end{aligned}\tag{1.4.1}$$

Theorem 1.4.4 (Green Correspondence). *If $M \in \Gamma G - \mathbf{mod}$ is indecomposable with vertex $Q \in \mathfrak{Z}$, then $\text{res}_H^G(M)$ has a unique (up to isomorphism) component $\mathbf{f}(M)$ with the same vertex Q . Moreover, $\mathbf{f}(M)$ has multiplicity one in $\text{res}_H^G(M)$ and the other components of $\text{res}_H^G(M)$ have vertices in $\mathfrak{Y}$. If $N \in \Gamma H - \mathbf{mod}$ is indecomposable with vertex $Q \in \mathfrak{Z}$, then $\text{ind}_H^G(N)$ has a unique (up to isomorphism) component $\mathbf{g}(N)$ with the same vertex. Moreover, $\mathbf{g}(N)$ has multiplicity one in $\text{ind}_H^G(N)$ and the other components of $\text{ind}_H^G(N)$ have vertices in $\mathfrak{X}$.*

This gives mutually inverse bijections between isomorphism classes of indecomposable ΓG -modules with vertex in $\mathfrak{Z}$ and isomorphism classes of indecomposable ΓH -modules with vertex in $\mathfrak{Z}$ preserving vertices and sources.

The following theorem, due to D. W. Burry, J. F. Carlson (cf. [BC82]) and L. Puig (cf. [Pui81]), gives us more information about the situation where Green correspondence holds.

Theorem 1.4.5. *Suppose that H is a subgroup of G containing $N_G(D)$. Let V be an indecomposable ΓG -module such that $\text{res}_H^G(V)$ has a direct summand M with vertex D . Then V has vertex D , and V is the Green correspondent $\mathbf{g}(M)$.*

Proof. See [Ben98, Theorem 3.12.3]. □

1.5 Trivial and linear source lattices

Trivial and linear source lattices will play a central role in this thesis; in this section the definition and the main properties of these families of lattices will be introduced (cf. [BK00] and [Bro85]). Let $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ be a splitting p -modular system.

Definition 1.5.1. Let $M \in \mathcal{O}G - \mathbf{lat}$, M is called a *permutation lattice* if M has a G -stable finite $\mathcal{O}$ -basis.

The category $\mathcal{O}G - \mathbf{per}$ of permutation lattices is a full subcategory of $\mathcal{O}G - \mathbf{lat}$. Moreover, the class of permutation lattices is stable under conjugation, restriction, induction and closed with respect to direct sum and tensor product.

Definition 1.5.2. A lattice $T \in \mathcal{O}G - \mathbf{lat}$ is called *trivial source $\mathcal{O}G$ -lattice* if all its indecomposable direct summands have the trivial module as a source.

Proposition 1.5.3. Let $T \in \mathcal{O}G - \mathbf{lat}$ and $P \in \text{Syl}_p(G)$. Then the following are equivalent:

- (i) The lattice T is a trivial source lattice.
- (ii) The module $\text{res}_P^G(T)$ is a permutation $\mathcal{O}P$ -module.
- (iii) The module T is isomorphic to a direct summand of a permutation $\mathcal{O}G$ -module.

Proof. See [Bro85]. □

Because of Proposition 1.5.3.(ii), trivial source lattices are also called *p-permutation lattices*.

Proposition 1.5.4. Let H be a subgroup of G , let T and R be trivial source $\mathcal{O}G$ -lattices and V a trivial source $\mathcal{O}H$ -lattice. Then:

- (i) The lattices $T \oplus R$, $T \otimes_{\mathcal{O}} R$ and $\hat{T} = \text{Hom}_{\mathcal{O}}(T, \mathcal{O})$ are trivial source $\mathcal{O}G$ -lattices.
- (ii) The lattice $\text{res}_H^G(T)$ is a trivial source $\mathcal{O}H$ -lattice.
- (iii) The lattice $\text{ind}_H^G(V)$ is a trivial source $\mathcal{O}G$ -lattice.
- (iv) Every direct summand of V is a trivial source $\mathcal{O}G$ -lattice.

Proof. See [Bro85]. □

Definition 1.5.5. An $\mathcal{O}G$ -lattice M is called *monomial* if M is a direct sum of $\mathcal{O}G$ -lattices isomorphic to $\mathcal{O}G$ -lattices of the form $\text{ind}_H^G(W)$ for a subgroup $H \leq G$ and an $\mathcal{O}H$ -lattice W of $\mathcal{O}$ -rank 1.

Definition 1.5.6. Let M be indecomposable $\mathcal{O}G$ -lattice and let V be a vertex of M . The $\mathcal{O}G$ -lattice M is said to be an indecomposable *linear source $\mathcal{O}G$ -lattice*, if $S \in \mathbf{sv}(M)$ has $\mathcal{O}$ -rank 1, i.e., there exists $\varphi \in \text{Hom}_G(V, \mathcal{O}^*)$ such that $S \simeq \mathcal{O}_{\varphi}$, where $\mathcal{O}_{\varphi}$ is the $\mathcal{O}V$ -lattice with V -action given by φ . An $\mathcal{O}G$ -lattice $M = \coprod_{1 \leq j \leq r} M_j$, M_j indecomposable, is called a *linear source $\mathcal{O}G$ -module*, if every component M_j is a linear source $\mathcal{O}G$ -lattice.

It follows from the definition that every trivial source lattice is a linear source lattice. In analogy with Proposition 1.5.3 and Proposition 1.5.4, the following results hold.

Proposition 1.5.7. *Let $L \in \mathcal{OG} - \mathbf{lat}$ and $P \in \text{Syl}_p(G)$. Then the following are equivalent:*

- (i) *The lattice L is a linear source lattice.*
- (ii) *The module $\text{res}_P^G(T)$ is a monomial $\mathcal{OP}$ -module.*
- (iii) *The module L is isomorphic to a direct summand of a monomial $\mathcal{OG}$ -module.*

Proof. See [Bol98b, Proposition 1.2(a)] □

Proposition 1.5.8. *Let $H \leq G$, let T and R be linear source $\mathcal{OG}$ -lattices and V a linear source $\mathcal{OH}$ -lattice. Then:*

- (i) *The lattices $T \oplus R$, $T \otimes_{\mathcal{O}} R$ and $\hat{T} = \text{Hom}_{\mathcal{O}}(T, \mathcal{O})$ are linear source $\mathcal{OG}$ -lattices.*
- (ii) *The lattice $\text{res}_H^G(T)$ is a linear source $\mathcal{OH}$ -lattice.*
- (iii) *The lattice $\text{ind}_H^G(V)$ is a linear source $\mathcal{OG}$ -lattice.*
- (iv) *Every direct summand of V is a linear source $\mathcal{OG}$ -lattice.*

Proof. See [Bol98b, Proposition 1.2(b)] □

Let $\mathcal{OG} - \mathbf{triv}$ and $\mathcal{OG} - \mathbf{lin}$ denote the fully subcategory of $\mathcal{OG} - \mathbf{lat}$ which contain $\mathcal{OG} - \mathbf{per}$ and whose objects are trivial source $\mathcal{OG}$ -lattices and linear source $\mathcal{OG}$ -lattices respectively, then the following inclusions hold:

$$\mathcal{OG} - \mathbf{per} \subseteq \mathcal{OG} - \mathbf{triv} \subseteq \mathcal{OG} - \mathbf{lin} \subseteq \mathcal{OG} - \mathbf{lat}. \quad (1.5.1)$$

1.5.1 Grothendieck rings and groups

Let $\text{IL}_{\mathcal{O}}(G)$ and $\text{ITr}_{\mathcal{O}}(G)$ denote the set of isomorphism classes of indecomposable linear source $\mathcal{OG}$ -lattices and the subset of isomorphism classes of indecomposable trivial source $\mathcal{OG}$ -lattices, respectively, and let

$$\text{IL}_{\mathcal{O}}^{\text{mx}}(G) \subseteq \text{IL}_{\mathcal{O}}(G) \quad (1.5.2)$$

and

$$\text{ITr}_{\mathcal{O}}^{\text{mx}}(G) \subseteq \text{ITr}_{\mathcal{O}}(G) \quad (1.5.3)$$

denote, respectively, the set of isomorphism classes of indecomposable linear and trivial source $\mathcal{OG}$ -lattices with maximal vertex.

Then it is possible to define

$$\text{IL}_{\mathcal{O}}^{\prec}(G) = \text{IL}_{\mathcal{O}}(G) \setminus \text{IL}_{\mathcal{O}}^{\text{mx}}(G) \quad (1.5.4)$$

and

$$\mathrm{ITr}_{\mathcal{O}}^{\prec}(G) = \mathrm{ITr}_{\mathcal{O}}(G) \setminus \mathrm{ITr}_{\mathcal{O}}^{\mathrm{mx}}(G) \quad (1.5.5)$$

as, respectively, the set of isomorphism classes of indecomposable linear and trivial source lattices with vertex set strictly contained in $\mathrm{Syl}_p(G)$.

Let $\mathbf{A}_{\mathcal{O}}(G)$ denote the *Green ring* of $\mathcal{O}G$, i.e., the free abelian group on the set of isomorphism classes of indecomposable $\mathcal{O}G$ -lattices where the ring structure is given by the tensor product. Let $\mathbf{T}_{\mathcal{O}}(G)$ and $\mathbf{L}_{\mathcal{O}}(G)$ denote the *Grothendieck ring* of linear source and trivial source $\mathcal{O}G$ -lattices with respect to direct sum decomposition, i.e.,

$$\begin{aligned} \mathbf{L}_{\mathcal{O}}(G) &= \mathrm{span}_{\mathbb{Z}}\{[L] \mid [L] \in \mathrm{IL}_{\mathcal{O}}(G)\} \\ \mathbf{T}_{\mathcal{O}}(G) &= \mathrm{span}_{\mathbb{Z}}\{[T] \mid [T] \in \mathrm{ITr}_{\mathcal{O}}(G)\}. \end{aligned} \quad (1.5.6)$$

Analogously, let

$$\begin{aligned} \mathbf{L}_{\mathcal{O}}^{\prec}(G) &= \mathrm{span}_{\mathbb{Z}}\{[L] \mid [L] \in \mathrm{IL}_{\mathcal{O}}^{\prec}(G)\} \\ \mathbf{L}_{\mathcal{O}}^{\mathrm{mx}}(G) &\simeq \mathbf{L}_{\mathcal{O}}(G)/\mathbf{L}_{\mathcal{O}}^{\prec}(G) \end{aligned} \quad (1.5.7)$$

and

$$\begin{aligned} \mathbf{T}_{\mathcal{O}}^{\prec}(G) &= \mathrm{span}_{\mathbb{Z}}\{[T] \mid [T] \in \mathrm{ITr}_{\mathcal{O}}^{\prec}(G)\} \\ \mathbf{T}_{\mathcal{O}}^{\mathrm{mx}}(G) &\simeq \mathbf{T}_{\mathcal{O}}(G)/\mathbf{T}_{\mathcal{O}}^{\prec}(G). \end{aligned} \quad (1.5.8)$$

Note that, in contrast with $\mathbf{A}_{\mathcal{O}}(G)$, the rings $\mathbf{T}_{\mathcal{O}}(G)$ and $\mathbf{L}_{\mathcal{O}}(G)$ are always finitely generated. In particular, one obtains the following rings' inclusions:

$$\mathbf{T}_{\mathcal{O}}(G) \subseteq \mathbf{L}_{\mathcal{O}}(G) \subseteq \mathbf{A}_{\mathcal{O}}(G). \quad (1.5.9)$$

Let B be an $\mathcal{O}G$ -block, as done for the group G it is possible to define the sets $\mathrm{IL}_{\mathcal{O}}(B)$ and $\mathrm{ITr}_{\mathcal{O}}(B)$ of isomorphism classes of indecomposable linear and trivial source $\mathcal{O}B$ -lattices, i.e., $\mathcal{O}G$ -lattices belonging to B and the sets $\mathrm{IL}_{\mathcal{O}}(B)$, $\mathrm{IL}_{\mathcal{O}}^{\prec}(B)$ and $\mathrm{IL}_{\mathcal{O}}^{\mathrm{mx}}(B)$.

Let $\mathbf{T}_{\mathcal{O}}(B)$ and $\mathbf{L}_{\mathcal{O}}(B)$ denote respectively the *Grothendieck group* of linear source and trivial source $\mathcal{O}B$ -lattices with respect to direct sum decomposition, i.e.,

$$\begin{aligned} \mathbf{L}_{\mathcal{O}}(B) &= \mathrm{span}_{\mathbb{Z}}\{[L] \mid [L] \in \mathrm{IL}_{\mathcal{O}}(B)\} \\ \mathbf{T}_{\mathcal{O}}(B) &= \mathrm{span}_{\mathbb{Z}}\{[T] \mid [T] \in \mathrm{ITr}_{\mathcal{O}}(B)\}. \end{aligned} \quad (1.5.10)$$

As done in (1.5.7) and (1.5.8), also the sets $\mathbf{L}_{\mathcal{O}}^{\prec}(B)$, $\mathbf{L}_{\mathcal{O}}^{\mathrm{mx}}(B)$, $\mathbf{T}_{\mathcal{O}}^{\prec}(B)$, $\mathbf{T}_{\mathcal{O}}^{\mathrm{mx}}(B)$ can be defined considering only the $\mathcal{O}G$ -lattices belonging to the block B .

Let $\mathbf{R}_{\mathbb{K}}(G)$ and $\mathbf{R}_{\mathbb{F}}(G)$ denote the *Grothendieck ring* of finitely generated left $\mathbb{K}G$ -modules and $\mathbb{F}G$ -modules respectively, i.e., $\mathbf{R}_{\mathbb{K}}(G)$ and $\mathbf{R}_{\mathbb{F}}(G)$ are the free $\mathbb{Z}$ -modules spanned by the set of isomorphism classes of irreducible left $\mathbb{K}G$ -modules and $\mathbb{F}G$ -modules respectively (cf. [Ser77, Proposition 40]). In particular, in the zero characteristic case the Grothendieck ring $\mathbf{R}_{\mathbb{K}}(G)$ coincides with the ring of the $\mathbb{K}G$ -representations, while in the positive characteristic case it is the quotient of the ring of the modular representations

and the ideal of the short exact sequences (cf. [Ben98, Chapter 5]).

As well known, in the ordinary case one can compute with representation easily considering their (ordinary) characters; the *Brauer characters* play a similar role in the modular context; for a complete reference see [Isa06], [Ben98, § 5.3] and others.

As before, if B is an $\mathcal{O}G$ -block, then $\mathbf{R}_{\mathbb{K}}(B)$ and $\mathbf{R}_{\mathbb{F}}(B)$ will denote the *Grothendieck group* of finitely generated $\mathbb{K}B$ -modules and $\mathbb{F}B$ -modules, i.e., the free $\mathbb{Z}$ -modules spanned by the set of isomorphism classes of irreducible $\mathbb{K}G$ -modules and $\mathbb{F}G$ -modules belonging to the block B .

1.5.2 Detecting trivial and linear source lattices

One major tool in the detection of trivial and linear source lattices is given by *Green correspondence*; in fact the bijections $\mathbf{f}$ and $\mathbf{g}$ defined in Theorem 1.4.4 map linear source lattices to linear source lattices and trivial source lattices to trivial source lattices.

A key result in the detection of trivial source $\mathcal{O}G$ -lattices is given by the following remark.

Remark 1.5.9. Let V be a fixed p -subgroup of G , then the set of isomorphism classes of indecomposable trivial source $\mathcal{O}G$ -lattices with vertex V is in bijective correspondence with the set of isomorphism classes of indecomposable projective $\mathbb{F}[N_G(V)/V]$ -modules (cf. [Bro85, Part 3.6])

The goal in this section is to obtain a similar result for linear source $\mathcal{O}G$ -lattices. The starting point of this consideration is the following *structure theorem* for indecomposable linear source lattices which has been established in a slightly different form by R. Boltje and B. Külshammer in [BK00, Theorem 3.3].

For a p -subgroup V of the group G , let $\hat{V} = \text{Hom}_{\text{grp}}(V, \mathcal{O}^*)$ denote the set of $\mathcal{O}$ -linear characters of V . In particular, $\hat{V}$ is a left $N_G(V)$ -set, where ${}^g\varphi$ is given by

$${}^g\varphi(x) = \varphi(g^{-1}xg), \quad (1.5.11)$$

for $g \in N_G(V)$, $\varphi \in \hat{V}$ and $x \in V$.

The stabilizer $I(\varphi) = \{g \in N_G(V) \mid {}^g\varphi = \varphi\}$ of φ coincides with the *inertia subgroup* of the $\mathcal{O}V$ -lattice $\mathcal{O}_\varphi$, where $\mathcal{O}_\varphi$ is - considered as $\mathcal{O}$ -module - isomorphic to $\mathcal{O}$ with left V -action given by φ . Moreover, for $\varphi \in \hat{V}$, $V \cdot C_G(V) \subseteq I(\varphi) \subseteq N_G(V)$.

By construction $\text{IL}_{\mathcal{O}}^{\text{mx}}(V) = \{[\mathcal{O}_\varphi] \mid \varphi \in \hat{V}\}$. Let $\text{ind}_V^{I(\varphi)}(\mathcal{O}_\varphi) = L_1 \oplus \cdots \oplus L_r$, where L_i are indecomposable $\mathcal{O}I(\varphi)$ -lattices. As $\ker(\varphi)$ is a normal subgroup of $I(\varphi)$, $\text{ind}_V^{I(\varphi)}(\mathcal{O}_\varphi)$ is an $\mathcal{O}I(\varphi)$ -lattice which is inflated from $\bar{I}(\varphi) = I(\varphi)/\ker(\varphi)$. Hence L_i are indecomposable $\mathcal{O}I(\varphi)$ -lattices which are inflated from $\bar{I}(\varphi)$. Let $\bar{V} = V/\ker(\varphi)$. Then $\bar{V}$ is a cyclic p -group contained in $Z(\bar{I}(\varphi))$, the center of $\bar{I}(\varphi)$. Let

$$\mathcal{O}\bar{I}(\varphi) = \text{ind}_{\ker(\varphi)}^{I(\varphi)}(\mathcal{O}) = Q_1 \oplus \cdots \oplus Q_s, \quad (1.5.12)$$

where Q_i are projective indecomposable $\mathcal{O}\bar{I}(\varphi)$ -lattices. For an $\mathcal{O}\bar{I}(\varphi)$ -lattice Q for which $\text{res}_{\bar{V}}^{\bar{I}(\varphi)}(Q)$ is a projective $\mathcal{O}\bar{V}$ -lattice, let

$$Q(\varphi) = Q / \langle (c - \varphi(c) \cdot \text{id}_Q) \cdot q \mid c \in \bar{V}, q \in Q \rangle, \quad (1.5.13)$$

e.g., $\text{ind}_V^{I(\varphi)}(\mathcal{O}_\varphi) = \text{ind}_{\ker(\varphi)}^{I(\varphi)}(\mathcal{O})(\varphi)$. As $\bar{V}$ is a normal p -subgroup of $\bar{I}(\varphi)$, $\bar{V}$ acts trivially on any simple $\mathbb{F}\bar{I}(\varphi)$ -module. Hence $\text{hd}(Q_j(\varphi)) = \text{hd}(Q_j)$, and thus $Q_j(\varphi)$ is an indecomposable $\mathcal{O}\bar{I}(\varphi)$ -module. This implies that $r = s$, and after renumbering one may also assume $L_j = Q_j(\varphi)$. By construction, $\tilde{Q}_j(\varphi) = \inf_{\bar{I}(\varphi)}^{I(\varphi)}(Q_j(\varphi))$ is an indecomposable linear source $\mathcal{O}I(\varphi)$ -lattice with vertex set $\{V\}$. The following theorem gives a complete description of indecomposable linear source lattices modulo Green correspondence.

Theorem 1.5.10. *Let V be a p -subgroup of G , and let M be an indecomposable linear source $\mathcal{O}G$ -lattice with vertex set ${}^G V$.*

- (a) *There exist an element $\varphi \in \hat{V}$ and a projective indecomposable $\mathcal{O}\bar{I}(\varphi)$ -lattice Q such that $\mathbf{f}(M) \simeq \text{ind}_{I(\varphi)}^{N_G(V)}(\tilde{Q}(\varphi))$.*
- (b) *If $\psi \in \hat{V}$ and Q_o is a projective indecomposable $\mathcal{O}\bar{I}(\psi)$ -lattice such that $\mathbf{f}(M) \simeq \text{ind}_{I(\psi)}^{N_G(V)}(\tilde{Q}_o(\varphi))$, then there exists $g \in N_G(V)$ such that $\psi = {}^g\varphi$ and $Q_o \simeq {}^gQ$.*
- (c) *Let $\varphi \in \hat{V}$, and let Q be a projective indecomposable $\mathcal{O}\bar{I}(\varphi)$ -lattice. Then $L = \text{ind}_{I(\varphi)}^{N_G(V)}(\tilde{Q}(\varphi))$ is an indecomposable linear source $\mathcal{O}N_G(V)$ -lattice, and $\mathbf{g}(L)$ is an indecomposable linear source $\mathcal{O}G$ -lattice.*

Proof. (a) Let $N = N_G(V)$, and let $L = \mathbf{f}(M)$ be its Green correspondent. Then L is an indecomposable linear source $\mathcal{O}N$ -lattice with vertex set $\{V\}$. Hence there exists $\varphi \in \hat{V}$ such that $\mathcal{O}_\varphi$ is a direct summand of $\text{res}_V^N(L)$. As V is normal in N , Clifford theory applies, i.e., there exists a direct summand $\tilde{Q}$ of $\text{ind}_V^{I(\varphi)}(\mathcal{O}_\varphi)$ such that $L \simeq \text{ind}_{I(\varphi)}^N(\tilde{Q})$ (see [Ben98, 3.13.2]). By the previously mentioned argument, one has $\tilde{Q} \simeq \tilde{Q}(\varphi)$ for some projective indecomposable $\mathcal{O}\bar{I}(\varphi)$ -lattice Q .

- (b) is a direct consequence of the fact that the source is uniquely determined modulo $N_G(V)$ -conjugation and the final remark of [Ben98, 3.13.2].
- (c) follows from [Ben98, 3.13.2] and Green correspondence.

□

Remark 1.5.11. Theorem 1.5.10 shows that there is a sets bijection

$$\begin{aligned} \chi : \text{IL}_{\mathcal{O}}(G) &\rightarrow \{ {}^G(V, \varphi, Q) \mid V \in \Sigma_p(G), \varphi \in \hat{V}, Q \text{ projective} \\ &\quad \text{indecomposable } \mathcal{O}\bar{I}(\varphi) \text{ -- lattice} \}. \end{aligned} \quad (1.5.14)$$

Another consequence of Theorem 1.5.10 is the following.

Theorem 1.5.12. *Let N be a finite group with a normal Sylow p -subgroup P . Then $-\mathbb{K}$ induces a bijection*

$$\tilde{\kappa}_N^\circ: \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(N) \rightarrow \text{Irr}_{\mathbb{K}}(N/P') \quad (1.5.15)$$

In particular, $\tilde{\kappa}_N: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(N) \rightarrow \mathbf{R}_{\mathbb{K}}(N/P')$ is an isomorphism of rings.

Proof. Let M be an indecomposable linear source $\mathcal{O}N$ -lattice with maximal vertex, i.e., $\chi([M]) =^N (P, \varphi, Q)$. The $\mathbb{K}P$ -module $\mathbb{K}_{\varphi} = (\mathcal{O}_{\varphi})_{\mathbb{K}} = \mathbb{K} \otimes_{\mathcal{O}} \mathcal{O}_{\varphi}$ is irreducible. Let W be an irreducible $\mathbb{K}N$ -submodule of $M_{\mathbb{K}}$ containing $\mathbb{K}_{\varphi}$. Then $I(\varphi) \subseteq N$ coincides with the inertia subgroup of the $\mathbb{K}P$ -submodule $\mathbb{K}_{\varphi}$ of W . Since $\bar{I}(\varphi)$ contains a subgroup $\bar{P} = P/\ker(\varphi)$ which is contained in the center of $\bar{I}(\varphi)$, and as $\bar{I}(\varphi)/\bar{P}$ is a p' -group, one has that $\bar{I}(\varphi) \simeq \bar{C} \times \bar{P}$ for $\bar{C} = O_{p'}(\bar{I}(\varphi))$. In particular, $\text{res}_{\bar{C}}^{\bar{I}(\varphi)}(Q(\varphi)_{\mathbb{K}})$ is an irreducible $\mathbb{K}\bar{C}$ -module. Moreover, by Clifford theory and Theorem 1.5.10 one has that

$$\dim_{\mathbb{K}}(W) = |N : I(\varphi)| \cdot \text{rk}_{\mathcal{O}}(Q(\varphi)) = \text{rk}_{\mathcal{O}}(M). \quad (1.5.16)$$

Hence $W = M_{\mathbb{K}}$, and $M_{\mathbb{K}}$ is an irreducible $\mathbb{K}N$ -module which is inflated from N/P' . Thus $-\mathbb{K}$ induces a map $\tilde{\kappa}_N^\circ: \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(N) \rightarrow \text{Irr}_{\mathbb{K}}(\bar{N})$, where $\bar{N} = N/P'$. Moreover, one has a commutative diagram

$$\begin{array}{ccc} & \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(N) & \\ i_N^N \nearrow & & \searrow \tilde{\kappa}_N^\circ \\ \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(\bar{N}) & \xrightarrow{\tilde{\kappa}_N^\circ} & \text{Irr}_{\mathbb{K}}(\bar{N}), \end{array} \quad (1.5.17)$$

where $i_N^N(-)$ is induced by inflation. Note that Theorem 1.5.10 implies that $i_N^N(-)$ is a bijection.

Let Y be an irreducible $\mathbb{K}\bar{N}$ -module, and let $\mathbb{K}_{\varphi} = \mathbb{K} \otimes_{\mathcal{O}} \mathcal{O}_{\varphi}$ be an irreducible constituent of $\text{res}_{P^{\text{ab}}}^{\bar{N}}(Y)$, where $P^{\text{ab}} = P/P'$ and $\varphi \in \bar{P}^{\text{ab}}$. Let $I = I(Y, \mathbb{K}_{\varphi})$ be the inertia subgroup with respect to $\mathbb{K}_{\varphi}$, and let $\tilde{\mathbb{K}}_{\varphi} \subseteq Y$ be the $\mathbb{K}_{\varphi}$ -homogeneous component of Y . Then, by elementary Clifford theory, $Y \simeq \text{ind}_I^{\bar{N}}(\tilde{\mathbb{K}}_{\varphi})$. Let $U = \ker(\varphi) \triangleleft I$. Then $I/U \simeq C \times \text{im}(\varphi)$, where $C = O_{p'}(I/U)$. Moreover, as U acts trivially on $\tilde{\mathbb{K}}_{\varphi}$, $\tilde{\mathbb{K}}_{\varphi}$ is an irreducible $\mathbb{K}I/U$ -module, and thus is isomorphic to $B \otimes_{\mathbb{K}} \mathbb{K}_{\varphi}$, where B is an irreducible $\mathbb{K}C$ -module. Let $\Omega \subseteq B$ be an $\mathcal{O}C$ -sublattice of B , and let $Q = \Omega \otimes_{\mathcal{O}} \mathcal{O}\text{im}(\varphi)$. Let L be the indecomposable linear source $\mathcal{O}\bar{N}$ -lattice satisfying

$$\chi([L]) = \bar{N}(P^{\text{ab}}, \varphi, Q). \quad (1.5.18)$$

It is straight forward to verify that $\rho_{\bar{N}}([L]) = [J]$ showing that ρ_N is surjective.

Elementary Clifford theory shows that the number of isomorphism types of irreducible $\mathbb{K}\bar{N}$ -modules coincides with the $\bar{N}$ -orbits on the set

$$\mathfrak{Y} = \{ (\varphi, [L]) \mid \varphi \in \hat{P}^{\text{ab}}, [L] \in \text{Irr}_{\mathbb{K}}(\bar{N}_{\varphi}/P^{\text{ab}}) \}, \quad (1.5.19)$$

where $\bar{N}_{\varphi} = \{ g \in \bar{N} \mid {}^g\varphi = \varphi \}$. Hence, by Theorem 1.5.10, one can conclude that $|\text{IL}(\bar{N})| = |\text{Irr}_{\mathbb{K}}(\bar{N})|$, and thus $\tilde{\kappa}_{\bar{N}}^{\circ}$ is a bijection. This yields the claim. $\square$

This theorem and the Green correspondence guarantee that the indecomposable linear source $\mathcal{O}G$ -lattice with vertex set ${}^G V$ are in bijective correspondence with the irreducible $\mathbb{K}N/V'$ -modules.

Remark 1.5.11 gives a complete description of linear source $\mathcal{O}G$ -lattice, instead the next result gives a criterion ensuring that a given linear source $\mathcal{O}G$ -lattice M has maximal vertex. Let P be a Sylow p -subgroup of G and let M be a linear source $\mathcal{O}G$ -lattice. Then $\text{res}_P^G(M)$ is a linear source $\mathcal{O}P$ -lattice (cf. Proposition 1.5.8(ii)) and $[\text{res}_P^G(M)] \in \mathbf{L}_{\mathcal{O}}(P)$ has a unique decomposition

$$[\text{res}_P^G(M)] = [\text{res}_P^G(M)]_{\text{mx}} + [\text{res}_P^G(M)]_{\prec}, \quad (1.5.20)$$

where $[\text{res}_P^G(M)]_{\text{mx}} \in \mathbf{L}_{\mathcal{O}}^{\text{mx}}(P)$ and $[\text{res}_P^G(M)]_{\prec} \in \mathbf{L}_{\mathcal{O}}^{\prec}(P)$.

Proposition 1.5.13. *Let G be a finite group, let $P \in \text{Syl}_p(G)$, and let M be an indecomposable linear source $\mathcal{O}G$ -lattice. Then M has maximal vertex if, and only if, $[\text{res}_P^G(M)]_{\text{mx}} \neq 0$.*

Proof. It follows from Green's indecomposability theorem, that for a finite p -group V an indecomposable linear source $\mathcal{O}V$ -module M has maximal vertex V if, and only if, it is linear, i.e., there exists $\varphi \in \hat{V}$ such that $M \simeq \mathcal{O}_{\varphi}$. Let M be an indecomposable linear source $\mathcal{O}G$ -lattice with vertex set $\mathbf{v}(M) = \text{Syl}_p(G)$, i.e., for $P \in \text{Syl}_p(G)$ one has $\chi([M]) = {}^G(P, \varphi, Q)$. By Theorem 1.5.10, one has for $N = N_G(P)$ that

$$[\text{res}_P^N(\mathbf{f}(M))] \simeq \sum_{\psi \in N_{\varphi}} \frac{\text{rk}_{\mathcal{O}}(M)}{|\text{im}(\varphi)|} \cdot [\mathcal{O}_{\psi}] \in \mathbf{L}_{\mathcal{O}}^{\text{mx}}(P) \setminus \{0\}. \quad (1.5.21)$$

In particular, as $\text{res}_P^N(\mathbf{f}(M))$ is a direct summand of $\text{res}_P^G(M)$, this yields that $[\text{res}_P^G(M)]_{\text{mx}} \neq 0$.

Let L be an indecomposable linear source $\mathcal{O}G$ -lattice with vertex set $\mathbf{v}(L) = {}^G V$ satisfying $[\text{res}_P^G(L)]_{\text{mx}} \neq 0$, i.e., there exists $\varphi \in \hat{P}$ such that $\mathcal{O}_{\varphi}$ is a direct summand of $\text{res}_P^G(L)$. Hence ${}^G P \preceq {}^G V$ (cf. [CR90, Thm. 19.14]), and thus ${}^G V = \text{Syl}_p(G)$. $\square$

Remark 1.5.14. Let N be a finite group with a unique Sylow p -subgroup, i.e., $\text{Syl}_p(N) = \{P\}$ and $P \in \text{Syl}_p(N)$ is normal. Note that $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(P)$ is a

subring of $\mathbf{L}_{\mathcal{O}}(P)$ isomorphic to the subring $\mathbb{Z}\mathrm{IL}_{\mathcal{O}}^{\mathrm{mx}}(P)$ of $\mathbf{L}_{\mathcal{O}}(P)$. . Let M be an indecomposable linear source $\mathcal{O}N$ -lattice with maximal vertex. Then, by (1.5.21), one has $[\mathrm{res}_P^N(M)]_{\prec} = 0$. Thus, as $\rho : [\mathrm{res}_P^G(-)]_{\mathrm{mx}} : \mathbf{L}_{\mathcal{O}}(N) \rightarrow \mathbf{L}_{\mathcal{O}}(P)$ is a ring homomorphism, $\mathbf{L}_{\mathcal{O}}^{\mathrm{mx}}(N)$ is a subring isomorphic to to the subring $\mathbb{Z}\mathrm{IL}_{\mathcal{O}}^{\mathrm{mx}}(N)$ of $\mathbf{L}_{\mathcal{O}}(N)$.

CHAPTER 2

The canonical sections in the representation theory of finite groups

In a paper of Robert Boltje (cf. [Bol98a]), the author introduced a canonical induction formula for the Grothendieck rings and the representation rings of a finite group G . As introduced in §1.5.1, let $\mathbf{R}_{\mathbb{F}}(G)$ and $\mathbf{R}_{\mathbb{K}}(G)$ denote the Grothendieck rings of the representations of $\mathbb{F}G$ and $\mathbb{K}G$, respectively, and $\mathbf{T}_{\mathcal{O}}(G)$ and $\mathbf{L}_{\mathcal{O}}(G)$ the Grothendieck rings of the indecomposable trivial source $\mathcal{O}G$ -lattices and linear source $\mathcal{O}G$ -lattices, respectively. The underlying idea of this thesis is that the rings $\mathbf{T}_{\mathcal{O}}(G)$ and $\mathbf{L}_{\mathcal{O}}(G)$ are a key tool in the study of the representation theory of a finite group G and to understand how they are linked to the representation rings $\mathbf{R}_{\mathbb{F}}(G)$ and $\mathbf{R}_{\mathbb{K}}(G)$ can be useful and interesting. It is in this context that the canonical maps $\mathbf{t}_G: \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}} \rightarrow \mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}}$ and $\mathbf{l}_G: \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}} \rightarrow \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$ arise. The main result of this chapter is the definition and construction of these canonical section considering the species of the Grothendieck rings involved and the canonical induction formulae defined by R. Boltje in [Bol98a].

Moreover, the canonical sections defined appear strictly linked to the (*extended*) *representation table* of trivial and linear source $\mathcal{O}G$ -lattices (see § 2.5.2 and § 2.6.2) which computation is an interesting challenge on its own.

In Chapter 4 a first non trivial and interesting application of these canonical sections will be presented (see Theorem D).

Through all the chapter G will be a finite group, p a prime and $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ a splitting p -modular system.

2.1 Setting

Many Grothendieck rings $\mathcal{A}(G)$ one usually considers for a finite group G share very similar finiteness properties, e.g.,

- (G₁) $\mathcal{A}(G)$ - considered as abelian group - is a finitely generated free $\mathbb{Z}$ -module with a canonical basis $\mathfrak{B}_{\mathcal{A}(G)}$;
- (G₂) $\mathcal{A}(G)$ is a commutative ring with unit, and its *complexification*, i.e., $\mathcal{A}(G)_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}(G)$, is an abelian semisimple $\mathbb{C}$ -algebra;
- (G₃) there exists a canonical involution $_{-}^{\star}: \mathcal{A}(G)^{\text{op}} \rightarrow \mathcal{A}(G)$ which is a ring isomorphism satisfying $_{-}^{\star\star} = \text{id}_{\mathcal{A}(G)}$ and $\mathfrak{B}_{\mathcal{A}(G)}^{\star} = \mathfrak{B}_{\mathcal{A}(G)}$.

The ring isomorphism $_{-}^{\star}: \mathcal{A}(G)^{\text{op}} \rightarrow \mathcal{A}(G)$ extends to a sesqui-linear ring isomorphism $_{-}^{\star}: \mathcal{A}(G)_{\mathbb{C}}^{\text{op}} \rightarrow \mathcal{A}(G)_{\mathbb{C}}$, and therefore one can use the same notation, i.e., $(\mathcal{A}(G)_{\mathbb{C}}, \star)$ is an *abelian semisimple $\star$ -algebra* (cf. §2.2.3). A $\star$ -homomorphism $\alpha: \mathcal{A}(G)_{\mathbb{C}} \rightarrow \mathbb{C}$ is usually called a *species* of $\mathcal{A}(G)$ (cf. § 2.3); the set of all species $\text{spec}(\mathcal{A}(G))$ will be called the *spectrum* of $\mathcal{A}(G)$. By hypothesis, it is a finite set. In analogy to $\mathbb{C}^*$ -algebras there is a kind of *generalized Gelfand correspondence* (cf. [Rud66, Exercise 18.21]), i.e., every $\star$ -homomorphism of Grothendieck rings $\phi: \mathcal{A}(G) \rightarrow \mathcal{B}(G)$ induces a mapping of finite sets $\phi^{\vee}: \text{spec}(\mathcal{B}(G)) \rightarrow \text{spec}(\mathcal{A}(G))$, and vice versa, every mappings of finite sets $\psi^{\vee}: \text{spec}(\mathcal{B}(G)) \rightarrow \text{spec}(\mathcal{A}(G))$ induces a $\star$ -homomorphism $\psi: \mathcal{A}(G)_{\mathbb{C}} \rightarrow \mathcal{B}(G)_{\mathbb{C}}$. Moreover, every $\mathbb{C}$ -linear map $\psi_{\mathbb{C}}^{\vee}: \mathbb{C}[\text{spec}(\mathcal{B}(G))] \rightarrow \mathbb{C}[\text{spec}(\mathcal{A}(G))]$ commuting with $\star$ induces a $\mathbb{C}$ -linear map $\psi: \mathcal{A}(G)_{\mathbb{C}} \rightarrow \mathcal{B}(G)_{\mathbb{C}}$ commuting with $\star$. In particular, this construction can be done considering the complexification of the representation rings $\mathbf{R}_{\mathbb{K}}(G)$ and $\mathbf{R}_{\mathbb{F}}(G)$ and the Grothendieck rings $\mathbf{T}_{\mathcal{O}}(G)$ and $\mathbf{L}_{\mathcal{O}}(G)$:

$$\begin{aligned} \mathbf{t}_G: \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}} &\longrightarrow \mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}}, \\ \mathbf{l}_G: \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}} &\longrightarrow \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}} \end{aligned} \tag{2.1.1}$$

The map $\mathbf{t}_G$ is induced from a canonical map $\mathbf{t}_G^{\vee}: \text{spec}(\mathbf{T}_{\mathcal{O}}(G)) \rightarrow \text{spec}(\mathbf{R}_{\mathbb{F}}(G))$ (cf. § 2.5.1) and, in particular, it is a homomorphism of $\star$ -algebras. On the other hand, $\mathbf{l}_G$ is induced from a mapping $\mathbf{l}_{G, \mathbb{C}}^{\vee}: \mathbb{C}[\text{spec}(\mathbf{L}_{\mathcal{O}}(G))] \rightarrow \mathbb{C}[\text{spec}(\mathbf{R}_{\mathbb{K}}(G))]$ and thus in general just a $\mathbb{C}$ -linear map commuting with $\star$. But in case that $P \in \text{Syl}_p(G)$ is abelian, it is also a mapping of $\star$ -algebras (cf. Remark 2.6.1).

On the other hand, as suggested by R. Boltje (cf. [Bol98b]), using canonical induction formulae (cf. § 2.7) for the Mackey functors defined by the representation rings $\mathbf{R}_{\mathbb{F}}$ and $\mathbf{R}_{\mathbb{K}}$ and considering the $\mathbb{Z}$ -restriction subfunctor given by their abelianizations, it is possible to define the canonical sections τ_G and λ_G for the surjective maps:

$$\begin{aligned} -_{\mathbb{F}} &= \mathbb{F} \otimes_{\mathcal{O}} -: \mathbf{T}_{\mathcal{O}}(G) \longrightarrow \mathbf{R}_{\mathbb{F}}(G), \\ -_{\mathbb{K}} &= \mathbb{K} \otimes_{\mathcal{O}} -: \mathbf{L}_{\mathcal{O}}(G) \longrightarrow \mathbf{R}_{\mathbb{K}}(G). \end{aligned} \tag{2.1.2}$$

The purpose of this chapter is to prove that

$$\begin{aligned} \mathbf{t}_G|_{\mathbf{T}_{\mathcal{O}}(G)} &= \tau_G \\ \mathbf{l}_G|_{\mathbf{L}_{\mathcal{O}}(G)} &= \lambda_G, \end{aligned} \quad (2.1.3)$$

i.e., the maps $\mathbf{t}_G^\vee$ and $\mathbf{l}_{G,\mathbb{C}}^\vee$ induce the maps τ_G and λ_G , respectively.

These equalities bring to an easier computation of the explicit values of the maps τ_G and λ_G in the case that $\mathbf{T}_{\mathcal{O}}(G)$ and $\mathbf{L}_{\mathcal{O}}(G)$ are known and a deeper knowledge of their properties. On the other hand, if the values of the map λ_G and τ_G are known, then (2.1.3) brings informations about the isomorphism classes of indecomposable trivial and linear source $\mathcal{O}G$ -lattices and their vertices.

2.2 Abelian semisimple $\star$ -rings

2.2.1 Abelian semisimple $\mathbb{C}$ -algebras

Let $\mathcal{A}$ be a finite-dimensional (associative) $\mathbb{C}$ -algebra. Then $\mathcal{A}$ is said to be *semisimple* if its *Jacobson radical* $J(\mathcal{A})$ is trivial (cf. [Ben06, §1.1]). From Wedderburn's theorem one concludes the following straightforward fact.

Fact 2.2.1. Let $\mathcal{A}$ be an abelian $\mathbb{C}$ -algebra of dimension d . Then the following are equivalent.

- (i) $\mathcal{A}$ is semisimple;
- (ii) $\mathcal{A}$ is isomorphic to $\prod_{i=1}^d \mathbb{C}$ as $\mathbb{C}$ -algebra;
- (iii) $\text{spec}(\mathcal{A}) = \text{Hom}_{\mathbb{C}\text{-alg.}}(\mathcal{A}, \mathbb{C})$ has cardinality d .

For an abelian semisimple $\mathbb{C}$ -algebra $\mathcal{A}$ an element $\alpha \in \text{spec}(\mathcal{A})$ is called a *species* of $\mathcal{A}$. The set $\text{spec}(\mathcal{A})$ may be considered as a subset of the dual space $\mathcal{A}^\vee = \text{Hom}_{\mathbb{C}}(\mathcal{A}, \mathbb{C})$ of $\mathcal{A}$, and by (ii), it is a basis of $\mathcal{A}^\vee$, i.e.,

$$\mathcal{A}^\vee = \text{span}_{\mathbb{C}} \text{spec}(\mathcal{A}). \quad (2.2.1)$$

By definition, the elements of its dual basis $\{e_\alpha \mid \alpha \in \text{spec}(\mathcal{A})\} \subseteq \mathcal{A}$ satisfy

$$\beta(e_\alpha^2) = \beta(e_\alpha)^2 = \delta_{\alpha,\beta}, \quad \alpha, \beta \in \text{spec}(\mathcal{A}), \quad (2.2.2)$$

where $\delta_{\alpha,\beta}$ is Kronecker's function. In particular, $e_\alpha = e_\alpha^2$, and e_α is an idempotent. Moreover, for $\alpha, \gamma \in \text{spec}(\mathcal{A})$, $\alpha \neq \gamma$, one has

$$e_\alpha e_\gamma = e_\gamma e_\alpha = 0. \quad (2.2.3)$$

2.2.2 Homomorphisms

The following fact is straightforward.

Fact 2.2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian semisimple $\mathbb{C}$ -algebras.

- (a) Every homomorphism of $\mathbb{C}$ -vector spaces $\phi: \mathcal{A} \rightarrow \mathcal{B}$ induces a map

$$\phi^\vee: \mathcal{B}^\vee \longrightarrow \mathcal{A}^\vee. \quad (2.2.4)$$

- (b) For every homomorphism of $\mathbb{C}$ -vector spaces $\psi: \mathcal{B}^\vee \rightarrow \mathcal{A}^\vee$ there exists a unique homomorphism of $\mathbb{C}$ -vector spaces $\phi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\psi = \phi^\vee$.
- (c) Every homomorphism of $\mathbb{C}$ -algebras $\phi: \mathcal{A} \rightarrow \mathcal{B}$ induces a map

$$\phi_*: \text{spec}(\mathcal{B}) \longrightarrow \text{spec}(\mathcal{A}). \quad (2.2.5)$$

- (d) For every map $\psi: \text{spec}(\mathcal{B}) \rightarrow \text{spec}(\mathcal{A})$ there exists a unique homomorphism of $\mathbb{C}$ -algebras $\phi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\psi = \phi_*$.

Proof. (a) The mapping $\phi^\vee$ is given by $\phi^\vee(\xi) = \xi \circ \phi$, $\xi \in \mathcal{B}^\vee$.

- (b) follows from the fact that $\eta: \text{id} \rightarrow -^{\vee\vee}$, given by $\eta_V(v)(w^\vee) = w^\vee(v)$, $v \in V$, $w \in V^\vee$, is a natural isomorphism of additive functors on the category of finite-dimensional $\mathbb{C}$ -vector spaces.

- (c) Note that $\phi_* = \phi^\vee|_{\text{spec}(\mathcal{B})}$ has the desired properties.

- (d) Let $\psi: \mathcal{B}^\vee \rightarrow \mathcal{A}^\vee$ also denote the induced map, and let $\phi: \mathcal{A} \rightarrow \mathcal{B}$ be the $\mathbb{C}$ -linear map such that $\psi = \phi^\vee$. For $\alpha \in \text{spec}(\mathcal{A})$, let

$$\psi^{-1}(\alpha) = \{ \beta \in \text{spec}(\mathcal{B}) \mid \psi(\beta) = \alpha \} \quad (2.2.6)$$

denote the ψ -fibre of α . Then for $\alpha \in \text{spec}(\mathcal{A})$ one has

$$\phi(e_\alpha) = \sum_{\beta \in \psi^{-1}(\alpha)} e_\beta. \quad (2.2.7)$$

In particular, by (2.2.3), $\phi(e_\alpha)$ is an idempotent or 0, and

$$\phi(e_\alpha) \phi(e_\gamma) = \phi(e_\gamma) \phi(e_\alpha) = 0 \quad (2.2.8)$$

for $\alpha, \gamma \in \text{spec}(\mathcal{A})$, $\alpha \neq \gamma$. Since $\{e_\alpha \mid \alpha \in \text{spec}(\mathcal{A})\}$ is a $\mathbb{C}$ -basis of $\mathcal{A}$, this yields that ϕ is a $\mathbb{C}$ -algebra homomorphism. $\square$

2.2.3 Abelian semisimple $\star$ -algebras

Let $\mathcal{A}$ be an abelian semisimple $\mathbb{C}$ -algebra, and let $_{-}^{\star}: \mathcal{A} \rightarrow \mathcal{A}$ be an automorphism of order 2, i.e., $_{-}^{\star\star} = \text{id}_{\mathcal{A}}$. Then $(\mathcal{A}, _{-}^{\star})$ will be called an *abelian semisimple $\star$ -algebra*. If $(\mathcal{A}, _{-}^{\star})$ and $(\mathcal{B}, _{-}^{\star})$ are abelian semisimple $\star$ -algebras, a homomorphism of $\mathbb{C}$ -algebras $\phi: \mathcal{A} \rightarrow \mathcal{B}$ will be called to be a *$\star$ -homomorphism*, if

$$\phi(a^{\star}) = \phi(a)^{\star} \quad \text{for all } a \in \mathcal{A}. \quad (2.2.9)$$

Remark 2.2.3. A $\mathbb{C}^{\star}$ -algebra $(A, \|_{-}\|, _{-}^{\star})$ is an (associative) $\mathbb{C}$ -Banach algebra together with an involution $_{-}^{\star}: A^{\text{op}} \rightarrow A$ satisfying

$$\|x^{\star}x\| = \|x\|^2 \quad (2.2.10)$$

for all $x \in A$ (cf. [Sak71, §1.1]).

Remark 2.2.4. The complexification $\mathcal{A}(G)_{\mathbb{C}}$ of a Grothendieck ring $\mathcal{A}(G)$ is an abelian semisimple $\star$ -algebra.

2.3 Species

Let $\mathcal{A}$ be a Grothendieck ring, a *species* s of $\mathcal{A}$ is a $\star$ -algebra homomorphism

$$s: \mathcal{A}(G)_{\mathbb{C}} \rightarrow \mathbb{C} \quad (2.3.1)$$

(cf. [Ben06, § 2.2]). The spectrum of $\mathcal{A}$, i.e., the set of species of $\mathcal{A}$, will be denoted $\text{spec}(\mathcal{A})$. A *vertex* of a species $s \in \text{spec}(\mathcal{A}(G))$ is a vertex of minimal size over indecomposable modules $V \in \mathcal{A}(G)$ such that $s(V) \neq 0$.

Remark 2.3.1. Let $\mathcal{A}(G)$ and $\mathcal{B}(G)$ be two semisimple $\star$ -algebras and let $\alpha: \mathcal{A}(G) \rightarrow \mathcal{B}(G)$ be a $\star$ -homomorphism of Grothendieck rings. In analogy with the abelian semisimple $\mathbb{C}$ -algebras, by the *generalized Gelfand correspondence* the homomorphism α induces a mapping of finite sets

$$\alpha^{\vee}: \text{spec}(\mathcal{A}(G)) \rightarrow \text{spec}(\mathcal{B}(G)). \quad (2.3.2)$$

Then, for every $a \in \mathcal{A}(G)$ and $s_{\mathcal{B}} \in \text{spec}(\mathcal{B}(G))$

$$\langle \alpha(a), s_{\mathcal{B}} \rangle_{\mathcal{B}(G)} = \langle a, \alpha^{\vee}(s_{\mathcal{B}}) \rangle_{\mathcal{A}(G)}, \quad (2.3.3)$$

where the angled brackets denote the evaluation, i.e.,

$$\langle \alpha(a), s_{\mathcal{B}} \rangle_{\mathcal{B}(G)} = s_{\mathcal{B}}(\alpha(a)) = \langle a, \alpha^{\vee}(s_{\mathcal{B}}) \rangle_{\mathcal{A}(G)} = \alpha^{\vee}(s_{\mathcal{B}})(a), \quad (2.3.4)$$

Even if the definition of a species of a Grothendieck ring is elementary, it is not so easy to deal with it. So, in the following a very useful characterization of the species of the Grothendieck rings considered will be presented for convenience of the reader; for further details see the references.

2.3.1 Species of $\mathbf{R}_{\mathbb{K}}(G)$

Let $g \in G$, then g defines a species s_g of $\mathbf{R}_{\mathbb{K}}(G)$ (cf. [Bol, § 2.3]). If M is an irreducible $\mathbb{K}G$ -module and χ is its (ordinary) character, then $s_g(M) = \chi(g)$. It follows from the construction that G -conjugated elements of G define the same species.

In particular all the species of $\mathbf{R}_{\mathbb{K}}(G)$ can be constructed as above. Then it is possible to conclude that

$$\text{spec}(\mathbf{R}_{\mathbb{K}}(G)) \simeq \{^G g \mid g \in G\}. \quad (2.3.5)$$

With the above notation,

$$\langle M, s_g \rangle_{\mathbf{R}_{\mathbb{K}}(G)} = \chi(g). \quad (2.3.6)$$

2.3.2 Brauer species of $\mathbf{R}_{\mathbb{F}}(G)$

Since $\mathbb{F}$ is a field of characteristic p and it is splitting for G and all its subgroups, then the Brauer species of $\mathbf{R}_{\mathbb{F}}(G)$ are in one-to-one correspondence with the p -regular conjugacy class of G .

In fact, since a Brauer species of $\mathbf{R}_{\mathbb{F}}(G)$ is a species whose vertex is the trivial subgroup, choosing an isomorphism between the $|G|$ -th roots of unity in $\mathbb{F}$ and in $\mathbb{C}$, the Brauer species are in one-to-one correspondence with conjugacy classes of elements of order prime to p (cf. [Ben98, § 2.11]). If $g \in G$ has order prime to p , then g defines a species $s_g: \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}} \rightarrow \mathbb{C}$ such that, if φ is the Brauer character of an indecomposable $\mathbb{F}G$ -module M , $s_g(M) = \varphi(g)$. With the above notation, $\langle M, s_g \rangle_{\mathbf{R}_{\mathbb{F}}(G)} = \varphi(g)$.

Then it is possible to conclude that

$$\text{spec}(\mathbf{R}_{\mathbb{F}}(G)) = \{^G g \mid g \in G, \text{ord}(g) \text{ is a } p' \text{ number}\}. \quad (2.3.7)$$

Example 2.3.1. Let $G = A_5$ and $p = 2$, then

$$\text{spec}(\mathbf{R}_{\mathbb{F}}(A_5)) = \{1A, 3A, 5A, 5B\}. \quad (2.3.8)$$

As in [CCN⁺85] $n*$ will denote a conjugacy class of elements of G of order n .

2.3.3 Species of $\mathbf{T}_{\mathcal{O}}(G)$

Let $O_p(G)$ denote the largest normal p -subgroup of G . A group G is called p -hypo-elementary if $G/O_p(G)$ is cyclic. In particular, $G/O_p(G)$ has order prime to p . Given a group G , let $\text{Hyp}_p(G)$ denote the set of all p -hypo-elementary subgroups of G .

Let $[V] \in \mathbf{T}_{\mathcal{O}}(G)$ and $H \in \text{Hyp}_p(G)$. Let $\text{res}_H^G(V) = V_1 \oplus V_2$, where V_1 is a direct sum of indecomposable modules with vertex $O_p(H)$ and V_2 is a direct sum of indecomposable modules whose vertex is properly contained

in $O_p(H)$. Then $O_p(H)$ acts trivially on V_1 , and so V_1 is a module for $H/O_p(H)$. Let b be a Brauer species of $H/O_p(H)$ (cf. [Ben06, § 2.11]), and define $(V, s_{(H,b)})_{\mathbf{T}_O(G)} = (V_1, b)_{\mathbf{R}_F(H/O_p(H))}$. Then $s_{(H,b)}$ is a species of $\mathbf{T}_O(G)$ (cf. [Ben06, § 2.13]).

In particular (cf. [Ben06, Corollary 2.13.2]), every species of $\mathbf{T}_O(G)$ arises in the way described above.

Thus every pair $(H, hO_p(H))$, where $H \in \text{Hyp}_p(G)$ and $h \in H$ such that $\langle hO_p(H) \rangle = H/O_p(H)$ determines a species of $\mathbf{T}_O(G)$. Let $W(G)$ be the set of all these pairs, then G acts on $W(G)$ via conjugation and G -conjugated pairs determine the same species. By [Bol98b, Proposition 2.6],

$$\text{spec}(\mathbf{T}_O(G)) \simeq \{ {}^G(H, hO_p(H)) \mid H \in \text{Hyp}_p(G), \langle hO_p(H) \rangle = H/O_p(H) \}. \quad (2.3.9)$$

Let $X(G)$ be the set of pairs (g, P) where P is a p -subgroup of G and g is an element of $N_G(P)_{p'}$, i.e. $g \in N_G(P)$ has order prime to p . Let $\mathfrak{X}(G) = X(G)/G$, then it is possible to prove (cf. [Bol, § 2.7]) that there exists a sets' isomorphism

$$\{ {}^G(H, hO_p(H)) \mid H \in \text{Hyp}_p(G), \langle hO_p(H) \rangle = H/O_p(H) \} \simeq \mathfrak{X}(G), \quad (2.3.10)$$

defined as follows:

$${}^G(g, P) \mapsto {}^G(\langle P, g \rangle, gP). \quad (2.3.11)$$

Then we have the following description of the species of $\mathbf{T}_O(G)$:

$$\text{spec}(\mathbf{T}_O(G)) \simeq \{ {}^G(g, P) \mid P \leq G \text{ } p\text{-subgroup}, g \in N_G(P)_{p'} \}. \quad (2.3.12)$$

In particular, fixed the vertex P , the species (g, P) and (h, P) define the same species if, and only if, g and h are $N_G(P)$ -conjugated.

Example 2.3.2. Let $G = A_5$ and $p = 2$, then there are three conjugacy classes of 2-subgroup of G which can be represented by the trivial group 1_G , a cyclic group C of order 2 and a Sylow 2-subgroup P . In particular, $N_G(C) \simeq P$ and $N_G(P) \simeq P \rtimes C_3$, then

$$\begin{aligned} \text{spec}(\mathbf{T}_O(A_5)) \simeq \{ & {}^G(1A, 1_G), {}^G(3A, 1_G), {}^G(5A, 1_G), {}^G(5B, 1_G), \\ & {}^G(1A, C), {}^G(1A, P), {}^G(3A, P), {}^G(3B, P) \}. \end{aligned} \quad (2.3.13)$$

2.3.4 Species of $\mathbf{L}_O(G)$

As before, let one consider a pair $(H, hO_p(H)')$, where $H \in \text{Hyp}_p(G)$ and $h \in H$ such that $\langle hO_p(H) \rangle = H/O_p(H)$. Such a pair defines a species of $\mathbf{L}_O(G)$ (cf. [Bol98b, § 2.2]).

Let $J(G)$ be the set of these pairs, then G acts via conjugation on $J(G)$. In particular G -conjugated pairs of $J(G)$ determine the same species and

every species of $\mathbf{L}_{\mathcal{O}}(G)$ arises from one of these pairs (cf. [Bol98b, Proposition 2.6])). Then

$$\text{spec}(\mathbf{L}_{\mathcal{O}}(G)) \simeq \{ {}^G(H, hO_p(H)') \mid H \in \text{Hyp}_p(G), \langle hO_p(H) \rangle = H/O_p(H) \}. \quad (2.3.14)$$

Trivially there is a natural embedding $i: \mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}} \rightarrow \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$, which induces a surjective map $i^{\vee}: \text{spec}(\mathbf{L}_{\mathcal{O}}(G)) \rightarrow \text{spec}(\mathbf{T}_{\mathcal{O}}(G))$ given by

$$(H, hO_p(H)') \mapsto (H, hO_p(H)). \quad (2.3.15)$$

As done before for the ring $\mathbf{T}_{\mathcal{O}}(G)$, the goal of this section is to characterize the set of species of $\mathbf{L}_{\mathcal{O}}(G)$ in terms of pairs (g, P) , where P is a p -subgroup of G and g is a suitable element of G . Given a p -subgroup P of G , a pair (g, P) define a species $(\langle P, g \rangle, gP')$ if and only if $g \in N_G(P)$ and $g_p \in P$, i.e., the p -part of G belongs to P . Let $Y(G)$ denote the set of all these pairs. As two pairs in $J(G)$ define the same species if, and only if, they are G -conjugated, two pairs $(g, P), (h, Q) \in Y(G)$ define the same species if and only if $(\langle P, g \rangle, gP')$ and $(\langle Q, h \rangle, hQ')$ are G -conjugated, i.e., there exists $t \in G$ such that the following conditions are satisfied:

- (i) ${}^tQ = P$
- (ii) ${}^th{}^tQ' = gP'$
- (iii) ${}^th_s = g_s$

where g_s is the semisimple part of the element g .

If the pairs (g, P) and (h, Q) satisfy the three above conditions, we will say that $(g, P) \sim (h, Q)$ and $\sim(g, P)$ will denote their equivalence class. Then the below characterisation of the set of species of $\mathbf{L}_{\mathcal{O}}(G)$ follows.

$$\text{spec}(\mathbf{L}_{\mathcal{O}}(G)) \simeq \{ \sim(g, P) \mid P \leq G \text{ } p\text{-subgroup}, g \in N_G(P), g_p \in P \} \quad (2.3.16)$$

In particular, if $(g, P), (h, P) \in Y(G)$, then $(g, P) \sim (h, P)$ if, and only if, there exists $t \in N_G(P)$ such that ${}^thP' = gP'$ and ${}^th_s = g_s$.

Let one observe that

$$\text{spec}(\mathbf{T}_{\mathcal{O}}(G)) \subseteq \text{spec}(\mathbf{L}_{\mathcal{O}}(G)). \quad (2.3.17)$$

Example 2.3.3. It follows from Example 2.3.2 that

$$\text{spec}(\mathbf{L}_{\mathcal{O}}(A_5)) = \text{spec}(\mathbf{T}_{\mathcal{O}}(A_5)) \cup \{(2A, C), (2A, P)\}. \quad (2.3.18)$$

2.4 The maps $\gamma^\vee$ and $\kappa^\vee$

From now on the sets of species of the Grothendieck rings considered will be identified with their characterization described in § 2.3.

Let $-\mathbb{F}$ and $-\mathbb{K}$ be the surjective maps defined as in (2.1.1), i.e.,

$$\begin{aligned} -\mathbb{F} &= \mathbb{F} \otimes_{\mathcal{O}} -: \mathbf{T}_{\mathcal{O}}(G) \longrightarrow \mathbf{R}_{\mathbb{F}}(G), \\ -\mathbb{K} &= \mathbb{K} \otimes_{\mathcal{O}} -: \mathbf{L}_{\mathcal{O}}(G) \longrightarrow \mathbf{R}_{\mathbb{K}}(G). \end{aligned} \quad (2.4.1)$$

Considering the spectrum of the Grothendieck rings involved, i.e., $\mathbf{L}_{\mathcal{O}}(G)$, $\mathbf{T}_{\mathcal{O}}(G)$, $\mathbf{R}_{\mathbb{K}}(G)$ and $\mathbf{R}_{\mathbb{F}}(G)$, it is possible to define

$$\begin{aligned} \gamma_G^\vee: \text{spec}(\mathbf{R}_{\mathbb{F}}(G)) &\rightarrow \text{spec}(\mathbf{T}_{\mathcal{O}}(G)) \\ {}^G g &\mapsto {}^G (g, 1). \end{aligned} \quad (2.4.2)$$

and

$$\begin{aligned} \kappa_G^\vee: \text{spec}(\mathbf{R}_{\mathbb{K}}(G)) &\rightarrow \text{spec}(\mathbf{L}_{\mathcal{O}}(G)) \\ {}^G h &\mapsto \sim(h, \langle h_p \rangle). \end{aligned} \quad (2.4.3)$$

The maps $\gamma^\vee$ and $\kappa^\vee$ are induced by the $\star$ -homomorphisms $-\mathbb{F}$ and $-\mathbb{K}$ respectively. Moreover, these maps induce the following $\star$ -homomorphisms:

$$\begin{aligned} \gamma_G: \mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}} &\rightarrow \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}} \\ \gamma_G([T]) &= [T \otimes_{\mathcal{O}} \mathbb{F}], \end{aligned} \quad (2.4.4)$$

where $[T] \in \text{ITr}_{\mathcal{O}}(G)$, and

$$\begin{aligned} \kappa_G: \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}} &\rightarrow \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}} \\ \kappa_G([L]) &= [L \otimes_{\mathcal{O}} \mathbb{K}], \end{aligned} \quad (2.4.5)$$

where $[L] \in \text{IL}_{\mathcal{O}}(G)$.

In particular, for $[T] \in \text{ITr}_{\mathcal{O}}(G)$ and the species ${}^G g \in \text{spec}(\mathbf{R}_{\mathbb{F}}(G))$, and for $[L] \in \text{IL}_{\mathcal{O}}(G)$ and the species ${}^G h \in \text{spec}(\mathbf{R}_{\mathbb{K}}(G))$, the following equalities hold (cf. 2.3.1).

$$\langle \gamma_G([T]), {}^G g \rangle_{\mathbf{R}_{\mathbb{F}}(G)} = \langle [T], \gamma_G^\vee({}^G g) \rangle_{\mathbf{T}_{\mathcal{O}}(G)} = \langle [T], {}^G (g, 1) \rangle_{\mathbf{T}_{\mathcal{O}}(G)} \quad (2.4.6)$$

and

$$\langle \kappa_G([L]), {}^G h \rangle_{\mathbf{R}_{\mathbb{K}}(G)} = \langle [L], \kappa_G^\vee({}^G h) \rangle_{\mathbf{L}_{\mathcal{O}}(G)} = \langle [L], \sim(h, \langle h_p \rangle) \rangle_{\mathbf{L}_{\mathcal{O}}(G)}. \quad (2.4.7)$$

It follows from the definitions in (2.4.4) and (2.4.5) that $\gamma_G|_{\mathbf{T}_{\mathcal{O}}(G)} = -\mathbb{F}$ and $\kappa_G|_{\mathbf{L}_{\mathcal{O}}(G)} = -\mathbb{K}$; then in the following γ_G and κ_G will be replaced by $-\mathbb{F}$ and $-\mathbb{K}$, respectively.

Remark 2.4.1. If $g \in G$ has order prime to p , then

$$\gamma_G^\vee({}^G g) = {}^G (g, 1) = \sim(g, 1) = \kappa_G^\vee({}^G g) \quad (2.4.8)$$

2.5 The map $\mathbf{t}_G$ and the representation table

2.5.1 Definition

Let $\mathbf{t}_G$ be the $\star$ -algebras homomorphism

$$\mathbf{t}_G: \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}} \rightarrow \mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}} \quad (2.5.1)$$

induced from the map

$$\begin{aligned} \mathbf{t}_G^{\vee}: \operatorname{spec}(\mathbf{T}_{\mathcal{O}}(G)) &\rightarrow \operatorname{spec}(\mathbf{R}_{\mathbb{F}}(G)) \\ G(g, P) &\mapsto^G g \end{aligned} \quad (2.5.2)$$

Let $M \in \mathbf{R}_{\mathbb{F}}(G)$ and ${}^G(g, P)$ a species of $\mathbf{T}_{\mathcal{O}}(G)$, then $\mathbf{t}_G$ is defined as follows:

$$\langle \mathbf{t}_G(M), {}^G(g, P) \rangle_{\mathbf{T}_{\mathcal{O}}(G)} = \langle M, \mathbf{t}_G^{\vee}({}^G(g, P)) \rangle_{\mathbf{R}_{\mathbb{F}}(G)} = \langle M, {}^G g \rangle_{\mathbf{R}_{\mathbb{F}}(G)} \quad (2.5.3)$$

2.5.2 The representation table

Let $\Gamma \in \{\mathbb{K}, \mathbb{F}\}$ and let $\mathcal{A}$ be a subalgebra of the complexification of the representation ring $\mathbf{R}_{\Gamma}(G)_{\mathbb{C}}$ or of the complexification of the Green ring $\mathbf{A}(G)_{\mathbb{C}}$ (cf. § 1.5.1). Let $s_1, \dots, s_n$ be its species and let $[V_1], \dots, [V_n]$ be the isomorphism classes of indecomposable modules freely spanning $\mathcal{A}$. Then the matrix

$$R_{ij} = s_j([V_i]) = \langle [V_i], s_j \rangle_{\mathcal{A}} \quad (2.5.4)$$

is called *representation table* of A (cf. [Ben98, Definition 2.21.5]).

In particular, we will call *representation table of G with respect to p* the representation table of $\mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}}$.

2.5.3 Properties

In this section the properties of the $\star$ -homomorphism $\mathbf{t}_G$ will be discussed. In order to deal with an easier notation, a module M will be “identified” with the isomorphism class $[M]$; thus, for example, $\mathbf{t}_G(M) = \mathbf{t}_G([M])$.

Proposition 2.5.1. *The homomorphism $\mathbf{t}_G: \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}} \rightarrow \mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}}$ has the following properties:*

1. $-\mathbb{F} \circ \mathbf{t}_G = \operatorname{Id}_{\mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}}}$
2. If $H \leq G$, then for every $M \in \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}}$, $\operatorname{res}_H^G(\mathbf{t}_G(M)) = \mathbf{t}_H(\operatorname{res}_H^G(M))$
3. If $N \triangleleft G$, then $\forall S \in \mathbf{R}_{\mathbb{F}}(G/N)_{\mathbb{C}}$, $\operatorname{inf}_{G/N}^G(\mathbf{t}_G(S)) = \mathbf{t}_G(\operatorname{inf}_{G/N}^G(S))$

Proof. 1. First of all let us observe that

$$\mathbf{t}_G^\vee \circ \gamma_G^\vee = \text{Id}_{\text{spec}(\mathbf{R}_\mathbb{F}(G))}, \quad (2.5.5)$$

in fact for every species ${}^G g \in \text{spec}(\mathbf{R}_\mathbb{F}(G))$ it turns out that

$$\mathbf{t}_G^\vee(\gamma_G^\vee({}^G g)) = \mathbf{t}_G^\vee({}^G(g, 1)) = {}^G g. \quad (2.5.6)$$

Then, for every $M \in \mathbf{R}_\mathbb{F}(G)_\mathbb{C}$ and for every species ${}^G g \in \text{spec}(\mathbf{R}_\mathbb{F}(G))$,

$$\begin{aligned} \langle \mathbf{t}_G(M)_\mathbb{F}, {}^G g \rangle_{\mathbf{R}_\mathbb{F}(G)} &= \langle \mathbf{t}_G(M), \gamma_G^\vee({}^G g) \rangle_{\mathbf{T}_\mathcal{O}(G)} = \\ &= \langle M, \mathbf{t}_G^\vee(\gamma_G^\vee({}^G g)) \rangle_{\mathbf{R}_\mathbb{F}(G)} = \langle M, {}^G g \rangle_{\mathbf{R}_\mathbb{F}(G)}, \end{aligned} \quad (2.5.7)$$

and this yields the claim.

2. The goal is to prove that the following diagram commutes

$$\begin{array}{ccc} \mathbf{R}_\mathbb{F}(G)_\mathbb{C} & \xrightarrow{\mathbf{t}_G} & \mathbf{T}_\mathcal{O}(G)_\mathbb{C} \\ \text{res}_H^G \downarrow & & \downarrow \text{res}_H^G \\ \mathbf{R}_\mathbb{F}(H)_\mathbb{C} & \xrightarrow{\mathbf{t}_H} & \mathbf{T}_\mathcal{O}(H)_\mathbb{C} \end{array} \quad (2.5.8)$$

First of all let us remember that the restriction is an homomorphism of $\mathbb{C}^*$ -algebras, then we can consider the dual diagram:

$$\begin{array}{ccc} \text{spec}(\mathbf{R}_\mathbb{F}(G)) & \xleftarrow{\mathbf{t}_G^\vee} & \text{spec}(\mathbf{T}_\mathcal{O}(G)) \\ \text{res}_H^{G^\vee} \uparrow & & \uparrow \text{res}_H^{G^\vee} \\ \text{spec}(\mathbf{R}_\mathbb{F}(H)) & \xleftarrow{\mathbf{t}_H^\vee} & \text{spec}(\mathbf{T}_\mathcal{O}(H)) \end{array} \quad (2.5.9)$$

Diagram (2.5.9) commutes; in fact for every species ${}^H(s, V)$ of $\mathbf{T}_\mathcal{O}(H)$,

$$\text{res}_H^{G^\vee}(\mathbf{t}_H^\vee({}^H(s, V))) = \mathbf{t}_G^\vee(\text{res}_H^{G^\vee}({}^H(s, V))), \quad (2.5.10)$$

as $\text{res}_H^{G^\vee}({}^H s) = {}^G s$ and $\text{res}_H^{G^\vee}({}^H(s, V)) = {}^G(s, V)$. But then it follows that for every $M \in \mathbf{R}_\mathbb{F}(G)_\mathbb{C}$ and for every ${}^H(s, V) \in \text{spec}(\mathbf{T}_\mathcal{O}(H))$,

$$\begin{aligned} \langle \text{res}_H^G(\mathbf{t}_G(M)), {}^H(s, V) \rangle_{\mathbf{T}_\mathcal{O}(H)} &= \langle \mathbf{t}_G(M), \text{res}_H^{G^\vee}({}^H(s, V)) \rangle_{\mathbf{T}_\mathcal{O}(G)} = \\ &= \langle M, \mathbf{t}_G^\vee(\text{res}_H^{G^\vee}({}^H(s, V))) \rangle_{\mathbf{R}_\mathbb{F}(G)} = \langle M, \text{res}_H^{G^\vee}(\mathbf{t}_H^\vee({}^H(s, V))) \rangle_{\mathbf{R}_\mathbb{F}(G)} = \\ &= \langle \text{res}_H^G(M), \mathbf{t}_H^\vee({}^H(s, V)) \rangle_{\mathbf{R}_\mathbb{F}(H)} = \langle \mathbf{t}_H(\text{res}_H^G(M)), {}^H(s, V) \rangle_{\mathbf{T}_\mathcal{O}(H)}. \end{aligned} \quad (2.5.11)$$

Then diagram (2.5.8) commutes.

3. Let us consider the following diagram

$$\begin{array}{ccc} \mathbf{R}_{\mathbb{F}}(G/N)_{\mathbb{C}} & \xrightarrow{t_{G/N}} & \mathbf{T}_{\mathcal{O}}(G/N)_{\mathbb{C}} \\ \inf_{G/N}^G \downarrow & & \downarrow \inf_{G/N}^G \\ \mathbf{R}_{\mathbb{F}}(G)_{\mathbb{C}} & \xrightarrow{t_G} & \mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}} \end{array} \quad (2.5.12)$$

As in the previous cases, it is enough to prove that the dual diagram commutes:

$$\begin{array}{ccc} \mathrm{spec}(\mathbf{R}_{\mathbb{F}}(G/N)) & \xleftarrow{t_{G/N}^{\vee}} & \mathrm{spec}(\mathbf{T}_{\mathcal{O}}(G/N)) \\ \inf_{G/N}^G \uparrow & & \uparrow \inf_{G/N}^G \\ \mathrm{spec}(\mathbf{R}_{\mathbb{F}}(G)) & \xleftarrow{t_G^{\vee}} & \mathrm{spec}(\mathbf{T}_{\mathcal{O}}(G)) \end{array} \quad (2.5.13)$$

For every species ${}^G(s, V) \in \mathrm{spec}(\mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}})$

$$\inf_{G/N}^G \vee (t_G^{\vee}({}^G(s, V))) = \inf_{G/N}^G \vee ({}^G s) = {}^G s N, \quad (2.5.14)$$

and

$$t_{G/N}^{\vee}(\inf_{G/N}^G \vee ({}^G(s, V))) = t_{G/N}^{\vee}({}^G(sN, VN/N)) = {}^G s N. \quad (2.5.15)$$

Then diagram (2.5.13) commutes and this implies the thesis. $\square$

Remark 2.5.2. For a finite group N with normal Sylow p -subgroup P , there is a bijective correspondence between the set $\mathrm{ITr}_{\mathcal{O}}^{\mathrm{mx}}(N)$ and the set of irreducible Brauer characters $\mathrm{IBr}_{\mathbb{F}}(N/P) \simeq \mathrm{IBr}_{\mathbb{F}}(N)$ (cf [Bro85, Part 3.6]). Thus, if W is an irreducible $\mathbb{F}N$ -module with Brauer character $\varphi \in \mathrm{IBr}_{\mathbb{F}}(N)$ and $T \in \mathrm{ITr}_{\mathcal{O}}^{\mathrm{mx}}(G)$ is such that $T_{\mathbb{F}} = W$, then Proposition 2.5.1.(1) implies that

$$t_N(W) = T + \delta, \quad (2.5.16)$$

where $\delta \in \mathbf{T}_{\mathcal{O}}^{\prec}(N)$ and $\delta \in \ker(-_{\mathbb{F}})$.

Proposition 2.5.3. *Let $M \in \mathbf{R}_{\mathbb{F}}(G)$ be an irreducible $\mathbb{F}G$ -module, and let $t_G(M) = \sum_{i=1}^n a_i T_i$, where $a_i \in \mathbb{C}$, $T_i \in \mathrm{ITr}_{\mathcal{O}}(G)$. Let T_i for $i \in \{1, \dots, m\}$ have maximal vertex, for $m \leq n$. Then, for $i \in \{1, \dots, m\}$, the coefficients a_i are integers.*

Proof. Let $P \in \mathrm{Syl}_p(G)$ and let $N = N_G(P)$ be its normalizer in G ; thanks to Proposition 2.5.1.2, $t_N(\mathrm{res}_N^G(M)) = \mathrm{res}_N^G(t_G(M)) \in \mathbf{T}_{\mathcal{O}}(N)_{\mathbb{C}}$. By Green correspondence

$$\mathrm{res}_N^G(t_G(M)) = \sum_{i=1}^n a_i \cdot \mathrm{res}_N^G(T_i) = \sum_{i=1}^m a_i \cdot f(T_i) \oplus \delta, \quad (2.5.17)$$

where $\delta \in \mathbf{T}_{\mathcal{O}}^{\prec}(N)$ and, for $i \in \{1, \dots, m\}$, $f(T_i)$ is an indecomposable trivial source $\mathcal{O}N$ -lattice with maximal vertex.

On the other hand, $\mathbf{t}_N(\text{res}_N^G(M)) = \mathbf{t}_N(\sum_{i=1}^k c_i M_i) = \sum_{i=1}^k c_i \cdot \mathbf{t}_N(M_i)$, where $c_i \in \mathbb{Z}$ and $M_i \in \text{IBr}_{\mathbb{F}}(N)$. In particular the indecomposable trivial source $\mathcal{O}G$ -lattices with maximal vertex are in bijective correspondence with the irreducible $\mathbb{F}[N/P]$ -modules. Let $\pi: \mathbf{T}_{\mathcal{O}}(G) \rightarrow \mathbf{T}_{\mathcal{O}}^{\text{mx}}(G) \simeq \mathbf{T}_{\mathcal{O}}(G)/\mathbf{T}_{\mathcal{O}}^{\prec}(G)$ be the canonical projection. Then, without loss of generality, $\pi(\mathbf{t}_N(M_i)) = f(T_i)$ (cf. Remark 2.5.2), for $i \in \{1, \dots, m\}$, $m \leq k$.

Then it follows that

$$\mathbf{t}_N(\text{res}_N^G(M)) = \sum_{i=1}^m c_i f(T_i) \oplus \delta, \quad (2.5.18)$$

where $\delta \in \mathbf{T}_{\mathcal{O}}^{\prec}(N)_{\mathbb{C}}$. This implies $c_i = a_i \in \mathbb{Z}$, for $i \in \{1, \dots, m\}$, concluding the proof. $\square$

In § 2.8 it will be clear that all the coefficients are integers.

2.6 The map $\mathbf{l}_G$ and the extended representation table

2.6.1 Definition

The aim of this section is to define a $\mathbb{C}$ -vector spaces homomorphism from $\mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}}$ to $\mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$. Let $g \in G$, then g_s and g_p will be, respectively, the semisimple and unipotent component of g . In this section $\sim_G$ will denote the equivalence relation over $Y(G)$ defined in § 2.3.4.

Let $\mathbf{l}_G$ be the $\mathbb{C}$ -vector spaces homomorphism

$$\mathbf{l}_G: \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}} \rightarrow \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}} \quad (2.6.1)$$

induced from the mapping

$$\mathbf{l}_{G,\mathbb{C}}^{\vee}: \mathbb{C}[\text{spec}(\mathbf{L}_{\mathcal{O}}(G))] \rightarrow \mathbb{C}[\text{spec}(\mathbf{R}_{\mathbb{K}}(G))] \quad (2.6.2)$$

defined as follows

$$\mathbf{l}_{G,\mathbb{C}}^{\vee}(\sim^G(g, P)) = \frac{1}{|C_{P'}(g_s)|} \sum_{v \in C_{P'}(g_s)} \sim^G(g \cdot v). \quad (2.6.3)$$

In particular $\mathbf{l}_{G,\mathbb{C}}^{\vee}$ is a $\mathbb{C}$ -linear map commuting with $\star$. Let $M \in \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}}$ and let $\sim^G(g, P)$ be a species of $\mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$, then $\mathbf{l}_G$ is such that

$$\begin{aligned} \langle \mathbf{l}_G(M), \sim^G(g, P) \rangle_{\mathbf{L}_{\mathcal{O}}(G)} &= \langle M, \mathbf{l}_{G,\mathbb{C}}^{\vee}(\sim^G(g, P)) \rangle_{\mathbf{R}_{\mathbb{K}}(G)} \\ &= \frac{1}{|C_{P'}(g_s)|} \cdot \sum_{v \in C_{P'}(g_s)} \langle M, \sim^G(g \cdot v) \rangle_{\mathbf{R}_{\mathbb{K}}(G)}. \end{aligned} \quad (2.6.4)$$

Remark 2.6.1. Let $P \in \text{Syl}_p(G)$ be abelian, then $\mathfrak{l}_G$ is a $\star$ -algebras homomorphism. In fact in this case $\mathfrak{l}_G: \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}} \rightarrow \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$ is induced by $\mathfrak{l}_G^{\vee}: \text{spec}(\mathbf{L}_{\mathcal{O}}(G)) \rightarrow \text{spec}(\mathbf{R}_{\mathbb{K}}(G))$, since $P' = 1_G$.

Remark 2.6.2. Let us observe that $\text{spec}(\mathbf{R}_{\mathbb{F}}(G)) \subseteq \text{spec}(\mathbf{R}_{\mathbb{K}}(G))$ and, if $P \in \text{Syl}_p(G)$ is abelian, then

$$\mathfrak{l}_{G,\mathbb{C}}^{\vee}|_{\text{spec}(\mathbf{T}_{\mathcal{O}}(G))} = \mathfrak{t}^{\vee}. \quad (2.6.5)$$

Remark 2.6.3. Let $L \in \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$ and $\sim^G(g, D) \in \text{spec}(\mathbf{L}_{\mathcal{O}}(G))$ with D abelian. Then

$$\langle L, \sim(g, D) \rangle_{\mathbf{L}_{\mathcal{O}}(G)} = \left\langle ((\text{res}_{N_G(D)}^G(L)|_D)_{\mathbb{K}}, g) \right\rangle_{\mathbf{R}_{\mathbb{K}}(G)}, \quad (2.6.6)$$

where $\text{res}_{N_G(D)}^G(L)|_D$ denote the sum of indecomposable $\mathcal{O}N_G(D)$ -lattices with vertex D which are components of $\text{res}_{N_G(D)}^G(L)$.

2.6.2 The extended representation table

In the literature the representation table usually denotes the matrix defined in § 2.5.2 where $\mathbf{T}_{\mathcal{O}}(G)_{\mathbb{C}}$ is the algebra considered. It is also possible to construct a representation table starting from the algebra $\mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$. Let us define this matrix as the *extended representation table of G with respect to p* . As in the trivial source lattices case the representation table of G with respect to p allows us to compute the values of the homomorphism $\mathfrak{t}_G$, the extended one will allow us to find the explicit values of the map $\mathfrak{l}_G$.

Let us observe that for a fixed finite group G and a given prime p , the extended representation table restricted to the trivial source $\mathcal{O}G$ -lattices and their species coincides with the representation table.

2.6.3 Properties

In this section $\kappa_{G,\mathbb{C}}^{\vee}$ will denote the $\mathbb{C}$ -vector spaces homomorphism defined from $\mathbb{C}[\text{spec}(\mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}})]$ to $\mathbb{C}[\text{spec}(\mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}})]$ whose definition on the element of the basis is given in (2.4.3).

Also in this case, in order to deal with an easier notation, a module M will be “identified” with the isomorphism class $[M]$; thus, for example, $\mathfrak{l}_G(M) = \mathfrak{l}_G([M])$.

Proposition 2.6.4. *The homomorphism $\mathfrak{l}_G: \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}} \rightarrow \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$ has the following properties:*

1. $-\mathbb{K} \circ \mathfrak{l}_G = \text{Id}_{\mathbf{R}_{\mathbb{K}}(G)}$
2. If $H \leq G$, then for every $M \in \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}}$ $\text{res}_H^G(\mathfrak{l}_G(M)) = \mathfrak{l}_H(\text{res}_H^G(M))$

Proof. 1. It is enough to show that $\mathfrak{l}_{G,\mathbb{C}}^\vee \circ \kappa_{G,\mathbb{C}}^\vee = \text{Id}_{\mathbb{C}[\text{spec}(\mathbf{R}_{\mathbb{K}}(G))]}$. In fact, in this case, for every $M \in \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}}$ and for every ${}^G g \in \text{spec}(\mathbf{R}_{\mathbb{K}}(G))$

$$\langle \mathfrak{l}_G(M)_{\mathbb{K}}, {}^G g \rangle_{\mathbf{R}_{\mathbb{K}}(G)} = \langle M, {}^G g \rangle_{\mathbf{R}_{\mathbb{K}}(G)}. \quad (2.6.7)$$

Let ${}^G g \in \text{spec}(\mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}})$, then

$$\mathfrak{l}_{G,\mathbb{C}}^\vee \circ \kappa_{G,\mathbb{C}}^\vee ({}^G g) = \mathfrak{l}^\vee(\sim(g, \langle g_p \rangle)) = {}^G g \quad (2.6.8)$$

and this yields the claim.

2. As in Proposition 2.5.1.(2), the goal is to prove that the diagram below commutes

$$\begin{array}{ccc} \mathbf{R}_{\mathbb{K}}(G)_{\mathbb{C}} & \xrightarrow{\mathfrak{t}_G} & \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}} \\ \text{res}_H^G \downarrow & & \downarrow \text{res}_H^G \\ \mathbf{R}_{\mathbb{K}}(H)_{\mathbb{C}} & \xrightarrow{\mathfrak{t}_H} & \mathbf{L}_{\mathcal{O}}(H)_{\mathbb{C}} \end{array} \quad (2.6.9)$$

and, also in this case, it is enough to check that the dual diagram commutes.

$$\begin{array}{ccc} \mathbb{C}[\text{spec}(\mathbf{R}_{\mathbb{K}}(G))] & \xleftarrow{\mathfrak{l}_{H,\mathbb{C}}^\vee} & \mathbb{C}[\text{spec}(\mathbf{L}_{\mathcal{O}}(G))] \\ \text{res}_H^{G^\vee} \uparrow & & \uparrow \text{res}_H^{G^\vee} \\ \mathbb{C}[\text{spec}(\mathbf{R}_{\mathbb{K}}(H))] & \xleftarrow{\mathfrak{l}_{G,\mathbb{C}}^\vee} & \mathbb{C}[\text{spec}(\mathbf{L}_{\mathcal{O}}(H))] \end{array} \quad (2.6.10)$$

So, it is enough to prove that for every species $\sim^H(h, P)$ of $\mathbf{L}_{\mathcal{O}}(H)_{\mathbb{C}}$

$$\mathfrak{l}_{G,\mathbb{C}}^\vee(\text{res}_H^{G^\vee}(\sim^H(h, P))) = \text{res}_H^{G^\vee}(\mathfrak{l}_{H,\mathbb{C}}^\vee(\sim^H(h, P))). \quad (2.6.11)$$

First of all, let us observe that

$$\begin{aligned} \text{res}_H^{G^\vee} : \text{spec}(\mathbf{L}_{\mathcal{O}}(H)) &\rightarrow \text{spec}(\mathbf{L}_{\mathcal{O}}(G)) \\ \sim^H(h, P) &\mapsto \sim^G(h, P) \end{aligned} \quad (2.6.12)$$

and,

$$\begin{aligned} \text{res}_H^{G^\vee} : \text{spec}(\mathbf{R}_{\mathbb{K}}(H)) &\rightarrow \text{spec}(\mathbf{R}_{\mathbb{K}}(G)). \\ \sim^H h &\mapsto \sim^G h \end{aligned} \quad (2.6.13)$$

In this case we should define the dual maps of the restriction considering the $\mathbb{C}$ -vector spaces spanned by the set of the species and not just

between the spectra of the Grothendieck rings. In order to have an easier notation we will denote by $\text{res}_H^{G^\vee}$ the maps between the $\mathbb{C}$ -vector spaces, too. Then it holds that

$$\mathfrak{l}_{G,\mathbb{C}}^\vee(\text{res}_H^{G^\vee}(\sim^H(h, P))) = \mathfrak{l}_{G,\mathbb{C}}^\vee(G(h, P)) = \frac{1}{|C_{P'}(h_s)|} \sum_{v \in C_{P'}(h_s)} G(h \cdot v) \quad (2.6.14)$$

and

$$\begin{aligned} \text{res}_H^{G^\vee}(\mathfrak{l}_{H,\mathbb{C}}^\vee(\sim^H(h, P))) &= \text{res}_H^{G^\vee} \left(\frac{1}{|C_{P'}(h_s)|} \sum_{v \in C_{P'}(h_s)} H(h \cdot v) \right) \\ &= \frac{1}{|C_{P'}(h_s)|} \sum_{v \in C_{P'}(h_s)} G(h \cdot v) \end{aligned} \quad (2.6.15)$$

So, diagram (2.6.10) commutes and this implies the thesis. $\square$

Proposition 2.5.1.(1) and Proposition 2.6.4.(1) allow us to refer to the maps $\mathfrak{t}_G$ and $\mathfrak{l}_{G,\mathbb{C}}$ as *sections* of the surjective maps $\text{—}_{\mathbb{F}}: \mathbf{T}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{F}}(G)$ and $\text{—}_{\mathbb{K}}: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G)$.

Remark 2.6.5. Let us observe that the sections $\mathfrak{t}_H$ and $\mathfrak{l}_H$ do not commute with the induction to every overgroup G of a finite group H . In fact, for example, if $H \leq G$ such that $\text{Syl}_p(G) \cap \text{Syl}_p(H) = \emptyset$ and $(1, V) \in \text{spec}(\mathbf{T}_{\mathcal{O}}(G))$ for $V \in \text{Syl}_p(G)$, then

$$\mathfrak{t}_H^\vee(\text{ind}^\vee(G(1, V))) = 0 \quad (2.6.16)$$

while, on the other hand,

$$\text{ind}^\vee(\mathfrak{t}_G^\vee(G(1, V))) = 1. \quad (2.6.17)$$

Proposition 2.6.6. *Let P be a p -group and M an irreducible $\mathbb{K}P$ -module such that p divides the dimension of M over $\mathbb{K}$. Then $\mathfrak{l}_P(M) \in \mathbf{L}_{\mathcal{O}}^\sim(P)_{\mathbb{C}}$.*

Proof. It is enough to prove that for every $\sim(g, P) \in \text{spec}(\mathbf{L}_{\mathcal{O}}(P))$,

$$\langle \mathfrak{l}_G(M), \sim(g, P) \rangle_{\mathbf{L}_{\mathcal{O}}(P)} = 0.$$

In particular, if ψ is the irreducible $\mathbb{K}P$ -character associated to M , then

$$\begin{aligned} \langle \mathfrak{l}_P(M), \sim(g, P) \rangle_{\mathbf{L}_{\mathcal{O}}(P)} &= \langle M, \mathfrak{l}_{P,\mathbb{C}}^\vee(\sim(g, P)) \rangle_{\mathbf{R}_{\mathbb{K}}(P)} \\ &= \langle M, \frac{1}{|P'|} \sum_{v \in P'} ({}^P(g \cdot v)) \rangle_{\mathbf{R}_{\mathbb{K}}(P)} = \frac{1}{|P'|} \sum_{v \in P'} \psi(g \cdot v). \end{aligned} \quad (2.6.18)$$

Let $\mathbf{1}_{P'} = \text{ind}_{P'}^P(\mathbf{1}_P)$, where $\mathbf{1}_P$ denotes the character associated to the trivial $\mathbb{K}P$ -module. Then $\mathbf{1}_{P'} = \frac{1}{|P:P'|} \sum_{\rho \in \text{Irr}_{\mathbb{K}}(P/P')} \rho$. Thus the *Generalized Orthogonality Relation* (cf. [Isa06, Theorem 2.13]) implies that

$$\frac{1}{|P|} \sum_{v \in P'} \psi(gv) = \frac{1}{|P|} \sum_{v \in P} \psi(gv) \mathbf{1}_{P'}(v^{-1}) = 0. \quad (2.6.19)$$

Then the thesis holds. $\square$

Given a group G let $\pi_G: \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}} \rightarrow (\mathbf{L}_{\mathcal{O}}(G)/\mathbf{L}_{\mathcal{O}}^{\prec}(G))_{\mathbb{C}} \simeq \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G)_{\mathbb{C}}$ be the canonical projection. Then the following results hold.

Proposition 2.6.7. *Let N be a group with normal Sylow p -subgroup P . Let $L \in \text{IL}_{\mathcal{O}}^{\text{mx}}(N)$ and let $\text{res}_{\text{mx}_P}^N(L) = L_1 \oplus \cdots \oplus L_r$, where $L_i \in \text{IL}_{\mathcal{O}}^{\text{mx}}(P)$. Then for every $i, j \in \{1, \dots, r\}$, there exists $g \in N$ such that L_i is isomorphic to ${}^g L_j$.*

Proof. Let $\varphi \in \hat{P}$ be such that $\mathcal{O}_{\varphi}$ is a source of P and let I denote $I(\varphi) = \{g \in N \mid \varphi = {}^g \varphi\}$. Let Q_{φ} be the projective indecomposable $\mathcal{O}(I/\ker(\varphi))$ -lattice such that $L \simeq \text{ind}_I^N(\tilde{Q}_{\varphi})$, where $\tilde{Q}_{\varphi}$ is the indecomposable linear source $\mathcal{O}I$ -lattice with vertex set $\{P\}$ inflated from Q_{φ} (cf. 1.5.2). But then, it follows from the Mackey decomposition's formula (cf. [Ben06, Lemma 2.14]) that:

$$\begin{aligned} \text{res}_P^N(L) &= \text{res}_P^N(\text{ind}_I^N(\tilde{Q}_{\varphi})) \\ &= \bigoplus_{g \in N/PI} \text{ind}_{gI \cap P}^P(\text{res}_{gI \cap P}^{gI}({}^g \tilde{Q}_{\varphi})) \\ &= \bigoplus_{g \in N/PI} \bigoplus_{P \subseteq {}^g I} \text{res}_{gI \cap P}^{gI}({}^g \tilde{Q}_{\varphi}) \oplus \bigoplus_{g \in N/PI} \bigoplus_{P \not\subseteq {}^g I} \text{ind}_{gI \cap P}^P(\text{res}_{gI \cap P}^{gI}({}^g \tilde{Q}_{\varphi})). \end{aligned} \quad (2.6.20)$$

Then,

$$\text{res}_{\text{mx}_P}^N(L) = \bigoplus_{g \in N/PI} \bigoplus_{P \subseteq {}^g I} \text{res}_{gI \cap P}^{gI}({}^g \tilde{Q}_{\varphi}) \quad (2.6.21)$$

In particular $\text{res}_P^{gI}(Q_{\varphi}) = |{}^g I : P|^G \mathcal{O}_{\varphi}$, where $\mathcal{O}_{\varphi}$ is an indecomposable linear source $\mathcal{O}P$ -lattice with maximal vertex. In particular, the Green Indecomposability Theorem (cf. [CR90, Corollary 19.24]) implies that the indecomposable $\mathcal{O}({}^g I \cap P)$ -lattices induced to $\mathcal{O}P$ -lattices are still indecomposable lattices. $\square$

Also the following stronger reformulation holds.

Proposition 2.6.8. *Let N be a group with normal Sylow p -subgroup P and let $L \in \text{IL}_{\mathcal{O}}^{\text{mx}}(N)$. Then every indecomposable direct summand of $\text{res}_P^N(L)$ is of the form $\mathcal{O}_{\varphi}$ for some source $\mathcal{O}_{\varphi}$ of L .*

Proof. Let $\mathcal{O}_\psi$ be a source of $L \in \text{IL}_{\mathcal{O}}^{\text{mx}}(N)$. Then, $L|\text{ind}_P^N(\mathcal{O}_\psi)$ and in particular $\text{res}_P^N(L)|\text{res}_P^N(\text{ind}_P^N(\mathcal{O}_\psi))$. Applying the Mackey decomposition's formula (cf. [Ben06, Lemma 2.14]):

$$\text{res}_P^N(\text{ind}_P^N(\mathcal{O}_\psi)) = \oplus_{g \in N/P} {}^g \mathcal{O}_\psi. \quad (2.6.22)$$

□

In order to prove an analogous of Proposition 2.6.6 in the case of a finite group with normal Sylow p -subgroup, the following technical result is necessary.

Proposition 2.6.9. *Let G be a finite group and N a normal subgroup. Let $\chi \in \text{Irr}(G)$. If there exist $g \in G$ such that $\chi(g) \neq 0$ and $\chi(gn) = \chi(g)$ for each $n \in N$, then χ is inflated from G/N .*

Proof. Let $\psi = \sum_i \psi_i(1)\psi_i = \text{ind}_N^G(\mathbf{1})$, where $\mathbf{1}$ is the trivial $\mathbb{K}N$ -character. Then

$$\begin{aligned} \frac{1}{|G|} \sum_{x \in G} \chi(gx)\psi(x^{-1}) &= \sum_i \frac{\psi_i(1)}{|G|} \sum_{x \in G} \chi(gx)\psi_i(x^{-1}) \\ &= \sum_i \frac{\psi_i}{|G|} \cdot \frac{\chi(g)}{\chi(1)} \delta_{\chi, \psi_i} \\ &= \sum_i \frac{\psi_i}{|G|} \cdot \frac{\partial_{\chi, \psi_i}}{\chi(1)}. \end{aligned} \quad (2.6.23)$$

On the other hand,

$$\begin{aligned} \frac{1}{|G|} \sum_{x \in G} \chi(gx)\psi(x^{-1}) &= \frac{|G/N|}{|G|} \sum_{x \in N} \chi(gx) \\ &= \frac{|G/N|}{|G|} \cdot |N|\chi(g) \neq 0. \end{aligned} \quad (2.6.24)$$

The thesis follows. □

Fact 2.6.10. Let $G = \langle g \rangle P$, where g is an element of G of p' -order and P is a p -group. Let $\psi = \text{res}_P^G(\chi)$, then, by Clifford Theory, either $\psi \in \text{Irr}_{\mathbb{K}}(P)$ or $\chi(gv) = 0$ for every $v \in P$. In fact let $\text{res}_N^G(\chi) = \sum_i \phi_i = \varphi$, where $\phi_i \in \text{Irr}(P)$. If the inertia group $I(\varphi)$ of φ is different from G , then for every $v \in P$, $gv \notin I(\varphi)$. Since $P \trianglelefteq I(\psi) \trianglelefteq G$, χ is induced from a $\mathbb{K}I(\varphi)$ -character and $\chi(gv) = 0$. On the other hand if $I(\varphi) = G$, then χ is $\langle g \rangle$ -stable and thus irreducible (cf. [Isa06, ?]).

Proposition 2.6.11. *Let $G = \langle g_s \rangle P$, where g_s is an element of G with p' -order and P is a p -group. If M is an irreducible $\mathbb{K}G$ -module with character $\chi \in \text{Irr}_{\mathbb{K}}(G)$ such that $\chi \notin \text{Irr}_{\mathbb{K}}(G/P')$. Let $(g = g_s g_p = g_p g_s, P) \in \text{spec}(\mathbf{L}_{\mathcal{O}}(G))$, then $\mathfrak{l}_G(M)(g, P) = 0$.*

Proof. By definition it is enough to show that $\sum_{v \in C_{P'}(g_s)} \chi(g \cdot v) = 0$ for every irreducible characters $\chi \in \text{Irr}(G)$.

If $\text{res}_P^G(\chi) \notin \text{Irr}(P)$, then by Fact 2.6.10, $\chi(gv) = 0$ for every $v \in C_{P'}(g_s)$ and thus the thesis follows trivially.

Let us suppose $\text{res}_P^G(\chi)$ irreducible. If $g_s = 1$, then Proposition 2.6.6 imply the thesis. Otherwise, [Gla68, Theorem 3] implies that there exists $\lambda \in \text{Irr}(C_P(g_s))$ such that

$$\chi(gv) = \chi(g_s uv) = \varepsilon \lambda(uv), \quad (2.6.25)$$

where $\varepsilon = \pm 1$ and $v \in C_{P'}(g_s)$. Let us suppose $\sum_{v \in C_{P'}(g_s)} \chi(gv) \neq 0$, then also $\sum_{v \in C_{P'}(g_s)} \lambda(uv) \neq 0$. So, if $\mu = \text{ind}_{C_P(g_s) \cap P'}^{C_P(g_s)}(\mathbf{1})$ where $\mathbf{1}$ is the trivial $\mathbb{K}[C_P(g_s) \cap P']$ -character,

$$\sum_{v \in C_P(g_s)} \lambda(uv) \mu(v^{-1}) \neq 0. \quad (2.6.26)$$

Therefore there exists a linear character $\theta \in \text{Irr}(C_P(g_s)/C_P(g_s) \cap P')$ which is an irreducible constituent of μ and such that $\sum_{v \in C_{P'}(g_s)} \lambda(uv) \theta(v^{-1}) \neq 0$. The *Generalized orthogonal relation* implies that $\lambda = \theta$ is linear. So, $\lambda(ux) = \lambda(u)$ for every $x \in P'$ and thus

$$\chi(gv) = \chi(g_s uv) = \varepsilon \lambda(uv) = \varepsilon \lambda(u) = \chi(g), \quad (2.6.27)$$

for every $v \in C_{P'}(g_s)$.

Thanks to [Gla68, Lemma 2] for every $x \in P'$, there exists $t \in C_{P'}(g_s)$ such that $g_s x$ is conjugate to $g_s t$. Then $\chi(g_s x) = \chi(g_s t) = \varepsilon \lambda(t) = \varepsilon$ and thus $\chi(g_s v) = \chi(g_s)$ for every $v \in P'$, and Proposition 2.6.9 implies that χ is inflated from a G/P' -character, which contradicts the hypothesis. $\square$

Remark 2.6.12. Let N be a finite group with normal Sylow p -subgroup P . Theorem 1.5.12 ensures that for $L \in \text{IL}_{\mathcal{O}}^{\prec}(N)$, $L_{\mathbb{K}} \notin \text{Irr}_{\mathbb{K}}(N/P')$.

Let us observe that $L_{\mathbb{K}}$ can belong to $\mathbf{R}_{\mathbb{K}}(N/P')$. In fact, for example, if $N = P$, $L = \text{ind}_{P'}^P(\mathcal{O}) \in \text{IL}_{\mathcal{O}}^{\prec}(N)$ then $L_{\mathbb{K}}$ is a sum of linear characters.

Proposition 2.6.13. *Let G be a finite group with a Sylow p -subgroup P . Let $M \in \mathbf{R}_{\mathbb{K}}(G)$ be an irreducible $\mathbb{K}G$ -module, and let $\mathbf{l}_G(M) = \sum_{i=1}^n a_i L_i \in \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}}$, where $a_i \in \mathbb{C}$ and L_i are indecomposable linear source $\mathcal{O}G$ -lattices. Without lost of generality let L_i be $\mathcal{O}G$ -lattices maximal vertex for an integer $m \leq n$ and every $i \in \{1, \dots, m\}$. Then for $i \in \{1, \dots, m\}$, the coefficients a_i are integers.*

Proof. Let $P \in \text{Syl}_p(G)$ and let $N = N_G(P)$, thanks to Proposition 2.6.4.2, it holds that $\mathbf{l}_N(\text{res}_N^G(M)) = \text{res}_N^G(\mathbf{l}_G(M)) \in \mathbf{L}_{\mathcal{O}}(N)_{\mathbb{C}}$. By Green correspondence

$$\text{res}_N^G(\mathbf{l}_G(M)) = \sum_{i=1}^n a_i \cdot \text{res}_N^G(L_i) = \sum_{i=1}^m a_i \cdot f(L_i) \oplus \delta, \quad (2.6.28)$$

where $\delta \in \mathbf{L}_{\mathcal{O}}^{\prec}(N)$ and, for $i \in \{1, \dots, m\}$, $f(L_i)$ is an indecomposable linear source $\mathcal{O}N$ -lattice with maximal vertex.

On the other hand,

$$\mathfrak{l}_N(\text{res}_N^G(M)) = \mathfrak{l}_N\left(\sum_{i=1}^k c_i M_i\right) = \sum_{i=1}^k c_i \cdot \mathfrak{l}_N(M_i), \quad (2.6.29)$$

where $c_i \in \mathbb{Z}$ and $M_i \in \text{Irr}_{\mathbb{K}}(N)$.

Proposition 2.6.11 implies that an indecomposable linear source $\mathcal{O}N$ -lattice with maximal vertex is in the image via the map $\mathfrak{l}_N$ of an irreducible $\mathbb{K}[N/P']$ -module. But then, without loss of generality, thanks to Remark 2.6.12 we can conclude that $\pi(\mathfrak{l}_N(M_i)) = f(L_i)$, for $i \in \{1, \dots, m\}, m \leq k$.

Then we can conclude that

$$\mathfrak{l}_N(\text{res}_N^G(M)) = \sum_{i=1}^m c_i f(L_i) \oplus \delta, \quad (2.6.30)$$

where $\delta \in \mathbf{L}_{\mathcal{O}}^{\prec}(N)$. This implies $c_i = a_i \in \mathbb{Z}$, for $i \in \{1, \dots, m\}$. $\square$

In § 2.8 it will be clear that all the coefficients are integers.

2.7 Canonical induction formulae and sections

In the previous sections we have constructed sections of the surjective maps $-\mathbb{K} : \mathbf{R}_{\mathbb{K}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}(G)$ and $-\mathbb{F} : \mathbf{R}_{\mathbb{F}}(G) \rightarrow \mathbf{T}_{\mathcal{O}}(G)$ considering the set of species of these Grothendieck rings. Of course this approach allows us to compute explicitly the values of these maps, but it does not ensure that the sections are integral maps, i.e., $\text{im}(\mathfrak{t}_G) \subseteq \mathbf{T}_{\mathcal{O}}(G)$ and $\text{im}(\mathfrak{l}_G) \subseteq \mathbf{L}_{\mathcal{O}}(G)$.

As suggested by R. Boltje (cf. [Bol98b]), it is possible to define canonical sections of the surjective maps considered, using the canonical induction formulae for the Mackey functors defined by the Grothendieck rings $\mathbf{R}_{\mathbb{K}}$ and $\mathbf{R}_{\mathbb{F}}$ and considering the $\mathbb{Z}$ -restriction subfunctor given by their abelianizations. For convenience of the reader in this section the construction of the canonical induction formulae for $\mathbf{R}_{\mathbb{K}}$ and $\mathbf{R}_{\mathbb{F}}$ (cf. [Bol98a, Theorem 9.3]) is (briefly) presented in order to fix the notation and to allow a clearer definition of the canonical sections presented in § 2.8. For a more precise and complete reference see [Bol98a].

Let $\mathcal{M}(G) = \{(H, \varphi) \mid H \leq G, \varphi \in \text{Hom}(H, \mathbb{K}^*)\}$ be the set of *monomial pairs*; $\mathcal{M}(G)$ is a partially ordered set with $(I, \psi) \leq (H, \varphi)$ if, and only if, $I \leq H$ and $\psi|_H = \varphi$. Clearly, the group G acts on $\mathcal{M}(G)$ via conjugation

$${}^g(H, \varphi) = ({}^gH, {}^g\varphi), \quad (2.7.1)$$

where ${}^g\varphi(x) = \varphi(g^{-1}xg)$. Let $\mathcal{D}(G)$ denote the free abelian group on the G -conjugacy classes $[H, \varphi]_G$ of $\mathcal{M}(G)$. In particular $\mathcal{D}(G)$ is a commutative

ring with product given by

$$[H, \varphi]_G \cdot [I, \psi]_G = \sum_{g \in H \backslash G / I} [H \cap^g I, \varphi|_{H \cap^g I} \cdot^g \psi|_{H \cap^g I}]_G. \quad (2.7.2)$$

Let us consider the mapping

$$\begin{aligned} b_G^{\mathbf{R}_{\mathbb{K}}}: \mathcal{D}(G) &\rightarrow \mathbf{R}_{\mathbb{K}}(G) \\ [H, \varphi]_G &\mapsto \text{ind}_H^G(\varphi). \end{aligned} \quad (2.7.3)$$

A *canonical induction formula* for $\mathbf{R}_{\mathbb{K}}(G)$ from $\mathcal{D}(G)$ is an injective map

$$a_G^{\mathbf{R}_{\mathbb{K}}}: \mathbf{R}_{\mathbb{K}}(G) \rightarrow \mathcal{D}(G) \quad (2.7.4)$$

such that

$$b_G^{\mathbf{R}_{\mathbb{K}}} \circ a_G^{\mathbf{R}_{\mathbb{K}}} = \text{Id}_{\mathbf{R}_{\mathbb{K}}(G)}. \quad (2.7.5)$$

Analogously, let $\mathcal{M}(G)_p$ denote the subset of $\mathcal{M}(G)$ of elements (H, φ) where φ has p' -order and $\mathcal{D}_p(G)$ denote the free abelian group on the G -conjugacy classes $[H, \varphi]_G$ of elements of $\mathcal{M}(G)_p$. As before, let us consider the mapping

$$\begin{aligned} b_G^{\mathbf{R}_{\mathbb{F}}}: \mathcal{D}_p(G) &\rightarrow \mathbf{R}_{\mathbb{F}}(G) \\ [H, \varphi]_G &\mapsto \text{ind}_H^G(\varphi) \end{aligned} \quad (2.7.6)$$

A canonical induction formula for $\mathbf{R}_{\mathbb{F}}(G)$ from $\mathcal{D}_p(G)$ is an injective map

$$a_G^{\mathbf{R}_{\mathbb{F}}}: \mathbf{R}_{\mathbb{F}}(G) \rightarrow \mathcal{D}_p(G) \quad (2.7.7)$$

such that

$$b_G^{\mathbf{R}_{\mathbb{F}}} \circ a_G^{\mathbf{R}_{\mathbb{F}}} = \text{Id}_{\mathbf{R}_{\mathbb{F}}(G)}. \quad (2.7.8)$$

2.7.1 The canonical induction formula for $\mathbf{R}_{\mathbb{K}}$

Let $H \leq G$ be a subgroup of G and let $\mathbf{R}_{\mathbb{K}}^{\text{ab}}(H)$ the span over $\mathbb{Z}$ of the linear characters of G . Let one define a map

$$p_H: \mathbf{R}_{\mathbb{K}}(H) \rightarrow \mathbf{R}_{\mathbb{K}}^{\text{ab}}(H) \quad (2.7.9)$$

such that, if $\chi \in \text{Irr}_{\mathbb{K}}(H)$

$$p_H(\chi) = \begin{cases} \chi & \text{if } \chi(1) = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2.7.10)$$

Let $\mathfrak{B}(H) = \hat{H}$, then $\mathfrak{B}(H)$ is such that for all $K \leq H \leq G$, the elements $\text{res}_K^H(\varphi) \in \mathbf{R}_{\mathbb{K}}^{\text{ab}}(K)$ are a linear combination

$$\text{res}_K^H(\varphi) = \sum_{\psi \in B(K)} m_{(K, \psi)}^{(H, \varphi)} \cdot \psi \quad (2.7.11)$$

of the basis elements $\psi \in \mathfrak{B}(K)$ with non-negative coefficients $m_{(K,\psi)}^{(H,\varphi)} \in \mathbb{N}_0$. In particular, $\mathcal{M}(H) = \{(K, \varphi) | K \leq G, \varphi \in \mathfrak{B}(K)\}$. Let $\Delta(\mathcal{M}(H))$ denote the set of ascending chains in the partially order set $\mathcal{M}(H)$.

Then,

$$p_H(\chi) = \sum_{\varphi \in \mathfrak{B}(H)} m_\varphi(\chi) \cdot \varphi, \quad (2.7.12)$$

where $m_\varphi(\chi) \in \mathbb{Z}$ is the multiplicity of φ in χ .

In this setting, it is possible to define a canonical induction formula $a_G^{\mathbf{R}_\mathbb{K}} : \mathbf{R}_\mathbb{K}(G) \rightarrow \mathcal{D}(G)$ as follows,

$$a_G^{\mathbf{R}_\mathbb{K}}(\chi) = \sum_{\sigma \in G \setminus \Delta(\mathcal{M}(G))} (-1)^n \cdot m_{\varphi_n}(\text{res}_{G_n}^G(\chi)) [G_0, \varphi_0]_G \quad (2.7.13)$$

where $\sigma = ((G_0, \varphi_0) < \dots < (G_n, \varphi_n))$ and the sum runs over a set of representatives for the G -orbits of $\mathcal{M}(G)$.

Remark 2.7.1. Let us observe that m_{φ_n} depends on p .

Remark 2.7.2. In particular $a_G^{\mathbf{R}_\mathbb{K}}$ is $\mathbf{R}_\mathbb{K}^{\text{ab}}(G)$ -linear, but not a ring homomorphism since p_G is not a ring homomorphism, but a $\mathbf{R}_\mathbb{K}^{\text{ab}}(G)$ -linear map. Moreover, if $H \leq G$ and $\varphi \in \mathbf{R}_\mathbb{K}^{\text{ab}}(H)$, then

$$a_H^{\mathbf{R}_\mathbb{K}}(\varphi) = [H, \varphi]_H. \quad (2.7.14)$$

Remark 2.7.3. The canonical induction formula defined commutes with the restriction, i.e., for all subgroup $H \leq G$ one has

$$\text{res}_H^+ \circ a_G^{\mathbf{R}_\mathbb{K}} = a_H^{\mathbf{R}_\mathbb{K}} \circ \text{res}_H^G, \quad (2.7.15)$$

where $\text{res}_H^+ : \mathcal{D}(G) \rightarrow \mathcal{D}(H)$ is defined as in [Bol98a, §10.2] (see [Bol98a, Proposition 10.3]).

2.7.2 The canonical induction formula for $\mathbf{R}_\mathbb{F}$

As before, it is necessary to introduce a suitable setting to define a canonical induction formula for $\mathbf{R}_\mathbb{F}$. Let $H \leq G$ be a subgroup of G and let $\mathbf{R}_\mathbb{F}^{\text{ab}}(H)$ the span over $\mathbb{Z}$ of the isomorphism classes of $\mathbb{F}H$ -modules of dimension 1 over $\mathbb{F}$. Let us define a map

$$p_H : \mathbf{R}_\mathbb{F}(H) \rightarrow \mathbf{R}_\mathbb{F}^{\text{ab}}(H) \quad (2.7.16)$$

such that, if M is simple

$$p_H([M]) = \begin{cases} [M] & \text{if } \dim_\mathbb{F}([M]) = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2.7.17)$$

Let $\mathfrak{B}(H) = \hat{H}(\mathbb{F}) = \text{Hom}(H, \mathbb{F}^*)$, then $\mathfrak{B}(H)$ is such that for all subgroups $K \leq H \leq G$, the elements $\text{res}_K^H(\varphi) \in \mathbf{R}_{\mathbb{F}}^{\text{ab}}(K)$ are a linear combination

$$\text{res}_K^H(\varphi) = \sum_{\psi \in B(K)} m_{(K, \psi)}^{(H, \varphi)} \cdot \psi \quad (2.7.18)$$

of the basis elements $\psi \in \mathfrak{B}(K)$ with non-negative coefficients $m_{(K, \psi)}^{(H, \varphi)} \in \mathbb{N}_0$.

In particular, $\mathcal{M}(H)_p = \{(K, \varphi) | K \leq G, \varphi \in \mathfrak{B}(K)\}$. Let $\Delta(\mathcal{M}(H)_p)$ denote the set of ascending chains in $\mathcal{M}(H)_p$. Given a subgroup H of G and $[V] \in \mathbf{R}_{\mathbb{F}}(H)$, let $m_{\varphi}([V])$ be the integer counting the multiplicity of $\mathbb{F}_{\varphi}$ as composition factor in V and $\varphi \in \hat{H}(\mathbb{F})$ (see [Bol98a, Example 9.8]).

In this setting, it is possible to define a canonical induction formula $a_G^{\mathbf{R}_{\mathbb{F}}}: \mathbf{R}_F(G) \rightarrow \mathcal{D}_p(G)$ such that

$$a_G^{\mathbf{R}_{\mathbb{F}}}([M]) = \sum_{\sigma \in G \backslash \Delta(\mathcal{M}(G)_p)} (-1)^n \cdot m_{\varphi_n}(\text{res}_{G_n}^G([M])) [G_0, \varphi_0]_G \quad (2.7.19)$$

where $\sigma = ((G_0, \varphi_0) < \dots < (G_n, \varphi_n))$ and the sum runs over a set of representatives for the G -orbits of $\mathcal{M}(G)$.

In analogy with the case presented in § 2.7.1, the following remarks can be stated.

Remark 2.7.4. Let us observe that m_{φ_n} depends on p .

Remark 2.7.5. In particular $a_G^{\mathbf{R}_{\mathbb{F}}}$ is $\mathbf{R}_{\mathbb{F}}^{\text{ab}}(G)$ -linear, but not a ring homomorphism since p_G is not a ring homomorphism, but a $\mathbf{R}_{\mathbb{F}}(G)$ -linear map. Moreover, if $H \leq G$ and $\varphi \in \hat{H}(\mathbb{F})$, then

$$a_H^{\mathbf{R}_{\mathbb{F}}}([\mathbb{F}_{\varphi}]) = [H, [\mathbb{F}_{\varphi}]]_H. \quad (2.7.20)$$

Remark 2.7.6. The canonical induction formula defined commutes with the restriction, i.e., for all subgroup $H \leq G$ one has

$$\text{res}_H^G \circ a_G^{\mathbf{R}_{\mathbb{F}}} = a_H^{\mathbf{R}_{\mathbb{F}}} \circ \text{res}_H^G \quad (2.7.21)$$

(cf. [Bol98a, Proposition 10.3]).

2.8 Canonical sections

At this point the main goal is to construct canonical sections τ_G and λ_G for the surjective maps

$$\begin{aligned} -_{\mathbb{F}}: \mathbf{T}_{\mathcal{O}}(G) &\rightarrow \mathbf{R}_{\mathbb{F}}(G) \\ -_{\mathbb{K}}: \mathbf{L}_{\mathcal{O}}(G) &\rightarrow \mathbf{R}_{\mathbb{K}}(G) \end{aligned} \quad (2.8.1)$$

using the canonical induction formulae $a_G^{\mathbf{R}_{\mathbb{K}}}$ and $a_G^{\mathbf{R}_{\mathbb{F}}}$ defined in the previous sections.

First of all, let one define the homomorphism

$$\begin{aligned}\beta_G: \mathcal{D}(G) &\rightarrow \mathbf{L}_{\mathcal{O}}(G). \\ [H, \varphi]_G &\mapsto [\text{ind}_H^G(\mathcal{O}_\varphi)]\end{aligned}\tag{2.8.2}$$

In particular,

$$\beta_G|_{\mathcal{D}_p(G)}: \mathcal{D}_p(G) \rightarrow \mathbf{T}_{\mathcal{O}}(G),\tag{2.8.3}$$

in fact, if φ has p' -order, then $\mathcal{O}_\varphi$ has trivial source.

From now on let $\beta_{G,p}$ denote the restriction of β_G to $\mathcal{D}_p(G)$. The maps β_G and $\beta_{G,p}$ are surjective; there exists, in fact, a canonical induction formula for $\mathbf{L}_{\mathcal{O}}(G)$ and $\mathbf{T}_{\mathcal{O}}(G)$ using $\mathcal{D}(G)$ and $\mathcal{D}_p(G)$ respectively.

In particular, the maps β_G and $\beta_{G,p}$ commute with the restriction, i.e., for all subgroups H of G one has

$$\begin{aligned}\text{res}_H^G \circ \beta_G &= \beta_H \circ \text{res}_H^G \quad \text{and} \\ \text{res}_H^G \circ \beta_{G,p} &= \beta_{H,p} \circ \text{res}_H^G.\end{aligned}\tag{2.8.4}$$

Thus one can define the maps

$$\begin{aligned}\lambda_G &= \beta_G \circ a_G^{\mathbf{R}_{\mathbb{K}}}: \mathbf{R}_{\mathbb{K}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}(G) \quad \text{and} \\ \tau_G &= \beta_{G,p} \circ a_G^{\mathbf{R}_{\mathbb{F}}}: \mathbf{R}_{\mathbb{F}}(G) \rightarrow \mathbf{T}_{\mathcal{O}}(G).\end{aligned}\tag{2.8.5}$$

Explicitly, for $\sigma = ((G_0, \varphi_0) < \dots < (G_n, \varphi_n))$

$$\lambda_G(\chi) = \sum_{\sigma \in G \setminus \Delta(\mathcal{M}(G))} (-1)^n \cdot m_{\varphi_n}(\text{res}_{G_n}^G(\chi)) [\text{ind}_{G_0}^G(\mathcal{O}_{\varphi_0})].\tag{2.8.6}$$

and,

$$\tau_G([M]) = \sum_{\sigma \in G \setminus \Delta(\mathcal{M}(G)_p)} (-1)^n \cdot m_{\varphi_n}(\text{res}_{G_n}^G([M])) [\text{ind}_{G_0}^G(\mathcal{O}_{\varphi_0})].\tag{2.8.7}$$

The definition of the canonical section τ_G was presented in the preprint [Bol]. It has been repeated in this chapter for completeness, as the characterization of the set of the species considered.

2.8.1 The canonical section λ_G

Proposition 2.8.1. *Let $\lambda_G: \mathbf{R}_{\mathbb{K}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}(G)$ be defined as in (2.8.6), then λ_G has the following properties:*

- (i) *the map λ_G commutes with restriction, i.e., for all $H \leq G$ one has $\text{res}_H^G \circ \lambda_G = \lambda_H \circ \text{res}_H^G$.*
 - (ii) *The map λ_G is a canonical section for $\text{--}_{\mathbb{K}}: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G)$, i.e., $\text{--}_{\mathbb{K}} \circ \lambda_G = \text{Id}_{\mathbf{R}_{\mathbb{K}}(G)}$.*
-

Proof.

- (i) As β_G and $a_G^{\mathbf{R}_{\mathbb{K}}}$ commute with restriction, then also the map λ_G commutes with restriction.
- (ii) Every character is uniquely determined by its restriction to cyclic subgroups, then it is enough to show that $\text{res}_H^G \circ -_{\mathbb{K}} \circ \lambda_G = \text{res}_H^G : \mathbf{R}_{\mathbb{K}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(H)$ for every cyclic subgroups H of G . Both $-_{\mathbb{K}}$ and λ_G commute with restriction, then it is enough to prove that $-_{\mathbb{K}} \circ \lambda_H = \text{Id}_{\mathbf{R}_{\mathbb{K}}(H)}$ for a cyclic group H . The group H is cyclic, thus $\mathbf{R}_{\mathbb{K}}(H) = \text{span}_{\mathbb{Z}}\{\varphi | \varphi \in \hat{H}\}$ and

$$(\lambda_H(\varphi))_{\mathbb{K}} = (\beta_G([H, \varphi]_H))_{\mathbb{K}} = ([\mathcal{O}_{\varphi}])_{\mathbb{K}} = \varphi, \quad (2.8.8)$$

and this implies the thesis. $\square$

Proposition 2.8.2. *Let P be a p -group and $\psi \in \text{Irr}_{\mathbb{K}}(P)$ such that $p | \psi(1)$. Then $\lambda_P(\psi) \in \mathbf{L}_{\mathcal{O}}^{\sim}(P)$.*

Proof. First of all,

$$\lambda_P(\psi) = \sum_{\sigma \in P \setminus \Delta(\mathcal{M}(P))} (-1)^n \cdot m_{\varphi_n}(\text{res}_{P_n}^P(\psi)) \cdot [\text{ind}_{P_0}^P(\mathcal{O}_{\varphi_0})], \quad (2.8.9)$$

where $\sigma = ((P_0, \varphi_0) < \dots < (P_n, \varphi_n))$. If $P_0 \neq P$, then $[\text{ind}_{P_0}^P(\mathcal{O}_{\varphi_0})]$ has no components with maximal vertex.

On the other hand, if $P_0 = P$, then $\sigma = (P, \varphi)$ for some $\varphi \in \hat{P}$ and, as ψ is a non linear irreducible character, then $m_{\varphi_0}(\psi) = 0$. $\square$

The following result has been proved by R. Boltje.

Proposition 2.8.3. *Let N be a group with normal Sylow p -subgroup P . Let $\chi \in \text{Irr}_{\mathbb{K}}(N)$ such that $p | \chi(1)$. Then $\lambda_N(\chi) \in \mathbf{L}_{\mathcal{O}}^{\sim}(N)$.*

Fact 2.8.4. Let G be a finite group and $N \trianglelefteq G$. Then in general $\lambda_G \circ \text{inf}_{G/N}^G \neq \text{inf}_{G/N}^G \circ \lambda_{G/N}$. Let $\chi = \text{inf}_{G/N}^G(\bar{\chi})$, for $\bar{\chi} \in \text{Irr}(G/N)$. Then

$$\lambda_G(\chi) - \text{inf}_{G/N}^G(\lambda_{G/N}(\bar{\chi})) = \sum_{\sigma \in \mathfrak{D}} (-1)^n \cdot m_{\varphi_n}(\text{res}_{G_n}^G(\chi)) [\text{ind}_{G_0}^G(\mathcal{O}_{\varphi_0})] \quad (2.8.10)$$

where $\mathfrak{D} \subseteq G \setminus \Delta(\mathcal{M}(G))$ is the set of chains $\sigma = (G_0, \varphi_0) < \dots < (G_n, \varphi_n)$ such that $N \not\subseteq G_0$.

Proposition 2.8.5. *The map $(\lambda_G)_{\mathbb{C}}$ is induced by $\mathfrak{l}_{G, \mathbb{C}}^{\vee}$.*

Proof. The map $(\lambda_G)_{\mathbb{C}}$ is induced by $\mathfrak{l}_{G, \mathbb{C}}^{\vee}$ if and only if $\mathfrak{l}_G(\chi)(g, V) = \lambda_G(\chi)(g, V)$ for every $\chi \in \text{Irr}(G)$ and $(g, V) \in \text{spec}(\mathbf{L}_{\mathcal{O}}(G))$. Let $H = \langle g_s \rangle V$, where $g = g_s u$ for g_s of p' -prime order and $u \in C_P(g_s)$. In particular $\mathfrak{l}_G(\chi)(g, V) = \text{res}_H^G(\mathfrak{l}_G(\chi))(g, V) = \mathfrak{l}_H(\text{res}_H^G(\chi))(g, V)$ and $\lambda_G(\chi)(g, V) = \text{res}_H^G(\lambda_G(\chi))(g, V) = \lambda_H(\text{res}_H^G(\chi))(g, V)$. Then it is enough to prove that for every $\psi \in \text{Irr}(H)$,

$$\mathfrak{l}_H(\psi)(g, V) = \lambda_H(\psi)(g, V). \quad (2.8.11)$$

Case 1: If ψ is not inflated from an irreducible $\mathbb{K}H/V'$ -character, then Proposition 2.6.11 and Proposition 2.8.3 imply $\lambda_H(\psi)(g, V) = 0 = \mathfrak{l}_H(\psi)(g, V)$.

Case 2: If $\psi(1) = 1$, then $\lambda_H(\psi)(g, V) = \mathfrak{l}_H(\psi)(g, V) = \psi(g)$.

Case 3: Let ψ be inflated from an irreducible $\mathbb{K}H/V'$ and $1 \neq \psi(1)$. Then $\psi(g) = \psi(gv)$ for every $v \in C_{V'}(g_s)$, thus $\mathfrak{l}_H(\psi)(g, V) = \psi(g)$. Let $\psi = \inf_{H/V'}^H(\bar{\psi})$, where $\bar{\psi} \in \text{Irr}(H/V')$. By definition,

$$\lambda_H(\psi) = \sum_{\sigma \in H \setminus \Delta(\mathcal{M}(H))} (-1)^n \cdot m_{\varphi_n}(\text{res}_{G_n}^G(\psi)) [\text{ind}_{G_0}^G(\mathcal{O}_{\varphi_0})]. \quad (2.8.12)$$

Let a_σ be the term of the alternating sum in (2.8.12) which corresponds to the chain $\sigma \in H \setminus \Delta(\mathcal{M}(H))$ and let $(G_0, \varphi_0)_\sigma$ be its first term. It is clear that if $V \not\subseteq G_0$, then $a_\sigma = 0$. Then, Fact 2.8.4, implies that in

$$\inf_{H/V'}^H(\lambda_{H/V'}(\bar{\psi})) = \lambda_H(\psi), \quad (2.8.13)$$

and thus

$$\lambda_H(\psi)(g, V) = \inf_{H/V'}^G(\lambda_{H/V'}(\bar{\psi}))(g, V) = \lambda_{H/V'}(\bar{\psi})(gV', V/V') \quad (2.8.14)$$

Let us proceed by steps.

[1] From now on let $\bar{H} = H/V' = \langle g_s \rangle \bar{V}$, where $\bar{V} = V/V'$ is abelian and $g_s \neq 1$. Since $p \nmid \bar{\psi}(1) \neq 1$, by Clifford theory $\bar{\psi}(g_s u) = 0$ for every $u \in \bar{V}$. In fact $\text{res}_{\bar{V}}^{\bar{H}}(\bar{\psi}) = \oplus_{1 \leq i \leq r} \mu_i$, for μ_i linear pairwise distinct characters. For every $i \in \{1, \dots, r\}$, $I = I_{\bar{H}}(\mu_i) \subset \bar{H}$. Moreover $\bar{H}\mu_i = \{\mu_1, \dots, \mu_r\}$ (and thus $r = |\bar{H} : I|$).

[2] Let (K, φ) be a pair in $\mathcal{D}(\bar{H})$ such that $m_\varphi(\text{res}_{\bar{K}}^{\bar{H}}(\bar{\psi})) \neq 0$. Then there exists $i \in \{1, \dots, r\}$ such that $\text{res}_{\bar{V}}^{\bar{K}}(\varphi) = \mu_i$. In fact, $\varphi \mid \text{res}_{\bar{K}}^{\bar{H}}(\bar{\psi})$ and thus $\text{res}_{\bar{V}}^{\bar{K}}(\varphi) \mid \text{res}_{\bar{V}}^{\bar{H}}(\bar{\psi}) = \sum_{1 \leq i \leq r} \mu_i$. Then, considering that $K/\ker(\mu_i)$ is cyclic, ${}^k\mu_i = \mu_i$ for every $k \in K$, and thus $K \subseteq I_{\bar{H}}(\mu_i) = I$.

Let $\sigma = (H_0, \varphi_0) < \dots < (H_n, \varphi_n)$, let us observe that if (H_n, φ_n) satisfies [2], then also (H_0, φ_0) satisfies [2].

[3] Let $J \subset \bar{H}$ such that $\bar{V} \subseteq J \subseteq I$ and let $\rho \in \hat{J}$ be such that $\text{res}_{\bar{V}}^J(\rho) \in \{\mu_1, \dots, \mu_r\}$. We want to prove $\text{ind}_J^{\bar{H}}(\rho)(g) = 0$. Since J is normal in $\bar{H}$, $I_{\bar{H}}(\rho) \subseteq I_{\bar{H}}(\text{res}_{\bar{V}}^J(\rho)) = I_{\bar{H}}(\mu_i) = I$. Moreover,

$I/\ker(\mu_i)$ is cyclic and thus $I_{\bar{H}}(\rho) = I$. Now,

$$\begin{aligned} \text{ind}_J^{\bar{H}}(\rho) &= \text{ind}_I^{\bar{H}}(\text{ind}_J^I \rho) \\ &= \text{ind}_I^{\bar{H}} \left(\sum_{\phi \in \hat{I}, \text{res}(\phi)=\rho} \phi \right) \\ &= \sum_{\phi \in \hat{I}, \text{res}(\phi)=\lambda} \text{ind}_I^{\bar{H}} \phi. \end{aligned} \quad (2.8.15)$$

In particular, $\text{ind}_I^{\bar{H}} \phi$ are irreducible characters and $I_{\bar{H}}(\phi) = I$. Since $g_s \notin I$, even $g_s u \notin I$, and thus $\text{ind}_I^{\bar{H}} \phi(g) = 0$ and this implies [3].

[4] The goal now is to show that $\text{ind}_J^{\bar{H}}(\mathcal{O}_\rho)(g, \bar{V}) = \text{ind}_J^{\bar{H}}(\rho)(g)$. Applying the Mackey formula,

$$\text{res}_{\bar{V}}^{\bar{H}}(\text{ind}_J^{\bar{H}}(\mathcal{O}_\rho)) = \oplus_{x \in \bar{H}/J} \text{res}_V^{xJ} \mathcal{O}_{x\rho} \quad (2.8.16)$$

In general if W is an $\mathcal{O}\bar{H}$ -lattice and $\text{res}_{\bar{V}}^{\bar{H}} W = W^{\text{mx}} + W^{\prec}$, where $W^{\text{mx}} \in \mathbf{L}_{\mathcal{O}}^{\text{mx}}(\bar{V})$ and $W^{\prec} \in \mathbf{L}_{\mathcal{O}}^{\prec}(\bar{V})$, then $W(g, \bar{V}) = (W^{\text{mx}})_{\mathbb{K}}(g)$. Then, since for every $x \in \bar{H}/J$, $\text{res}_V^{xJ} \mathcal{O}_{x\rho}$ have maximal vertex, [4] follows.

Finally, [3] implies $\text{ind}_J^{\bar{H}}(\mathcal{O}_\rho)(g, \bar{V}) = 0$, and [1] and [2] imply that

$$\lambda_{\bar{H}}(\bar{\psi})(g, \bar{V}) = 0 = \bar{\psi}(g). \quad (2.8.17)$$

□

Remark 2.8.6. Let G be a finite group with normal Sylow p -subgroup P and let $L \in \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(N)$. Let $\chi \in \text{Irr}_{p'}(G)$ such that $L_{\mathbb{K}} = \chi$. Then $\mathfrak{l}_G(\chi) = \lambda_G(\chi) = L$.

Remark 2.8.7. In particular, Proposition 2.8.1.(ii) implies that λ_G is an injective map, and Proposition 2.8.5 implies that

$$\lambda_G = \mathfrak{l}_G \mid_{\mathbf{R}_{\mathbb{K}}(G)}. \quad (2.8.18)$$

2.8.2 The canonical section τ_G

The proof of the following result is given also in [Bol]. It is repeated here for convenience of the reader.

Proposition 2.8.8. *Let $\tau_G: \mathbf{R}_{\mathbb{F}}(G) \rightarrow \mathbf{T}_{\mathcal{O}}(G)$ be defined as in (2.8.7), then τ_G has the following properties:*

- (i) *the map τ_G commutes with restriction, i.e., for all $H \leq G$ one has $\text{res}_H^G \circ \tau_G = \tau_H \circ \text{res}_H^G$.*

(ii) The map τ_G is a ring homomorphism.

(iii) The map τ_G is a canonical section for $\text{--}\mathbb{F} : \mathbf{T}_{\mathcal{O}}(G) \rightarrow \mathbf{R}_{\mathbb{F}}(G)$, i.e., $\text{--}\mathbb{F} \circ \tau_G = \text{Id}_{\mathbf{R}_{\mathbb{F}}(G)}$.

(iv) The map τ_G is induced by $\mathbf{t}^\vee$.

Proof.

- (i) The maps $\beta_{G,p}$ and $a_G^{\mathbf{R}_{\mathbb{F}}}$ commute with restriction, then also the mapping τ_G commutes with restriction.
- (ii) Trivial source $\mathcal{O}G$ -lattices are completely determined by their restrictions to p -hypo-elementary subgroups of G , then it is enough to prove that $\text{res}_H^G \circ \tau_G$ is a ring homomorphism for every p -hypo-elementary subgroup H of G . Moreover, (i) implies that it is enough to prove that τ_H is a ring homomorphism for every p -hypo-elementary group H . Let H be a p -hypo-elementary group, then $\mathbf{R}_{\mathbb{F}}(H) = \text{span}_{\mathbb{Z}}\{[\mathbb{F}_{\varphi}] | \varphi \in \mathbb{F}^*\}$. But

$$\tau_H([\mathbb{F}_{\varphi}]) = \beta_{G,p}([H, [\mathbb{F}_{\varphi}]]_H) = [\mathcal{O}_{\varphi}] \in \mathbf{T}_{\mathcal{O}}(G). \quad (2.8.19)$$

Then τ_G is a ring homomorphism.

- (iii) It is enough to show that $\text{res}_H^G \circ \text{--}\mathbb{F} \circ \tau_G = \text{res}_H^G : \mathbf{R}_{\mathbb{F}}(G) \rightarrow \mathbf{R}_{\mathbb{F}}(H)$ for every cyclic p' -subgroup H of G . As both $\text{--}\mathbb{F}$ and τ_G commute with restrictions, then it is enough to prove that $\text{--}\mathbb{F} \circ \tau_H = \text{Id}_{\mathbf{R}_{\mathbb{F}}(H)}$ for a cyclic group H of p' order. In this case $\mathbf{R}_{\mathbb{F}}(H) = \text{span}_{\mathbb{Z}}\{[\mathbb{F}_{\varphi}] | \varphi \in \mathbb{F}^*\}$ and $(\tau_H([\mathbb{F}_{\varphi}]))_{\mathbb{F}} = (\beta_{G,p}([H, [\mathbb{F}_{\varphi}]]_H))_{\mathbb{F}} = ([\mathcal{O}_{\varphi}])_{\mathbb{F}} = [\mathbb{F}_{\varphi}]$.
- (iv) Every trivial source lattice is determined by its restriction to p -hypo-elementary subgroups and τ commutes with restriction, then it is enough to prove the statement for every p -hypo-elementary group H . The mapping τ_H is induced by $\mathbf{t}_H^\vee$, if, and only if, for every $\varphi \in \hat{H}$ and every species $H(g, Q) \in \text{spec}(\mathbf{T}_{\mathcal{O}}(G))$,

$$\langle \tau_H(\mathbb{F}_{\varphi}), {}^H(h, Q) \rangle_{\mathbf{T}_{\mathcal{O}}(H)} = \langle \mathbf{t}_H(\mathbb{F}_{\varphi}), {}^H(h, Q) \rangle_{\mathbf{T}_{\mathcal{O}}(H)} \quad (2.8.20)$$

If Q is not contained in a vertex of $\mathbb{F}_{\varphi}$, then

$$\langle \tau_H(\mathbb{F}_{\varphi}), {}^H(h, Q) \rangle_{\mathbf{T}_{\mathcal{O}}(H)} = 0 = \langle \mathbf{t}_H(\mathbb{F}_{\varphi}), {}^H(h, Q) \rangle_{\mathbf{T}_{\mathcal{O}}(H)}. \quad (2.8.21)$$

So, let $\mathbb{F}_{\varphi} \in \mathbf{R}_{\mathbb{F}}(H)$ and ${}^H(h, Q) \in \text{spec}(\mathbf{T}_{\mathcal{O}}(G))$ such that Q is contained in a vertex of $\mathbb{F}_{\varphi}$. Then

$$\langle \tau_H(\mathbb{F}_{\varphi}), {}^H(h, Q) \rangle_{\mathbf{T}_{\mathcal{O}}(H)} = \langle \mathcal{O}_{\varphi}, {}^H(h, Q) \rangle_{\mathbf{T}_{\mathcal{O}}(H)} \quad (2.8.22)$$

and

$$\langle \mathbf{t}_H(\mathbb{F}_{\varphi}), {}^H(h, Q) \rangle_{\mathbf{T}_{\mathcal{O}}(H)} = \langle \mathbb{F}_{\varphi}, {}^H h \rangle_{\mathbf{R}_{\mathbb{F}}(H)} \quad (2.8.23)$$

which imply the thesis. $\square$

Remark 2.8.9. In particular, Proposition 2.8.8.(iii) implies that τ_G is injective and Proposition 2.8.8.(iv) implies that

$$\tau_G = \mathbf{t}_G \mid_{\mathbf{R}_{\mathbb{F}}(G)} . \quad (2.8.24)$$

2.9 Examples

(1) Let $G = A_5$, $p = 2$ and let χ_i for $i \in \{1, \dots, 5\}$ be its irreducible characters (in [CCN⁺85] notation). By M_j , $1 \leq j \leq 5$, the corresponding irreducible $\mathbb{K}G$ -modules will be denoted. In this case there are 4 isomorphism types of irreducible $\mathbb{F}G$ -modules which will be denoted by 1, 2₁, 2₂ and 4 reflecting their corresponding degrees, i.e., 4 coincides with the Steinberg module of $A_5 \simeq L_2(4)$, and thus is projective. Let St be the projective $\mathcal{O}G$ -lattice such that $St_{\mathbb{F}} = 4$. The projective covers $P(1)$, $P(2_1)$, $P(2_2)$ of the non-projective irreducible $\mathbb{F}G$ -modules have dimensions 12, 8 and 8, respectively. Moreover, one knows from the decomposition matrix for $p = 2$ (see Table 2.1) that

$$\begin{aligned} P(1)_{\mathbb{K}} &\simeq M_1 + M_2 + M_3 + M_5, \\ P(2_1)_{\mathbb{K}} &\simeq M_2 + M_5, \\ P(2_2)_{\mathbb{K}} &\simeq M_3 + M_5. \end{aligned} \quad (2.9.1)$$

Table 2.1: Decomposition matrix A_5 , $p = 2$

Block 1:	$1 = \varphi_1$	$2_1 = \varphi_2$	$2_2 = \varphi_3$
$1_1 = \chi_1$	1	0	0
$3_1 = \chi_2$	1	0	1
$3_2 = \chi_3$	1	1	0
$5_1 = \chi_5$	1	1	1
Block 2:	$4 = \varphi_4$		
$4_1 = \chi_4$	1		

There are three isomorphism classes of indecomposable trivial source lattices T_1 , T_2 and T_3 with maximal vertex which correspond to three 1-dimensional representations ρ_1 , ρ_2 , ρ_3 of $\mathbb{F}[C_3] \simeq \mathbb{F}[N/P]$. We suppose that ρ_1 is the trivial representation, i.e., T_1 is the trivial $\mathcal{O}G$ -lattice. One verifies easily using the explicit values of χ_5 that for $J_i = \text{ind}_N^G(\mathcal{O}_{\rho_i})$ for $i = 2, 3$ one has $(J_i)_{\mathbb{K}} \simeq M_5$. Hence $T_i \simeq J_i$ and $(T_i)_{\mathbb{K}} \simeq M_5$.

Let $C_2 \subset P$ be a cyclic group of order 2. In particular, $N_G(C_2) = C_G(C_2) = P$. Hence, as $\text{ind}_{C_2}^P(\mathcal{O})$ is indecomposable, there exists a unique trivial source lattice T_4 with vertex C_2 . By Green correspondence T_4 is a direct summand of $\text{ind}_{V_4}^{C_2}(\text{ind}_{C_2}^{V_4}(\mathcal{O}))$ which has rank 30. Let us consider the subgroup D_{10} of G . It has two trivial source lattices $\mathfrak{T}_1$, $\mathfrak{T}_2$ with vertex C_2 .

Considering their induction $\text{ind}_{D_{10}}^{V_4}(\mathfrak{T}_i)$, it turns out that $\text{ind}_{V_4}^{C_2}(\text{ind}_{C_2}^{V_4}(\mathcal{O})) \simeq T_4 + P(2_1) + P(2_1) + 2 \cdot St$. Thus $(T_4)_{\mathbb{K}} \simeq M_1 + M_5$.

Let θ be the non-trivial element in $\widehat{C}_2$, and let L_2 be the indecomposable linear source lattice with vertex C_2 and source $\mathcal{O}_\theta$. Let $\bar{\theta}$ be the non-trivial element in $\widehat{H}$. By the previously mentioned argument, L_1 must be isomorphic to a direct summand of $\text{ind}_H^G(\mathcal{O}_{\bar{\theta}})$ of even rank. By (2.9.1), $\text{ind}_H^G(\mathbb{K}_{\bar{\theta}}) \simeq M_2 + M_3$. Hence $L_1 \simeq \text{ind}_H^G(\mathcal{O}_{\bar{\theta}})$.

Let $\varphi \in \widehat{P}$ be a non-trivial. Then there exists an indecomposable linear source $\mathcal{O}G$ -lattice L_1 with vertex P and source $\mathcal{O}_\varphi$, i.e., $J = \text{ind}_P^N(\mathcal{O}_\varphi)$ is its Green correspondent. An elementary calculation shows that $\text{ind}_N^G(J_{\mathbb{K}}) \simeq M_2 + M_3 + M_4 + M_5$. Thus, as $P(4)$ is relative injective and since $(L_1)_{\mathbb{K}}$ is self-dual and of odd degree, one has either $(L_1)_{\mathbb{K}} \simeq M_2 + M_3 + M_5$ or $(L_1)_{\mathbb{K}} \simeq M_5$. Note that $(L_1)_{\mathbb{F}} \simeq (T_1 + T_2 + T_3)_{\mathbb{F}}$, thus this excludes the second possibility, and hence $(L_1)_{\mathbb{K}} \simeq M_2 + M_3 + M_5$. In particular Proposition 2.5.2 in [Ben06] ensures that if M is an indecomposable module and $s(M) \neq 0$ for a species s , then every vertex of s is contained in a vertex of M . Then, considering Example 2.3.2 and Example 2.3.3, it is possible to conclude that the representation table (cf. Table 2.2) and the extended representation table (cf. Table 2.3) of A_5 with respect to 2 are the following.

Table 2.2: Representation table $A_5, p = 2$

Lattices	1A	3A	5A	5B	1A, C	1A, P	3A, P	3B, P
$P(1)$	12	0	2	0	0	0	0	0
$P(2_1)$	8	-1	*	-b5	0	0	0	0
$P(2_2)$	8	-1	-b5	*	0	0	0	0
St	4	1	-1	-1	0	0	0	0
T_4	6	0	1	1	2	0	0	0
T_3	5	-1	0	0	1	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$
T_2	5	-1	0	0	1	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$
T_1	1	1	1	1	1	1	1	1

Table 2.3: Extended representation table $A_5, p = 2$

Lattices	1A	3A	5A	5B	1A, C	2A, C	1A, P	2A, P	3A, P	3B, P
$P(1)$	12	0	2	0	0	0	0	0	0	0
$P(2_1)$	8	-1	*	-b5	0	0	0	0	0	0
$P(2_2)$	8	-1	-b5	*	0	0	0	0	0	0
St	4	1	-1	-1	0	0	0	0	0	0
T_4	6	0	1	1	2	2	0	0	0	0
L_2	6	0	1	1	2	-2	0	0	0	0
L_1	11	-1	1	1	3	-1	3	-1	0	0
T_3	5	-1	0	0	1	1	1	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$
T_2	5	-1	0	0	1	1	1	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$
T_1	1	1	1	1	1	1	1	1	1	1

Let φ_i , $1 \leq i \leq 4$, denote the irreducible Brauer characters of A_5 with respect to the prime 2 (cf. 2.9) and let U_i be the corresponding $\mathbb{F}G$ -module. Then, in this case,

$$\begin{aligned}
t_G(U_1) &= T_1 \\
t_G(U_2) &= T_2 + T_3 - P(2_1) \\
t_G(U_3) &= T_2 + T_3 - P(2_2) \\
t_G(U_4) &= St + T_2 + 2T_1 + T_3 - P(1) \\
l_G(M_1) &= T_1 \\
l_G(M_2) &= L_1 - P(2_1) \\
l_G(M_3) &= L_1 - P(2_2) \\
l_G(M_4) &= St + T_1 + L_1 - P(1) \\
l_G(M_5) &= L_1 + T_2 + T_3 - P(2_1) - P(2_2).
\end{aligned} \tag{2.9.2}$$

(2) Let $G = A_5$ and $p = 5$ and, as before, let χ_i for $i \in \{1, \dots, 5\}$ be its irreducible characters (in [CCN⁺85] notation) and M_j , $1 \leq j \leq 5$ be the corresponding irreducible $\mathbb{K}G$ -modules.

Table 2.4: Decomposition matrix $A_5, p = 5$

Block 1:	1 = φ_1	3 = φ_2
$1_1 = \chi_1$	1	0
$3_1 = \chi_2$	0	1
$3_2 = \chi_3$	0	1
$4_1 = \chi_4$	1	1
Block 2:	5 = φ_3	
$5_1 = \chi_5$	1	

Considering $p = 5$, there are 3 isomorphism types of irreducible $\mathbb{F}G$ -

modules which will be denoted by 1, 3 and 5 reflecting their corresponding degrees; in particular 5 is projective. Let P be the projective $\mathcal{O}G$ -lattice such that $P_{\mathbb{F}} = 5$. The projective covers $P(1)$ and $P(3)$ of the non-projective irreducible $\mathbb{F}G$ -modules have dimensions 5 and 10, respectively. Moreover, one knows from the decomposition matrix that

$$\begin{aligned} P(1)_{\mathbb{K}} &\simeq M_1 + M_4, \\ P(3)_{\mathbb{K}} &\simeq M_2 + M_3 + M_4. \end{aligned} \tag{2.9.3}$$

In this case the principal block has cyclic defect, then it is possible to associate to it the corresponding Brauer tree Γ_{B_1} .

$$\circ^{\chi_1} \xrightarrow[1]{} \circ^{\chi_4} \xrightarrow[3]{} \bullet^{\chi_2, \chi_3} \tag{2.9.4}$$

Then we can conclude that the principal block contains two isomorphism classes of indecomposable trivial source lattices with maximal vertex T_1 and T_2 such that $(T_1)_{\mathbb{K}} = \chi_1$ and $(T_2)_{\mathbb{K}} = \chi_2 + \chi_3$ (see [KK10]). Moreover, let Q be a Sylow 5-subgroup of G , then $N/Q' = N \simeq Q \rtimes C_2$, where C_2 is a cyclic group of order 2. Then there are two isomorphism classes of indecomposable linear (non trivial) source lattices L_1 and L_2 with maximal vertex in the block B_1 . In particular, they are the Green correspondents of the projective indecomposable $\mathcal{O}N$ -lattices Q_1 and Q_2 of rank 5 and belonging to the principal $\mathcal{O}N$ -block b_1 . Then by Green correspondence it is possible to conclude that $(L_1)_{\mathbb{K}} = (L_2)_{\mathbb{K}} = \chi_2 + \chi_3 + \chi_4$.

The species of $\mathbf{T}_{\mathcal{O}}(G)$ correspond to the pairs $(1A, 1)$, $(2A, 1)$, $(3A, 1)$, $(1A, P)$ and $(2A, 1)$, and $\text{spec}(\mathbf{L}_{\mathcal{O}}(G)) \simeq \text{spec}(\mathbf{T}_{\mathcal{O}}(G)) \cup \{(5A, P), (5B, P)\}$. Then the representation table (see Table 2.5) and the extended representation table (see Table 2.6) of A_5 with respect to the prime 2 are the following.

Table 2.5: Representation table A_5 , $p = 5$

Lattices	1A	2A	3A	1A, P	2A, P
$P(3)$	10	-2	1	0	0
$P(1)$	5	1	2	0	0
P	5	1	-1	0	0
T_2	6	-2	0	1	-1
T_1	1	1	1	1	1

Table 2.6: Extended representation table $A_5, p = 5$

Lattices	$1A$	$2A$	$3A$	$1A, P$	$2A, P$	$5A, P$	$5B, P$
$P(3)$	10	-2	1	0	0	0	0
$P(1)$	5	1	2	0	0	0	0
P	5	1	-1	0	0	0	0
L_2	7	-1	1	2	0	b5	*
L_1	7	-1	1	2	0	*	b5
T_2	6	-2	0	1	-1	1	1
T_1	1	1	1	1	1	1	1

Let φ_i , $1 \leq i \leq 3$, denote the irreducible Brauer characters of A_5 and $p = 5$ (cf. Table 2.4) and let U_i be the corresponding $\mathbb{F}G$ -modules. Then, in this case,

$$\begin{aligned}
\mathfrak{l}_G(\chi_1) &= T_1 \\
\mathfrak{l}_G(\chi_2) &= T_2 + L_2 - P(3) \\
\mathfrak{l}_G(\chi_3) &= T_2 + L_1 - P(3) \\
\mathfrak{l}_G(\chi_4) &= L_1 + L_2 - P(3) \\
\mathfrak{l}_G(\chi_5) &= P + T_1 + L_1 + L_2 - P(3) - P(1)
\end{aligned} \tag{2.9.5}$$

$$\begin{aligned}
\mathfrak{t}_G(\varphi_1) &= T_1 \\
\mathfrak{t}_G(\varphi_2) &= T_1 + 2T_2 - P(3) \\
\mathfrak{t}_G(\varphi_3) &= 3T_1 + 2T_2 - P(1) - P(3) + P.
\end{aligned}$$

CHAPTER 3

Essential linear source lattices and particular cases

In the ordinary representation theory of finite groups the inner product between characters allows to establish the dimension of the space of the homomorphism between two representations. Studying the ring of linear source $\mathcal{O}G$ -lattice $\mathbf{L}_{\mathcal{O}}(G)$ a natural question is if it is possible to define a meaningful bilinear form between linear source $\mathcal{O}G$ -lattices. It is in this context that the ring $\mathbf{E}_{\mathcal{O}}(G)$ of *essential linear source* $\mathcal{O}G$ -lattices (cf. § 3.2) arises. In the first part of this chapter the ring $\mathbf{E}_{\mathcal{O}}(G)$ is introduced and some properties of the bilinear forms involved are investigated.

In the second part, the link between trivial source lattices with maximal vertex and irreducible characters will be investigated in two particular cases, i.e., groups with normal subgroups of index p and groups with cyclic Sylow p -subgroup of prime order.

As usual through all the chapter $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ will be a splitting p -modular system.

3.1 Bilinear forms

Let G be a finite group, p a prime which divides the order of G and $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ a splitting p -modular system. Let $\mathbf{L}_{\mathcal{O}}(G)$ be the Grothendieck ring of linear source $\mathcal{O}G$ -lattices. Then it is possible to define the $\mathbb{Z}$ -bilinear form

$$\langle \cdot, \cdot \rangle : \mathbf{L}_{\mathcal{O}}(G) \times \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbb{Z} \quad (3.1.1)$$

as follows. If L, M are linear source $\mathcal{O}G$ -lattices, then

$$\langle [L], [M] \rangle = \text{rk}_{\mathcal{O}}(\text{Hom}_G(L, M)) = \dim_{\mathbb{K}}(\text{Hom}_G(L_{\mathbb{K}}, M_{\mathbb{K}})). \quad (3.1.2)$$

It follows from the definition that the radical $\text{rad}(\langle \cdot, \cdot \rangle)$ of the bilinear form defined in (3.1.2) is an ideal of the ring $\mathbf{L}_{\mathcal{O}}(G)$ and $\text{rad}(\langle \cdot, \cdot \rangle) = \ker(-_{\mathbb{K}})$. In particular, canonically

$$\mathbf{L}_{\mathcal{O}}(G)/\text{rad}(\langle \cdot, \cdot \rangle) \simeq \mathbf{R}_{\mathbb{K}}(G). \quad (3.1.3)$$

Moreover, one can define a surjective map

$$\begin{aligned} \Delta: \mathbf{L}_{\mathcal{O}}(G) &\rightarrow \text{rad}(\langle \cdot, \cdot \rangle) \\ [M] &\mapsto [M] - \mathfrak{l}_G([M_{\mathbb{K}}]). \end{aligned} \quad (3.1.4)$$

Then $\Delta \circ \Delta = \text{Id}_{\text{rad}(\langle \cdot, \cdot \rangle)}$ and it follows from the definition of the canonical section $\mathfrak{l}_G$ (see §2.6.1) that $\ker(\Delta) = \text{im}(\mathfrak{l}_G)$. Then we can conclude that

$$\mathbf{L}_{\mathcal{O}}(G) \simeq \ker(-_{\mathbb{K}}) \oplus \text{im}(\mathfrak{l}_G). \quad (3.1.5)$$

Considering the ring of linear source lattices one has a $\mathbb{Z}$ -linear form

$$\eta: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbb{Z} \quad (3.1.6)$$

defined as follows. Let M be an indecomposable linear source $\mathcal{O}G$ -lattice, then:

$$\eta([M]) = \begin{cases} 1, & \text{for } [M] = [\mathcal{O}] \\ 0, & \text{for } [M] \in \text{IL}_{\mathcal{O}}(G) \setminus \{[\mathcal{O}]\} \end{cases} \quad (3.1.7)$$

The $\mathbb{Z}$ -linear form just defined can be used to define another $\mathbb{Z}$ -bilinear form

$$\langle\langle \cdot, \cdot \rangle\rangle: \mathbf{L}_{\mathcal{O}}(G) \times \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbb{Z}, \quad (3.1.8)$$

given by

$$\langle\langle [L], [M] \rangle\rangle = \eta([L^* \otimes_{\mathcal{O}} M]), \quad (3.1.9)$$

for $[L], [M] \in \text{IL}_{\mathcal{O}}(G)$.

Now that these bilinear forms have been defined, it is possible to state an immediate consequence of Theorem 1.5.12.

Corollary 3.1.1. *Let N be a finite group with a normal Sylow p -subgroup P , and let $[L], [M] \in \text{IL}_{\mathcal{O}}^{\text{mx}}(N)$. Then*

$$\langle\langle [L], [M] \rangle\rangle = \langle [L], [M] \rangle. \quad (3.1.10)$$

The following property is an immediate consequence of classical Green correspondence. In order to have an easier notation of the following result let $\bar{\mathbf{L}}_{\mathcal{O}}(G)$ denote the quotient $\mathbf{L}_{\mathcal{O}}(G)/\mathbf{L}_{\mathcal{O}}^{\prec}(G)$.

If necessary $-_G$ will be used to emphasize the group considered and avoid misunderstandings, otherwise it will be omitted in order to have an easier notation.

Theorem 3.1.2. *Let G be a finite group, let $P \in \text{Syl}_p(G)$ and let N be its normalizer in G . Then the restriction induces a $*$ -isomorphism of rings with antipode*

$$\bar{\mathbf{f}}: \bar{\mathbf{L}}_{\mathcal{O}}(G) \rightarrow \bar{\mathbf{L}}_{\mathcal{O}}(N), \quad (3.1.11)$$

where $\bar{\mathbf{f}}([M] + \mathbf{L}_{\mathcal{O}}^{\prec}(G)) = [\text{res}_N^G(M)] + \mathbf{L}_{\mathcal{O}}^{\prec}(N)$ for $[M] \in \mathbf{L}_{\mathcal{O}}^+(G)$ satisfying

$$\langle\langle [M_1], [M_2] \rangle\rangle_G = \langle\langle \bar{\mathbf{f}}([M_1]), \bar{\mathbf{f}}([M_2]) \rangle\rangle_N \quad (3.1.12)$$

for all $[M_1], [M_2] \in \mathbf{L}_{\mathcal{O}}^+(G)$, where $\mathbf{L}_{\mathcal{O}}^+(G)$ is the set of isomorphism classes of finitely generated linear source $\mathcal{O}G$ -lattices.

Proof. By construction, the map

$$\bar{\mathbf{f}}: \mathbf{L}_{\mathcal{O}}(G) \xrightarrow{[\text{res}_N^G(-)]} \mathbf{L}_{\mathcal{O}}(N) \xrightarrow{\pi} \bar{\mathbf{L}}_{\mathcal{O}}(N), \quad (3.1.13)$$

where π is the canonical projection. By Proposition 1.5.13, $\mathbf{L}_{\mathcal{O}}^{\prec}(G)$ is contained in the kernel of $\bar{\mathbf{f}}$. Hence the map $\bar{\mathbf{f}}$ induces a ring homomorphism $\bar{\mathbf{f}}: \bar{\mathbf{L}}_{\mathcal{O}}(G) \rightarrow \bar{\mathbf{L}}_{\mathcal{O}}(N)$.

It is easy to see, that for $[L] \in \text{IL}_{\mathcal{O}}^{\text{mx}}(N)$ one has

$$\bar{\mathbf{f}}([g(L)] + \mathbf{L}_{\mathcal{O}}^{\prec}(G)) = [L] + \mathbf{L}_{\mathcal{O}}^{\prec}(N). \quad (3.1.14)$$

Hence $\bar{\mathbf{f}}$ is surjective, and thus, as $\text{rk}_{\mathbb{Z}}(\bar{\mathbf{L}}_{\mathcal{O}}(G)) = \text{rk}_{\mathbb{Z}}(\bar{\mathbf{L}}_{\mathcal{O}}(N))$, $\bar{\mathbf{f}}$ is also injective. As $[\text{res}_N^G(-)]$ commutes with $-^*$, this yields the claim. $\square$

Remark 3.1.3. Let $\mathbf{g}: \text{IL}_{\mathcal{O}}^{\text{mx}}(N) \rightarrow \text{IL}_{\mathcal{O}}^{\text{mx}}(G)$ denote the Green correspondent and let $\mathbf{g}_*: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(N) \rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G)$ denote the $\mathbb{Z}$ -linear map given by

$$\mathbf{g}_*([L]) = \begin{cases} \mathbf{g}([L]), & \text{for } [L] \in \text{IL}_{\mathcal{O}}^{\text{mx}}(N), \\ 0, & \text{for } [L] \in \text{IL}_{\mathcal{O}}(N) \setminus \text{IL}_{\mathcal{O}}^{\text{mx}}(N). \end{cases} \quad (3.1.15)$$

The induced map $\bar{\mathbf{g}}: \bar{\mathbf{L}}_{\mathcal{O}}(N) \rightarrow \bar{\mathbf{L}}_{\mathcal{O}}(G)$, such that

$$\bar{\mathbf{g}}([L] + \mathbf{L}_{\mathcal{O}}^{\prec}(N)) = \mathbf{g}_*([L]) + \mathbf{L}_{\mathcal{O}}^{\prec}(G) \text{ for } [L] \in \text{IL}_{\mathcal{O}}(N), \quad (3.1.16)$$

is the inverse of $\bar{\mathbf{f}}: \bar{\mathbf{L}}_{\mathcal{O}}(G) \rightarrow \bar{\mathbf{L}}_{\mathcal{O}}(N)$.

In particular from (3.1.12) and Corollary 3.1.1, one can conclude the following.

Corollary 3.1.4. *Let $[L], [M] \in \text{IL}_{\mathcal{O}}(G)$. Then*

$$\langle\langle [L], [M] \rangle\rangle = \begin{cases} 1, & \text{if } [M] = [L] \text{ and } [L] \in \text{IL}_{\mathcal{O}}^{\text{mx}}(G) \\ 0, & \text{else.} \end{cases} \quad (3.1.17)$$

In particular, $\text{rad}(\langle\langle \cdot, \cdot \rangle\rangle) = \mathbf{L}_{\mathcal{O}}^{\prec}(G)$.

One can define a third $\mathbb{Z}$ -bilinear form

$$(\cdot, \cdot) : \mathbf{L}_{\mathcal{O}}(G) \times \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbb{Z} \quad (3.1.18)$$

as the difference of the two $\mathbb{Z}$ -bilinear form defined in (3.1.2) and (3.1.9), i.e.,

$$([L], [M]) = \langle [L], [M] \rangle - \langle\langle [L], [M] \rangle\rangle, \quad (3.1.19)$$

for $[L], [M] \in \mathbf{IL}_{\mathcal{O}}(G)$.

As already said, if necessary $_{-G}$ will be used to emphasize the group considered in the evaluation of all bilinear forms, if not necessary it will be omitted in order to have an easier notation.

It is possible to “translate” bilinear forms defined in terms of species of the rings involved. This can be useful for an easier computation.

Let $[M], [L] \in \mathbf{IL}_{\mathcal{O}}(G)$, then

$$\langle [M], [L] \rangle = \sum_{G(g, \langle g_p \rangle)} \frac{1}{|C_G(g)|} \langle M, {}^G(g, \langle g_p \rangle) \rangle_{\mathbf{L}_{\mathcal{O}}(G)} \cdot \langle [L], {}^G(g^{-1}, \langle g_p \rangle) \rangle_{\mathbf{L}_{\mathcal{O}}(G)}. \quad (3.1.20)$$

Analogously, if $[M], [L] \in \mathbf{IL}_{\mathcal{O}}(G)$ and $P \in \text{Syl}_p(G)$ and $N = N_G(P)$, then

$$\langle\langle [M], [L] \rangle\rangle = \sum_{\tilde{(g, P)}} \frac{1}{|C_{N/P'}(g, P)|} \langle [M], \tilde{(g, P)} \rangle_{\mathbf{L}_{\mathcal{O}}(G)} \cdot \langle [L], \tilde{(g^{-1}, P)} \rangle_{\mathbf{L}_{\mathcal{O}}(G)}. \quad (3.1.21)$$

The angled brackets $\langle \cdot, \cdot \rangle$ are used both to denote the bilinear form defined in (3.1.1) and the evaluation of a species (cf. § 2.3), by the way their meaning will be clear because of the context and the nature of the objects involved.

3.2 Essential linear source lattices and species

Considering the radical of the $\mathbb{Z}$ -bilinear forms defined in (3.1.2) and (3.1.9) one can define the finitely generated $\mathbb{Z}$ -module $\mathbf{E}_{\mathcal{O}}(G)$ of the *essential* linear source $\mathcal{O}G$ -lattices as follows

$$\begin{aligned} \mathbf{E}_{\mathcal{O}}(G) &= \mathbf{L}_{\mathcal{O}}(G) / (\text{rad}(\langle\langle \cdot, \cdot \rangle\rangle) \cap \text{rad}(\langle \cdot, \cdot \rangle)) \\ &\simeq \mathbf{L}_{\mathcal{O}}(G) / (\ker(-_{\mathbb{K}}) \cap \mathbf{L}_{\mathcal{O}}^{\sim}(G)). \end{aligned} \quad (3.2.1)$$

Moreover, the abelian group $\mathbf{E}_{\mathcal{O}}(G)$ has the structure of a ring in a natural way.

3.2.1 Species of $\mathbf{E}_{\mathcal{O}}(G)$

As done in § 2.3 for the Grothendieck rings considered till now, it is possible to describe explicitly the set of species of the essential linear source

$\mathcal{O}G$ -lattices. By definition there exists a natural surjection $\pi: \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}} \rightarrow \mathbf{E}_{\mathcal{O}}(G)_{\mathbb{C}}$ and then an inclusion of sets

$$\begin{aligned} \text{spec}(\mathbf{E}_{\mathcal{O}}(G)) &\subseteq \{\sim(g, P) \mid P \leq G \text{ } p\text{-subgroup}, g \in N_G(P), g_p \in P\} \\ &= \text{spec}(\mathbf{L}_{\mathcal{O}}(G)). \end{aligned} \quad (3.2.2)$$

Considering that $\text{rad}(\langle\langle \cdot, \cdot \rangle\rangle) \cap \text{rad}(\langle \cdot, \cdot \rangle) \simeq \ker(-_{\mathbb{K}}) \cap \mathbf{L}_{\mathcal{O}}^{\prec}(G)$ one can conclude that

$$\begin{aligned} \text{spec}(\mathbf{E}_{\mathcal{O}}(G)) &= \{\sim(g, P) \mid P \in \text{Syl}_p(G), g \in N_G(P), g_p \in P\} \\ &\cup \{^G(g, V) \mid V = \langle g_p \rangle\}. \end{aligned} \quad (3.2.3)$$

In order to compute the rank of $\mathbf{E}_{\mathcal{O}}(G)$ it is necessary to distinguish two different cases.

- (1) If $P \in \text{Syl}_p(G)$ is non-cyclic, then the union in (3.2.3) is disjoint. So,

$$|\text{spec}(\mathbf{E}_{\mathcal{O}}(G))| = |\text{spec}(\mathbf{L}_{\mathcal{O}}^{\text{mx}}(G))| + |\text{spec}(\mathbf{R}_{\mathbb{K}}(G))|, \quad (3.2.4)$$

and then

$$\text{rk}(\mathbf{E}_{\mathcal{O}}(G)) = \text{rk}(\mathbf{L}_{\mathcal{O}}^{\text{mx}}(G)) + \text{rk}(\mathbf{R}_{\mathbb{K}}(G)). \quad (3.2.5)$$

In particular, considering the following chain of isomorphisms

$$\mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \simeq \mathbf{L}_{\mathcal{O}}^{\text{mx}}(N_G(P)) \simeq \mathbf{R}_{\mathbb{K}}(N_G(P)/P'), \quad (3.2.6)$$

where the first one is given by the Green correspondence and the second one by Theorem 1.5.12, then it follows that

$$\begin{aligned} \text{rk}_{\mathcal{O}}(\mathbf{E}_{\mathcal{O}}(G)) &= \text{rk}_{\mathcal{O}}(\mathbf{L}_{\mathcal{O}}^{\text{mx}}(G)) + |\mathcal{C}l(G)| \\ &= |\mathcal{C}l(N_G(P)/P')| + |\mathcal{C}l(G)|. \end{aligned} \quad (3.2.7)$$

- (2) If $P \in \text{Syl}_p(G)$ is cyclic the relation $\sim$ defined in § 2.3.4: coincides with the conjugation with elements of G and then

$$\begin{aligned} \text{rk}_{\mathcal{O}}(\mathbf{E}_{\mathcal{O}}(G)) &= |\mathcal{C}l(N_G(P)/P')| + |\mathcal{C}l(G)| + \\ &\quad - |\{^G g \mid \langle g_p \rangle \in \text{Syl}_p(G)\}|. \end{aligned} \quad (3.2.8)$$

Considering the explicit description of the set of $\text{spec}(\mathbf{E}_{\mathcal{O}}(G))$ given in (3.2.3) it is possible to characterize the radical of the bilinear form $(\cdot, \cdot)$.

Let us consider a basis $\mathfrak{B}$ of $\mathbf{E}_{\mathcal{O}}(G)$ with dual basis $\mathfrak{B}^* = \text{spec}(\mathbf{E}_{\mathcal{O}}(G))$. Given $\sigma, \tau \in \text{spec}(\mathbf{E}_{\mathcal{O}}(G))$, then $\mathfrak{b}_{\sigma} \in \mathfrak{B}$ is such that $\tau(\mathfrak{b}_{\sigma}) = \delta_{\sigma, \tau}$. Moreover,

$$\langle \mathfrak{b}_{\sigma}, \mathfrak{b}_{\tau} \rangle = \begin{cases} 0, & \text{if } \sigma \neq \tau \\ \frac{1}{|{}^G(g)|}, & \text{if } \sigma = \tau = {}^G(g, \langle g_p \rangle) \end{cases} \quad (3.2.9)$$

Analogously,

$$\langle\langle \mathbf{b}_\sigma, \mathbf{b}_\tau \rangle\rangle = \begin{cases} 0, & \text{if } \sigma \neq \tau \\ \frac{1}{|C_{N_G(P)/P'}(g, P)|}, & \text{if } \sigma = \tau = \tilde{\cdot}(g, P) \end{cases} \quad (3.2.10)$$

where P is a Sylow p -subgroup of G . Let $\text{spec}(\mathbf{E}_\mathcal{O}(G)) = \Sigma_1 \cup \Sigma_2$, where

$$\begin{aligned} \Sigma_1 &= \{\tilde{\cdot}(g, P) \mid P \in \text{Syl}_p(G), g \in N_G(P), g_p \in P\}, \\ \Sigma_2 &= \{\tilde{\cdot}^G(g, V) \mid V = \langle g_p \rangle\}. \end{aligned} \quad (3.2.11)$$

If the Sylow p -subgroups are non-cyclic, then the intersection $\Sigma_1 \cap \Sigma_2$ is empty and then

$$(\mathbf{b}_\sigma, \mathbf{b}_\sigma) \neq 0, \quad \forall \sigma \in \text{spec}(\mathbf{E}_\mathcal{O}(G)). \quad (3.2.12)$$

In this case the bilinear form is non degenerate. On the other hand, if the Sylow p -subgroups are cyclic, then there exists $\sigma \in \Sigma_1 \cap \Sigma_2$ and thus

$$(\mathbf{b}_\sigma, \mathbf{b}_\sigma) = 0, \quad (3.2.13)$$

since in this case $\langle\langle \mathbf{b}_\sigma, \mathbf{b}_\sigma \rangle\rangle = \langle \mathbf{b}_\sigma, \mathbf{b}_\sigma \rangle$.

3.3 Restriction and induction

Let $\llbracket M \rrbracket_G \in \mathbf{E}_\mathcal{O}(G) \simeq \mathbf{L}_\mathcal{O}(G)/(\ker(-_\mathbb{K}) \cap \mathbf{L}_\mathcal{O}^\prec(G))$. Let $P \in \text{Syl}_p(G)$ and $N = N_G(P)$ be the normalizer of P in G . Then one can define the *restriction* in the ring $\mathbf{E}_\mathcal{O}(G)$.

Definition 3.3.1. Let H be a subgroup of G such that $N \subseteq H \subseteq G$. Then one can define the *restriction map*

$$\begin{aligned} \text{res}_H^G : \mathbf{E}_\mathcal{O}(G) &\rightarrow \mathbf{E}_\mathcal{O}(H). \\ \llbracket M \rrbracket_G &\mapsto \llbracket \text{res}_H^G M \rrbracket_H \end{aligned} \quad (3.3.1)$$

Analogously one can define the *induction* in the ring $\mathbf{E}_\mathcal{O}(G)$.

Definition 3.3.2. Let H be a subgroup of G such that $N \subseteq H \subseteq G$ and let $\llbracket M \rrbracket_H \in \mathbf{E}_\mathcal{O}(H) \simeq \mathbf{L}_\mathcal{O}(H)/(\ker(-_\mathbb{K}) \cap \mathbf{L}_\mathcal{O}^\prec(H))$, then the *induction map* is defined as follows.

$$\begin{aligned} \text{ind}_H^G : \mathbf{E}_\mathcal{O}(H) &\rightarrow \mathbf{E}_\mathcal{O}(G). \\ \llbracket M \rrbracket_H &\mapsto \llbracket \text{ind}_H^G M \rrbracket_G \end{aligned} \quad (3.3.2)$$

Remark 3.3.3. Let one consider the $\mathbb{Z}$ -bilinear forms defined before. Then the following equalities hold since the intersection $\ker(-_\mathbb{K}) \cap \mathbf{L}_\mathcal{O}^\prec$ is contained in the radical of both the bilinear forms.

$$\begin{aligned} \langle\langle \llbracket A \rrbracket, \llbracket B \rrbracket \rangle\rangle &= \langle\langle [A] + (\ker(-_\mathbb{K}) \cap \mathbf{L}_\mathcal{O}^\prec), [B] + (\ker(-_\mathbb{K}) \cap \mathbf{L}_\mathcal{O}^\prec) \rangle\rangle \\ &= \langle\langle [A], [B] \rangle\rangle \end{aligned} \quad (3.3.3)$$

$$\begin{aligned} \langle \llbracket A \rrbracket, \llbracket B \rrbracket \rangle &= \langle [A] + (\ker(-_\mathbb{K}) \cap \mathbf{L}_\mathcal{O}^\prec), [B] + (\ker(-_\mathbb{K}) \cap \mathbf{L}_\mathcal{O}^\prec) \rangle \\ &= \langle [A], [B] \rangle \end{aligned} \quad (3.3.4)$$

The above equalities tell us that the bilinear forms defined in § 3.1 can also be seen as defined over the ring $\mathbf{E}_{\mathcal{O}}(G)$ of essential linear source $\mathcal{O}G$ -lattices.

By definition $(\cdot, \cdot) = \langle \cdot, \cdot \rangle - \langle \langle \cdot, \cdot \rangle \rangle$. In particular, it follows from Definition 3.3.2, Definition 3.3.1 and (3.3.4) that

$$\begin{aligned} \langle \underline{\text{res}}_H^G([A]), [B] \rangle_H &= \langle [\underline{\text{res}}_H^G(A)], [B] \rangle_H = \langle [\text{res}_H^G(A)], [B] \rangle_H \\ &= \langle [A], [\text{ind}_H^G(B)] \rangle_G = \langle [A], [\underline{\text{ind}}_H^G(B)] \rangle_G \quad (3.3.5) \\ &= \langle [A], \underline{\text{ind}}_H^G([B]) \rangle_G \end{aligned}$$

Moreover considering (3.3.3) one can compute the following chains of equalities. Let $K = N_H(P)$

$$\begin{aligned} \langle \underline{\text{res}}_H^G([A]), [B] \rangle_H &= \langle [\underline{\text{res}}_H^G(A)], [B] \rangle_H = \langle [\text{res}_H^G(A)], [B] \rangle_H \\ &= \sum_{\sim^H(h, P)} \frac{1}{|C_{K/P'}(h, P)|} \langle \text{res}_H^G(A), \sim^H(h, P) \rangle_{\mathbf{E}_{\mathcal{O}}(H)} \cdot \langle B, \sim^H(h^{-1}, P) \rangle_{\mathbf{E}_{\mathcal{O}}(H)} \\ &= \sum_{\sim^H(h, P)} \frac{1}{|C_{K/P'}(h, P)|} \left\langle A, \text{res}_H^{G^\vee}(\sim^H(h, P)) \right\rangle_{\mathbf{E}_{\mathcal{O}}(G)} \cdot \langle B, \sim^H(h^{-1}, P) \rangle_{\mathbf{E}_{\mathcal{O}}(H)} \\ &= \sum_{\sim^H(h, P)} \frac{1}{|C_{K/P'}(h, P)|} \langle A, \sim^G(h, P) \rangle_{\mathbf{E}_{\mathcal{O}}(G)} \cdot \langle B, \sim^H(h^{-1}, P) \rangle_{\mathbf{E}_{\mathcal{O}}(H)} \quad (3.3.6) \end{aligned}$$

on the other hand,

$$\begin{aligned} \langle [A], [\underline{\text{ind}}_H^G(B)] \rangle_G &= \langle [A], [\text{ind}_H^G(B)] \rangle_G \\ &= \sum_{\sim^G(g, P)} \frac{1}{|C_{N/P'}(g, P)|} \langle A, \sim^G(g, P) \rangle_{\mathbf{E}_{\mathcal{O}}(G)} \cdot \langle \text{ind}_H^G(B), \sim^G(g^{-1}, P) \rangle_{\mathbf{E}_{\mathcal{O}}(G)} \\ &= \sum_{\sim^G(g, P)} \frac{1}{|C_{N/P'}(g, P)|} \langle A, \sim^G(g, P) \rangle_{\mathbf{E}_{\mathcal{O}}(G)} \cdot \left\langle B, \text{ind}_H^{G^\vee}(\sim^G(g^{-1}, P)) \right\rangle_{\mathbf{E}_{\mathcal{O}}(H)} \quad (3.3.7) \end{aligned}$$

3.4 Essential linear source lattices and linear source lattices

In this section maps linking the ring of essential linear source lattices and the ring of linear source lattices will be defined. The underlying idea is to know how to go from a Grothendieck ring to another one in order to take the most of what is known about them. A particular attention will be given to linear source lattices with maximal vertex.

As before, let $P \in \text{Syl}_p(G)$ be a Sylow p -subgroup of G , $N = N_G(P)$ its normalizer in G and H a subgroup of G such that $N \subseteq H \subseteq G$. The first map to be defined is the canonical projection.

Definition 3.4.1. Let

$$\begin{aligned} \pi: \mathbf{L}_{\mathcal{O}}(G) &\rightarrow \mathbf{E}_{\mathcal{O}}(G) \\ [M] &\mapsto \llbracket M \rrbracket \end{aligned} \quad (3.4.1)$$

be the canonical projection.

Another projection to be considered is the following.

Definition 3.4.2. Let

$$\bar{\sigma}: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \simeq \mathbf{L}_{\mathcal{O}}(G) / \mathbf{L}_{\mathcal{O}}^{\prec}(G) \quad (3.4.2)$$

such that, if $[M] \in \text{IL}_{\mathcal{O}}(G)$, then

$$\bar{\sigma}([M]) = \begin{cases} [M], & \text{if } [M] \in \text{IL}_{\mathcal{O}}^{\text{mx}}(G) \\ 0, & \text{otherwise} \end{cases} \quad (3.4.3)$$

Then, if $\llbracket M \rrbracket = [M] \oplus (\ker(-_{\mathbb{K}}) \cap \mathbf{L}_{\mathcal{O}}^{\prec}(G))$ we can define the map

$$\begin{aligned} \sigma: \mathbf{E}_{\mathcal{O}}(G) &\rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \\ \llbracket M \rrbracket &\mapsto \bar{\sigma}([M]) \end{aligned} \quad (3.4.4)$$

Let $\mathbf{K}_G$ denote the kernel of $\sigma: \mathbf{E}_{\mathcal{O}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G)$, the following propositions describe the behaviour of the kernel of σ under restriction and induction.

Proposition 3.4.3. *Let $H \subseteq G$, then $\underline{\text{res}}_H^G(\mathbf{K}_G) \subseteq \mathbf{K}_H$.*

Proof. Let $\llbracket L \rrbracket \in \mathbf{K}_G$, then $\llbracket L \rrbracket = [A] \oplus (\mathbf{L}_{\mathcal{O}}^{\prec}(G) \cap \ker(-_{\mathbb{K}})) \in \mathbf{K}_G$, where $[A] \in \mathbf{L}_{\mathcal{O}}^{\prec}(G)$. Moreover

$$\underline{\text{res}}_H^G(\llbracket L \rrbracket) = \llbracket \text{res}_H^G(L) \rrbracket = [\text{res}_H^G(A)] + (\mathbf{L}_{\mathcal{O}}^{\prec}(G) \cap \ker(-_{\mathbb{K}})) \quad (3.4.5)$$

and in particular $[\text{res}_H^G(A)] \notin \mathbf{L}_{\mathcal{O}}^{\text{mx}}(H)$. Then $\underline{\text{res}}_H^G(L) \in \mathbf{K}_H$. $\square$

Proposition 3.4.4. *Let $H \subseteq G$, then $\underline{\text{ind}}_H^G(\mathbf{K}_H) \subseteq \mathbf{K}_G$.*

Proof. It holds since $\underline{\text{ind}}_H^G(\ker(-_{\mathbb{K}}))$ is contained in the $\ker(-_{\mathbb{K}})$ over G and $\underline{\text{ind}}_H^G(\text{rad}(\langle\langle \cdot, \cdot \rangle\rangle_H)) \subseteq \text{rad}(\langle\langle \cdot, \cdot \rangle\rangle_G)$. $\square$

Remark 3.4.5. Let us observe that in the previous proposition it is not necessary to take $N \subseteq H \subseteq G$.

From now on the hypothesis $N \subseteq H \subseteq G$ is necessary since the *Green correspondence* (see § 1.4.1) is involved. In particular, given an indecomposable linear source $\mathcal{O}G$ -lattice L with maximal vertex, $\mathbf{f}([L])$ denotes the *Green correspondent*, i.e., $\mathbf{f}([L]) \in \text{IL}_{\mathcal{O}}^{\text{mx}}(N)$ is the unique direct summand of $[\text{res}_N^G(L)]$ which has maximal vertex.

Definition 3.4.6. Let $N \subseteq H \subseteq G$, then we can define a map

$$\begin{aligned} \phi: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \oplus \mathbf{K}_G &\rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(H) \oplus \mathbf{K}_H \\ [M] \oplus [S] &\mapsto [\mathbf{f}_*(M)] \oplus \underline{\text{res}}_H^G(S) = [\mathbf{f}_*(M)] \oplus [\text{res}_H^G(S)] \end{aligned} \quad (3.4.6)$$

where $\mathbf{f}_*: \mathbf{L}_{\mathcal{O}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}(N)$ denote the $\mathbb{Z}$ -linear map given by

$$\mathbf{f}_*([L]) = \begin{cases} \mathbf{f}([L]) & \text{for } [L] \in \text{IL}_{\mathcal{O}}^{\text{mx}}(G) \\ 0 & \text{for } [L] \in \text{IL}_{\mathcal{O}}(G) \setminus \text{IL}_{\mathcal{O}}^{\text{mx}}(G). \end{cases} \quad (3.4.7)$$

Let $[M] \in \text{IL}_{\mathcal{O}}^{\text{mx}}(G)$, then $\text{res}_N^G(M) = \mathbf{f}(M) \oplus \mathcal{R}_N(M)$, where $\mathbf{f}(M)$ is the Green correspondent and $\mathcal{R}_N(M) \in \mathbf{L}_{\mathcal{O}}^{\prec}(N)$. Then $[\mathcal{R}_N(M)] \in \mathbf{K}_N$.

Definition 3.4.7. For any subgroup H of G such that $N \subseteq H$ and for any $[M] \in \text{IL}_{\mathcal{O}}^{\text{mx}}(H)$ let $\mathcal{R}_H(M) \in \mathbf{L}_{\mathcal{O}}^{\prec}(H)$ let

$$\text{res}_N^H(\mathcal{R}_H(M)) = \mathcal{R}_N(M). \quad (3.4.8)$$

Let $N \subseteq H \subseteq G$ and, if $[L] \oplus (\mathbf{L}_{\mathcal{O}}(G) \cap \ker(-_{\mathbb{K}})) = [M] \in \mathbf{E}_{\mathcal{O}}(G)$, let

$$\tau([M]) = \sigma([M]) \oplus [\mathcal{R}_G(\sigma([M]))]. \quad (3.4.9)$$

By definition, if $\sigma([M]) = 0$, then $\mathcal{R}_G(\sigma([M])) = 0$.

Then one can construct the following diagram

$$\begin{array}{ccc} \mathbf{E}_{\mathcal{O}}(G) & \xrightarrow{\tau} & \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \oplus \mathbf{K}_G \\ \underline{\text{res}}_H^G \downarrow & & \downarrow \phi_H = \mathbf{f}_* \oplus \underline{\text{res}}_H^G \\ \mathbf{E}_{\mathcal{O}}(H) & \xrightarrow{\tau} & \mathbf{L}_{\mathcal{O}}^{\text{mx}}(H) \oplus \mathbf{K}_H \end{array} \quad (3.4.10)$$

Proposition 3.4.8. Diagram 3.4.10 commutes.

Proof. Let $[M] = [L] \oplus \mathbf{L}_{\mathcal{O}}(G) \cap \ker(-_{\mathbb{K}}) \in \mathbf{E}_{\mathcal{O}}(G)$, then

$$\begin{aligned} \phi_H \circ \tau([M]) &= \phi_H(\sigma([M]) \oplus [\mathcal{R}_G(\sigma([M]))]) \\ &= \begin{cases} [\mathbf{f}_*([L])] \oplus [\mathcal{R}_H(\mathbf{f}(L))] & \text{if } [L] \in \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (3.4.11)$$

and,

$$\begin{aligned} \tau \circ \underline{\text{res}}_H^G([M]) &= \sigma(\underline{\text{res}}_H^G([M]) \oplus [(\sigma(\underline{\text{res}}_H^G([M])))] \\ &= \begin{cases} [\mathbf{f}_*([L])] \oplus [\mathcal{R}_H(\mathbf{f}(L))] & \text{if } [L] \in \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (3.4.12)$$

□

3.5 Normal subgroups of index p

Let G be a finite group. If H is a subgroup of G , then it is well known that the restriction defines a ring homomorphism

$$\rho = \text{res}_G^H : \mathbf{R}_{\mathbb{K}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(H). \quad (3.5.1)$$

For normal subgroups of prime index p Clifford's theorem (see [CR90, Theorem 11.1]) specializes to the following fact for irreducible (ordinary) characters.

Fact 3.5.1. Let G be a finite group containing a normal subgroup H of prime index p .

- (a) If $\psi \in \text{Irr}_{\mathbb{K}}(H)$ satisfies ${}^z\psi \neq \psi$ for some $z \in G$, then $\chi = \text{ind}_H^G(\psi)$ is irreducible. Moreover,

$$\rho(\chi) = \sum_{y \in G/H} {}^y\psi. \quad (3.5.2)$$

- (b) If $\psi \in \text{Irr}_{\mathbb{K}}(H)$ satisfies ${}^g\psi = \psi$ for all $g \in G$, then there exists an irreducible character $\phi \in \text{Irr}_{\mathbb{K}}(G)$ such that $\rho(\phi) = \psi$. Moreover one has

$$\rho^{-1}(\{\psi\}) = \text{Lin}(G/H) \cdot \phi + \ker(\rho), \quad (3.5.3)$$

where $\text{Lin}(G/H)$ is the set of ordinary linear irreducible characters of G/H .

Proof. If ${}^z\psi \neq \psi$ holds for some $z \in G$, then the inertia group of ψ coincides with H , i.e., $I(\psi) = \{z \in G \mid {}^z\psi = \psi\} = H$. But then $\chi = \text{ind}_H^G(\psi)$ is irreducible and

$$\rho(\chi) = \sum_{y \in G/H} {}^y\psi. \quad (3.5.4)$$

If $\psi = {}^z\psi$ holds for all $z \in G$, then there exists $\phi \in \text{Irr}_{\mathbb{K}}(G)$ such that either $\rho(\phi) = \psi$, or $\chi = \text{ind}_H^G(\psi)$ is irreducible and $\rho(\chi) = p \cdot \psi$.

Let $\text{Irr}_{\mathbb{K}}(H)^{(a)}$ denote the set of irreducible characters for which case (a) holds, let $\text{Irr}_{\mathbb{K}}(H)^{(b)}$ denote the set of irreducible characters for which case (b) holds, and let $\text{Irr}_{\mathbb{K}}(H)^{(c)}$ denote the set of irreducible characters ψ satisfying $\psi = {}^z\psi$, $\text{ind}_H^G(\psi)$ irreducible. Then, by Clifford's theorem,

$$\begin{aligned} p \cdot |H| &= \sum_{\xi \in \text{Irr}_{\mathbb{K}}(G)} \xi(1)^2 \\ &= p \cdot \sum_{\chi \in \text{Irr}_{\mathbb{K}}(H)^{(a)}} \chi(1)^2 + p \cdot \sum_{\chi \in \text{Irr}_{\mathbb{K}}(H)^{(b)}} \chi(1)^2 + p^2 \cdot \sum_{\chi \in \text{Irr}_{\mathbb{K}}(H)^{(c)}} \chi(1)^2 \\ &= p \cdot |H| + (p^2 - p) \cdot \sum_{\chi \in \text{Irr}_{\mathbb{K}}(H)^{(c)}} \chi(1)^2. \end{aligned} \quad (3.5.5)$$

Hence $\text{Irr}_{\mathbb{K}}(H)^{(c)} = \emptyset$, and this yields the claim. $\square$

From now on, let $G = Z \rtimes H$, where Z is a cyclic group of order p and H is a subgroup of index p . Then $Z \in \text{Syl}_p(G)$ and

$$N = N_G(Z) = Z \times C, \quad (3.5.6)$$

where $C = C_H(Z)$.

3.5.1 Characters of $G = Z \rtimes H$

Let $\text{Lin}(G) \subseteq \text{Irr}_{\mathbb{K}}(G)$ denote the set of ordinary linear irreducible characters of G and $\text{Lin}_p(G) \subseteq \text{Lin}(G)$ the set of linear characters of G whose determinant order is a p -power. Let $\text{Irr}_p(G) \subseteq \text{Irr}_{\mathbb{K}}(G)$ denote the set of characters which degree is divisible by p and $\text{Irr}_{p'}(G) \subseteq \text{Irr}_{\mathbb{K}}(G)$ the set of characters with p' degree, then $\text{Irr}_{\mathbb{K}}(G) = \text{Irr}_p(G) \sqcup \text{Irr}_{p'}(G)$. Let $\text{SIrr}_{p'}(G)$ denote the set of p' -special characters of G , i.e.,

$$\text{SIrr}_{p'}(G) = \{\chi \in \text{Irr}_{p'}(G) \mid p \nmid o(\chi)\}. \quad (3.5.7)$$

Let us remember, that for p -solvable (and π -separable) groups, the *McKay conjecture* which asserts that G and N have equal numbers of irreducible characters with degrees not divisible by p has been proved. In particular, for this class of groups it can be formulated as follows.

Remark 3.5.2 (McKay conjecture for p -solvable groups). If G is a p -solvable (or a π -separable) group, then it holds ([IN01, Theorem 3.4(b)]) that

$$\begin{aligned} |\text{SIrr}_{p'}(G)| &= |\text{Irr}_{\mathbb{K}}(N/Z)| \text{ and} \\ |\text{SIrr}_{\pi'}(G)| &= |\text{Irr}_{\mathbb{K}}(N/Z)| \end{aligned} \quad (3.5.8)$$

Thanks to [IN01, Theorem 3.6], every character $\varphi \in \text{Irr}_{p'}(G)$ is a *satellite* of some unique character $\lambda \in \text{Lin}_p(G)$, i.e., $\varphi = \lambda \cdot \psi$ for some $\psi \in \text{SIrr}_{p'}(G)$ (see [IN01, § 3]). Then

$$\text{Irr}_{p'}(G) = \{\lambda \cdot \psi \mid \lambda \in \text{Lin}_p(G), \psi \in \text{SIrr}_{p'}(G)\}. \quad (3.5.9)$$

Moreover, for $\chi \in \text{Irr}_{\mathbb{K}}(G)$ one has that $p \mid \chi(1)$ if, and only if,

$$\text{Lin}_p(G) \cdot \chi = \{\chi\}. \quad (3.5.10)$$

3.5.2 Blocks and Brauer trees

The goal of this section is to give a description of the blocks B of $G = Z \rtimes H$ with maximal defect groups, i.e., $\text{df}(B) = {}^G Z$, and to construct the corresponding Brauer tree (cf. [HL89, Definition 2.1.19]).

Proposition 3.5.3. *The number of irreducible characters of $G = Z \rtimes H$ with degree not divisible by p is equal to the product of p and the number of p' -special characters, i.e.,*

$$|\text{Irr}_{p'}(G)| = p \cdot |\text{SIrr}_{p'}(G)|. \quad (3.5.11)$$

Proof. By [IN01, Theorem 2.1 and Corollary 2.5] it is possible to define a bijection

$$\varphi: \text{Irr}_{\mathbb{K}}(Z) \rightarrow \text{Lin}_p(G), \quad (3.5.12)$$

such that $\lambda \mapsto \bar{\lambda}$, where $\bar{\lambda}$ is the unique p -special extension of λ to G . Moreover, for all $\lambda \in \text{Lin}_p(G)$ one can define a map

$$\phi_\lambda: \text{SIrr}_{p'}(G) \rightarrow \text{Irr}_{p'}(G) \quad (3.5.13)$$

such that $\psi \mapsto \lambda \cdot \psi$, where $\lambda \cdot \psi$ is a satellite of λ . But then, by [IN01, Theorem 3.2] and the bijection defined in (3.5.12), the statement is proved. $\square$

Proposition 3.5.4. *The group $G = Z \rtimes H$ has $|\text{SIrr}_{p'}(G)| = \frac{|\text{Irr}_{p'}(G)|}{p}$ blocks B with cyclic defect groups $\text{df}(B) = {}^G Z$.*

Proof. Let $\alpha \in \text{SIrr}_{p'}(G)$ and M_α be the corresponding irreducible $\mathbb{K}G$ -module. Let L_λ be the irreducible $\mathbb{K}G$ -module which corresponds to the linear character $\lambda \in \text{Lin}_p(G) \setminus \{1_G\}$. Since the subgroup $H = O^p(G)$ acts trivially on the characters in $\text{Lin}_p(G)$, then for $\alpha \in \text{SIrr}_{p'}(G)$ the characters α and $\alpha \cdot \lambda$, with $\lambda \in \text{Lin}_p(G) \setminus \{1_G\}$, assume the same values on the p -regular classes. But then $\forall \lambda \in \text{Lin}_p(G)$ the modules M_α and $M_\alpha \otimes L_\lambda$ lie in the same block with cyclic defect group Z (cf. [HL89, Theorem 2.1.8(a)]) and a Brauer character in this block is given by $\phi = \alpha$. Moreover, by [Nav02, Theorem 2.1(e)] if α, β are distinct characters in $\text{SIrr}_{p'}(G)$, the irreducible modules M_α and M_β lie in two different blocks with cyclic defect groups ${}^G Z$. Then the number of block with defect group Z is at least $|\text{SIrr}_{p'}(G)| = \frac{|\text{Irr}_{p'}(G)|}{p}$. Thanks to the description of $|\text{Irr}_{p'}(G)|$ given in (3.5.9), it is possible to conclude that the number of blocks with defect groups ${}^G Z$ is exactly $|\text{SIrr}_{p'}(G)| = \frac{|\text{Irr}_{p'}(G)|}{p}$. $\square$

Fact 3.5.5. As a consequence of Proposition 3.5.4, it is possible to draw the Brauer tree of the blocks of G with defect 1. In fact, for $\alpha \in \text{SIrr}_{p'}(G)$, the Brauer tree that corresponds to the $\mathcal{O}G$ -block B_{M_α} containing M_α has the following shape (cf. [Ben98, Theorem 6.5.5]):

$$M_\alpha \circ \xrightarrow{\phi} \bullet M_\alpha \otimes L_\lambda \quad (3.5.14)$$

where $\phi \in \text{IBr}(G)$ is such that $\alpha = \phi$ (on the p -regular classes) and M_α is the module corresponding to the irreducible character α . In particular, the exceptional vertex (denoted by a black dot), has multiplicity $m = p - 1$, since λ varies in $\text{Lin}_p(G) \setminus \{1_G\}$.

3.5.3 Blocks of $N_G(Z)$

The Brauer's First Main Theorem (cf. Theorem 1.2.8) establishes a bijection between blocks of the group G with defect groups GZ and blocks of the subgroup $N = N_G(Z)$ with defect group Z . In this section the structure of the blocks of N and the corresponding Brauer trees are investigated. Since $G = Z \rtimes H$, $N_G(Z) = C_G(Z) = Z \times C$, where $C = C_H(Z)$. By [HL89, Theorem 4.2.1], the blocks of N are in bijective correspondence with the ordinary irreducible characters of C . In particular, if $\chi \in \text{Irr}_{\mathbb{K}}(C)$, and we define $\varphi_0 = 1_Z \otimes \chi$, where 1_Z is the principal character of Z , and $\varphi_i = \lambda_i \otimes \chi$, where $\lambda_i \in \text{Irr}_{\mathbb{K}}(Z) \setminus \{1_Z\}$ for $i = 1, \dots, p-1$. Then, for $i \in \{0, \dots, p-1\}$, the ordinary irreducible characters φ_i , belong to same block b . Moreover, all blocks of N have defect group Z .

This implies that there exists exactly one irreducible Brauer character ϕ_0 in b which is the restriction of φ_0 on the p -regular conjugacy classes of N . But then we can conclude that the Brauer tree corresponding to b has the following shape:

$$\varphi_0 \circ \xrightarrow{\phi_0} \bullet \varphi_1 \quad (3.5.15)$$

where again the exceptional vertex has multiplicity $m = p - 1$.

3.5.4 Trivial source lattices

Since each block with defect groups GZ contains at least a trivial source lattices with vertex set GZ (cf. [Ben98, Corollary 6.3.3]) and the number of trivial source lattice with vertex set GZ equals $|\text{Irr}_{\mathbb{K}}(N/Z)|$ (cf. 1.5.9), then Proposition 3.5.4 implies the following fact.

Fact 3.5.6. For each $\alpha \in \text{SIrr}_{p'}(G)$, the block B_{M_α} of defect 1 contains exactly one indecomposable trivial source lattice with maximal vertex.

Example 3.5.1. If $p = 2$ the only trivial source lattice with maximal vertex $\mathbb{Z}/2\mathbb{Z}$ of the dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, n odd, is the trivial module.

Proposition 3.5.7. *The block B_{M_α} , constructed as above, contains a projective module P_{M_α} that is a trivial source lattice with vertex $\{1\}$. Moreover, the dimension of $(P_{M_\alpha})_{\mathbb{K}}$ is equal to $p \cdot \alpha(1)$.*

Proof. Let us consider the Brauer tree drawn in (3.5.14), then the block B_{M_α} contains the projective $\mathbb{F}G$ -module $P(\Phi)$, which is the projective cover of the simple $\mathbb{F}G$ -module corresponding to the character $\phi \in \text{IBr}(G)$. But then it is enough to put $P_{M_\alpha} = P(\phi)$. In particular this module is a trivial source lattice with vertex $\{1\}$ (cf. Remark 1.5.9) and its dimension as $\mathbb{K}G$ -module is $\alpha(1) + (p-1) \cdot \alpha(1)$. $\square$

Theorem 3.5.8. *For each $[M] \in \text{ITr}_{\mathcal{O}}^{\text{mx}}(G)$ there exist $\varepsilon_M \in \{\pm 1\}$ and $\delta_M \in \mathbf{T}_{\mathcal{O}}^{\prec}(G)$ such that $t_{\varepsilon, \delta}([M]) = \varepsilon_M \cdot M_{\mathbb{K}} + (\delta_M)_{\mathbb{K}}$ is irreducible as $\mathbb{K}G$ -module. Moreover, the map*

$$t_{\varepsilon, \delta}: \text{ITr}_{\mathcal{O}}^{\text{mx}}(G) \rightarrow \text{Irr}_{\mathbb{K}}(G) \quad (3.5.16)$$

is injective.

In order to prove this result, the following fact is necessary.

Fact 3.5.9. Let $\underline{k} = (k_1, \dots, k_n)$, where $k_i \in \mathbb{Z}$. Let

$$\sigma(k_1, \dots, k_n) = \sum_{1 \leq i < j \leq n} (k_i - k_j)^2 \leq n - 1.$$

Then either

- (a) $\underline{k} = \underline{0}$ and $\sigma(k_1, \dots, k_n) = 0$, or
- (b) $\underline{k} = e \cdot (1, \dots, 1) \pm \delta_i$ and $\sigma(k_1, \dots, k_n) = n - 1$, where $e \in \mathbb{Z}$ and $\delta_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$ has the i -th component equal to 1 and the others equal to 0.

Proof. Let us proceed by induction on n . The result is trivially true for $n = 2$. Let us suppose it holds for $n - 1$. Without loss of generality we can suppose $k_n = 0$. Then

$$\sigma(k_1, \dots, k_n) = k_1^2 + \dots + k_{n-1}^2 + \sigma(k_1, \dots, k_{n-1}) \leq n - 1 \quad (3.5.17)$$

If $k_1 = \dots = k_{n-1} = 0$, then $\sigma(k_1, \dots, k_{n-1}) = 0$. If there exists $i \in \{1, \dots, n - 1\}$ such that $k_i \neq 0$, then $\sigma(k_1, \dots, k_{n-1}) \leq n - 2$; then, by induction, $k_1, \dots, k_{n-1} = e \cdot (1, \dots, 1) \pm \delta_i$ for $i \in \{1, \dots, n - 1\}$ and $\sigma(k_1, \dots, k_{n-1}) \leq n - 2$. But (3.5.17) implies $e = 0$, then $\underline{k} = \underline{0} \pm \delta_i$ and $\sigma(k_1, \dots, k_n) = n - 1$. $\square$

We can now prove Theorem 3.5.8.

Proof of Theorem 3.5.8. Let B be the block containing the trivial source lattice with maximal vertex M and let S be the unique simple $\mathbb{F}G$ -module contained in B . Let M_i , for $i \in \{1, \dots, p\}$ be the simple $\mathbb{K}G$ -modules contained in B . Since $M \in B$, then there exist $n_1, \dots, n_p \in \mathbb{N}_0$ such that

$$M_{\mathbb{K}} = n_1 M_1 \oplus \dots \oplus n_p M_p. \quad (3.5.18)$$

Moreover, $(\mathbb{F} \otimes_{\mathcal{O}} M) = (n_1 + \dots + n_p) \cdot S$. Let $\mathcal{C}l(G)$ denote the set of conjugacy class of G and $\mathcal{C}l(G)^{p'}$ the subset of conjugacy classes of p' elements.

Let $\mathcal{C}l(G)^p = \mathcal{C}l(G) \setminus \mathcal{C}l(G)^{p'}$. Then, it holds that

$$\begin{aligned}
n_1^2 + \dots + n_p^2 &= \langle [M], [M] \rangle \\
&= \sum_{g \in \mathcal{C}l(G)} \frac{1}{|C_G(g)|} \langle M, {}^G(g, \langle g_p \rangle) \rangle_{\mathbf{L}_{\mathcal{O}}(G)} \cdot \langle [M], {}^G(g^{-1}, \langle g_p \rangle) \rangle_{\mathbf{L}_{\mathcal{O}}(G)} \\
&= \sum_{g \in \mathcal{C}l(G)} \frac{1}{|C_G(g)|} \chi_{M_{\mathbb{K}}}(g) \cdot \chi_{M_{\mathbb{K}}}(g^{-1}) \\
&= \sum_{g \in \mathcal{C}l(G)^{p'}} \frac{1}{|C_G(g)|} \chi_{M_{\mathbb{K}}}(g) \cdot \chi_{M_{\mathbb{K}}}(g^{-1}) \\
&\quad + \sum_{g \in \mathcal{C}l(G)^p} \frac{1}{|C_G(g)|} \chi_{M_{\mathbb{K}}}(g) \cdot \chi_{M_{\mathbb{K}}}(g^{-1}) \\
&= \Sigma_1 + \Sigma_2,
\end{aligned} \tag{3.5.19}$$

where $\chi_{M_{\mathbb{K}}}$ is the character associated to the $\mathbb{K}G$ -module M .

Let N_1, N_2 be $\mathbb{F}G$ -modules and let φ_{N_i} be the corresponding Brauer character, then it is possible to define a bilinear form

$$\langle \cdot, \cdot \rangle_{p'} : \mathbf{P}_{\mathbb{F}}(G) \times \mathbf{R}_{\mathbb{F}}(G) \rightarrow \mathbb{Z} \tag{3.5.20}$$

as follows

$$\langle [N_1], [N_2] \rangle_{p'} = \frac{1}{|G|} \sum_{g \in \mathcal{C}l(G)^{p'}} \varphi_{N_1}(g) \cdot \varphi_{N_2}(g^{-1}). \tag{3.5.21}$$

In particular, $\Sigma_1 = \langle M_{\mathbb{F}}, M_{\mathbb{F}} \rangle_{p'} = (n_1 + \dots + n_p)^2 \cdot \langle S, S \rangle_{p'}$.

Let $P(S)$ be the projective cover of the simple $\mathbb{F}G$ -module S , then by Proposition 3.5.7

$$\begin{aligned}
1 &= \langle P(S), S \rangle_{p'} = \langle p \cdot S, S \rangle_{p'} \\
&= p \cdot \langle S, S \rangle_{p'}
\end{aligned} \tag{3.5.22}$$

since $\forall i \in \{1, \dots, p\}$ $(M_i)_{\mathbb{F}} = S$. But then $\Sigma_1 = \frac{1}{p} \cdot (n_1 + \dots + n_p)^2$. Let $\mathbf{f}(M)$ be the Green correspondent of M , since $C = C_H(Z)$ is a p' group, the indecomposable $\mathcal{O}C$ -module $\mathbf{f}(M)$ is also irreducible, but then $\langle \mathbf{f}(M), \mathbf{f}(M) \rangle_{\mathbf{L}_{\mathcal{O}}(C)} = 1$, so

$$\begin{aligned}
\Sigma_2 &= \sum_{g \in \mathcal{C}l(G)^p} \frac{1}{|C_G(g)|} \chi_{M_{\mathbb{K}}}(g) \cdot \chi_{M_{\mathbb{K}}}(g^{-1}) \\
&= \sum_{g \in \mathcal{C}l(C)} (p-1) \cdot \frac{1}{p \cdot |C_C(g)|} \chi_{\mathbf{f}(M)}(g) \cdot \chi_{\mathbf{f}(M)}(g^{-1}) \\
&= \frac{p-1}{p}.
\end{aligned} \tag{3.5.23}$$

Thus, we can conclude that

$$n_1^2 + \dots + n_p^2 = \frac{1}{p}(n_1 + \dots + n_p)^2 + \frac{p-1}{p} \quad (3.5.24)$$

and then

$$p-1 = \sum_{1 \leq i < j \leq p} (n_i - n_j)^2. \quad (3.5.25)$$

So, Fact 3.5.9 implies that

$$(n_1, \dots, n_p) = e \cdot (1, \dots, 1) \pm \delta_i. \quad (3.5.26)$$

But then $M_{\mathbb{K}} \simeq \pm M_i \oplus e \cdot P(S)_{\mathbb{K}}$. In particular, by dimension reason, $e \in \mathbb{N}_0$. Now, it is enough to define $t_{e,\delta}(M) = M_i$. $\square$

Example 3.5.2. Let $G = \mathbb{Z}/3\mathbb{Z} \ltimes Q_8$ and $p = 3$. The group G has two (isomorphism classes of) trivial source lattices with maximal vertex. Let $\chi \in \text{Irr}_{\mathbb{K}}(\mathbb{Z}/2\mathbb{Z})$ be the character corresponding to the signum representation ρ and T_ρ be the corresponding $\mathcal{O}N$ -lattice. Then $T_\chi = \text{ind}_N^G(T_\rho)$ is the trivial source $\mathcal{O}G$ -lattice which belongs to the non principal block B_M of defect 1, where $\dim_{\mathbb{K}}(M) = 2$. It holds that $\dim_{\mathbb{K}}((T_\chi)_{\mathbb{K}}) = 4$ and

$$(T_\chi)_{\mathbb{K}} \simeq -M_{\mathbb{K}} \oplus (P_M)_{\mathbb{K}}. \quad (3.5.27)$$

3.6 Cyclic Sylow subgroup of prime order

Let G be a finite group with a cyclic Sylow p -subgroup P of prime order and let $N = N_G(P)$ be its normalizer in G . The goal of this section is to define an injective map

$$\mathbf{m}: \mathbf{R}_{\mathbb{K}}(N/P) \rightarrow \mathbf{R}_{\mathbb{K}}(G) \quad (3.6.1)$$

involving the trivial source $\mathcal{O}G$ -lattices with maximal vertex, i.e., such that their vertex set coincides with the set $\text{Syl}_p(G)$ of Sylow p -subgroups. To do this, we will proceed by steps, and the map $\mathbf{m}$ will be defined as the composition of three maps,

$$\mathbf{m}: \mathbf{R}_{\mathbb{K}}(N/P) \leftrightarrow \mathbf{T}_{\mathcal{O}}^{\text{mx}}(N) \leftrightarrow \mathbf{T}_{\mathcal{O}}^{\text{mx}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G), \quad (3.6.2)$$

as follows

- the first map $\mathbf{R}_{\mathbb{K}}(N/P) \leftrightarrow \mathbf{T}_{\mathcal{O}}^{\text{mx}}(N)$ is a bijection induced by the tensor product (see Remark 1.5.9)
 - the second map $\mathbf{T}_{\mathcal{O}}^{\text{mx}}(N) \leftrightarrow \mathbf{T}_{\mathcal{O}}^{\text{mx}}(G)$ is given by *Green correspondence*
 - the third (injective) map $\mathbf{T}_{\mathcal{O}}^{\text{mx}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G)$ is constructed in Theorem 3.6.1).
-

The following result is the analogous of Theorem 3.5.8 in the case of Sylow subgroups of prime order. The strategy of the proof turns out to be completely different from the previous case.

In fact, as the Sylow p -subgroup is cyclic, we can also deal with a Brauer tree and the hypothesis of a Sylow with prime order allow us to use the Diederichsen's theorem (see [CR90, Theorem 34.31]), which plays a key role in the proof.

Theorem 3.6.1. *Let G be a finite group with a cyclic Sylow p -subgroup of order p . For each $[T] \in \text{ITr}_{\mathcal{O}}^{\text{mx}}(G)$ there exist $\varepsilon_T \in \{\pm 1\}$ and $\delta_T \in \mathbf{T}_{\mathcal{O}}^{\vee}(G)$ such that the $\mathbb{K}G$ -module $t_{\varepsilon, \delta}([T]) = \varepsilon_T \cdot (T)_{\mathbb{K}} + (\delta_T)_{\mathbb{K}}$ is irreducible. Moreover, the map*

$$t_{\varepsilon, \delta}: \mathbf{T}_{\mathcal{O}}^{\text{mx}}(G) \rightarrow \mathbf{R}_{\mathbb{K}}(G) \quad (3.6.3)$$

is injective.

Proof. Let $\mathbb{F} = \bar{\mathbb{F}}_p$ be the algebraic closure of the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and let $\mathcal{O}^{\text{un}} = W_p(\mathbb{F})$ be the ring of Witt vectors (cf. [Ser79, §II.6]) with coefficients in the field $\mathbb{F}$. Let $\mathbb{K}^{\text{un}} = \text{quot}(\mathcal{O}^{\text{un}})$, then the triple $(\mathbb{K}^{\text{un}}, \mathcal{O}^{\text{un}}, \mathbb{F})$ is a p -modular system.

Let ξ be a p -th primitive root of the unit, then if $\mathbb{K} = \mathbb{K}^{\text{un}}[\xi]$ and $\mathcal{O} = \mathcal{O}^{\text{un}}[\xi]$, the p -modular system $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ splits.

Let B be a block of defect 1. Let M be an irreducible $\mathbb{K}G$ -module in B such that, if χ_M is its irreducible characters, $\chi_M(g) \in \mathcal{O}^{\text{un}}$, for $g \in G$ element of order p . Then there exists an irreducible $\mathbb{K}^{\text{un}}G$ -module M_0 such that $M_0 \otimes_{\mathbb{K}^{\text{un}}} \mathbb{K} = M$. In particular, M corresponds to a non exceptional vertex of the Brauer tree Γ_B associated to the block B .

Let $T \subseteq M$ be an indecomposable $\mathcal{O}G$ -lattice; since M is not projective, the lattice T has maximal vertex, i.e., $\mathbf{v}(T) = \text{Syl}_p(G)$. The Diederichsen's theorem (see [CR90, Theorem 34.31]) implies that T has either a trivial source or a co-trivial source.

Let us define a vertex (and the corresponding irreducible module) of “+ type” if it contains an indecomposable trivial source lattice and of “− type” otherwise. In particular, if two vertices of the Brauer tree are linked by an edge, then they have different “sign type”.

Without lost of generality let M be of + type and for $i \in \{1, \dots, n\}$ let T_i be the trivial source lattices contained in M . Then each lattice T_i can be associated to the vertex v_M of Γ_B which corresponds to M and to the edge e_{S_i} of Γ_B which corresponds to the socle S_i of $(T_i)_{\mathbb{F}}$. Let v_M be the end of e_{S_i} and let v_{M_i} be its origin. If M_i denotes the irreducible character corresponding to the vertex v_{M_i} , then M_i is an irreducible module of − type. Moreover, M_i contains $\Omega(T_i) = -T_i + P(S_i)$, where Ω denotes the Heller operator and $P(S_i)$ the projective cover of S_i . Furthermore, $(T_i)_{\mathbb{K}} \simeq M$ and $\Omega(T_i)_{\mathbb{K}} \simeq M_i$, if v_{M_i} is a non exceptional vertex, otherwise $\Omega(T_i)_{\mathbb{K}} \simeq M_i + \Lambda$, where Λ is the sum of all irreducible characters corresponding to the exceptional vertex

except for M_i .

Then, if M_i corresponds to the exceptional vertex T_i will be mapped to M , otherwise T_i can be mapped to M or to $-(T_i)_{\mathbb{K}} + (P_{S_i})_{\mathbb{K}}$.

If the exceptional vertex is of $+$ type, it is enough to consider Λ instead of M in the construction above and to map T_i to $\Omega(T_i)_{\mathbb{K}}$.

The injectivity of the map depends on the choices above; but there exists a good choice that guarantees it. $\square$

3.6.1 Example

Let $G = L_2(7)$ and $p = 7$. The principal block B has defect 1 and it contains three isomorphism classes of indecomposable trivial source $\mathcal{O}G$ -lattices with maximal vertex $[T_1]$, $[T_2]$ and $[T_3]$ (cf. [KK10]). Let Γ_B be the Brauer tree associated to B , then the tree Γ_B has the following shape,

$$\circ\chi_1 \xrightarrow{\varphi_1} \circ\chi_4 \xrightarrow{\varphi_3} \circ\chi_6 \xrightarrow{\varphi_2} \bullet\chi_2, \chi_3 \quad (3.6.4)$$

where χ_i are the (ordinary) irreducible characters labelled as in [CCN⁺85] and $\varphi_1, \varphi_2, \varphi_3$ are the irreducible Brauer characters belonging to the principal block (cf. Table 3.1).

Table 3.1: Decomposition matrix $L_2(7)$, $p = 7$

Block 1:	1 = φ_1	3 = φ_2	5 = φ_3
$1_1 = \chi_1$	1	0	0
$3_1 = \chi_2$	0	1	0
$3_2 = \chi_3$	0	1	0
$6_1 = \chi_4$	1	0	1
$8_1 = \chi_6$	0	1	1

In particular, if M_i denotes the irreducible module corresponding to the irreducible character χ_i , then $(T_1)_{\mathbb{K}} = M_1$, $(T_2)_{\mathbb{K}} = M_6$ and $(T_3)_{\mathbb{K}} = M_6$. Let $P(1)$, $P(3)$ and $P(5)$ be the projective covers of the $\mathbb{F}G$ -modules of rank 1, 3 and 5 respectively, then from the decomposition matrix above one knows that

$$\begin{aligned} P(1)_{\mathbb{K}} &= M_1 + M_4, \\ P(3)_{\mathbb{K}} &= M_2 + M_3 + M_6, \\ P(5)_{\mathbb{K}} &= M_4 + M_6. \end{aligned} \quad (3.6.5)$$

Then $[T_1]$ can be mapped to M_1 or to $-M_1 + P(1)_{\mathbb{K}}$, $[T_2]$ can be mapped to M_6 or to $-M_6 + P(5)_{\mathbb{K}}$ and $[T_3]$ can be mapped to M_6 or to $-M_6 + P(3)_{\mathbb{K}}$. Since $-M_6 + P(3)_{\mathbb{K}}$ is not irreducible, the only choice that guarantees the

injectivity is the following

$$\begin{aligned} t_{\varepsilon, \delta}([T_1]) &= M_1 \\ t_{\varepsilon, \delta}([T_2]) &= -M_6 + P(5)_{\mathbb{K}} = M_4 \\ t_{\varepsilon, \delta}([T_3]) &= M_6. \end{aligned} \tag{3.6.6}$$

Moreover $N_G(P)/P \simeq C_3$, then $\mathbf{R}_{\mathbb{K}}(N/P) = \text{span}_{\mathbb{Z}}\{S_1, S_2, S_3\}$, where S_i are the linear irreducible representations of C_3 . In particular, N/P is a p' -group, then each irreducible module S_i corresponds exactly to one indecomposable trivial source $\mathcal{O}N$ -lattice $\bar{S}_i$ such that $(\bar{S}_i)_{\mathbb{K}} \simeq S_i$. By Green correspondence, $\bar{S}_i = \mathbf{f}(T_i)$. Then

$$\begin{aligned} \mathbf{m}([S_1]) &= M_1 \\ \mathbf{m}([S_2]) &= M_4 \\ \mathbf{m}([S_3]) &= M_6. \end{aligned} \tag{3.6.7}$$

The ideas behind the construction of this map will be much more investigated in the last chapter of this thesis and the result just proven will be developed. Theorem 3.6.1 and 3.5.8 have been presented since they are a first easy but not trivial example of the link between trivial source lattices and irreducible representations. The relationship between the Grothendieck rings defined in § 1.5.1 can be a useful and powerful tool in the study of *local-global conjectures*.

CHAPTER 4

A strong form of the Alperin-McKay conjecture

4.1 Introduction

For a splitting p -modular system $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ of a finite group G (cf. § 1.3) the category of (left) $\mathcal{O}G$ -lattices $\mathcal{O}G - \mathbf{lat}$ is in general extremely difficult to analyse. Nevertheless, it is quite important as it connects the characteristic 0 and the p -modular representation theory of the finite group G in a non-trivial way. Indeed, one has a “roof” of additive functors

$$\begin{array}{ccc} & \mathcal{O}G - \mathbf{lat} & \\ \swarrow \scriptstyle -_{\mathbb{K}} & & \searrow \scriptstyle -_{\mathbb{F}} \\ \mathbb{K}G - \mathbf{mod} & & \mathbb{F}G - \mathbf{mod}, \end{array} \quad (4.1.1)$$

where $-_{\mathbb{K}} = - \otimes_{\mathcal{O}} \mathbb{K}$ and $-_{\mathbb{F}} = - \otimes_{\mathcal{O}} \mathbb{F}$ (cf. (2.1.2)), and one may think of the category $\mathcal{O}G - \mathbf{lat}$ as a kind of “bridge” between $\mathbb{K}G - \mathbf{mod}$, the category of finitely generated (left) $\mathbb{K}G$ -modules, and $\mathbb{F}G - \mathbf{mod}$, the category of finitely generated (left) $\mathbb{F}G$ -modules. The roof (4.1.1) is used in the construction of the *decomposition homomorphism* quite essentially, e.g., the image $[L_{\mathbb{F}}] \in \mathbf{R}_{\mathbb{F}}(G)$ in the *Grothendieck ring* $\mathbf{R}_{\mathbb{F}}(G) = \mathbf{G}_0(\mathbb{F}G)$ of an $\mathcal{O}G$ -lattice $L \subseteq M$ inside the finitely generated $\mathbb{K}G$ -module M does not depend on the choice of the $\mathcal{O}G$ -lattice L , but only on the isomorphism type of the $\mathbb{K}G$ -module M . This fact enables one to define the decomposition homomorphism

$$\mathbf{d}: \mathbf{R}_{\mathbb{K}}(G) \longrightarrow \mathbf{R}_{\mathbb{F}}(G), \quad (4.1.2)$$

by $\mathbf{d}([M]) = [L_{\mathbb{F}}]$, where $\mathbf{R}_{\mathbb{K}}(G) = \mathbf{K}_0(\mathbb{K}G)$.

The starting point of this chapter is the *structure theorem* (see Theorem 1.5.10) for indecomposable linear source $\mathcal{O}G$ -lattices. Irreducible $\mathbb{K}G$ -modules and irreducible $\mathbb{F}G$ -modules in a block $B = \mathcal{O}G \cdot e_B$ with normal

defect are very well understood thanks to the work of W. F. Reynolds (see [Rey63]). For these blocks the structure theorem has the following important consequence (see Theorem 4.2.5).

Theorem A. Let G be a finite group, and let $B = \mathcal{O}G \cdot e_B$ be an $\mathcal{O}G$ -block with normal defect. Then for every indecomposable linear source B -lattice L with vertex set $\mathbf{v}(L) = \text{df}(B)$ the $\mathbb{K}G$ -module $L_{\mathbb{K}}$ is irreducible and has height zero. Moreover, the canonical map

$$-\mathbb{K}: \text{IL}_{\mathcal{O}}^{\text{mx}}(B) \longrightarrow \text{Irr}_0(B) \quad (4.1.3)$$

between the set $\text{IL}_{\mathcal{O}}^{\text{mx}}(B)$ of isomorphism types of linear source B -lattices with maximal vertex and the set $\text{Irr}_0(B)$ of irreducible $B_{\mathbb{K}}$ -modules of height zero is a bijection.

Let B be an $\mathcal{O}G$ -block with defect groups $\text{df}(B) = {}^G D = \{ {}^g D \mid g \in G \}$, and let $b = \mathcal{O}N_G(D)$ be the $\mathcal{O}N_G(D)$ -block in Brauer correspondence to B . In particular, Green correspondence yields canonical bijections

$$\begin{aligned} \mathbf{f}: \text{IL}_{\mathcal{O}}^{\text{mx}}(B) &\longrightarrow \text{IL}_{\mathcal{O}}^{\text{mx}}(b), \\ \mathbf{g}: \text{IL}_{\mathcal{O}}^{\text{mx}}(b) &\longrightarrow \text{IL}_{\mathcal{O}}^{\text{mx}}(B) \end{aligned} \quad (4.1.4)$$

(see [Ben06, Theorem 2.7.4(ii)]). For an $\mathcal{O}G$ -block B with defect groups $\text{df}(B) = {}^G D$ there are several Grothendieck groups of linear source lattices one may consider. Let $\mathbf{L}_{\mathcal{O}}(B)$ denote the abelian group being freely generated by the isomorphism types of indecomposable linear source B -lattices, then

$$\mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \simeq \mathbf{L}_{\mathcal{O}}(B) / \mathbf{L}_{\mathcal{O}}^{\prec}(B), \quad (4.1.5)$$

where $\mathbf{L}_{\mathcal{O}}^{\prec}(B)$ is the abelian group being freely generated by the isomorphism types of indecomposable linear source $\mathcal{O}G$ -lattices with vertex strictly smaller than ${}^G D$. With an abuse of notation $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(B)$ will also denote the quotient $\mathbf{L}_{\mathcal{O}}(B) / \mathbf{L}_{\mathcal{O}}^{\prec}(B)$.

For an irreducible $\mathbb{K}G$ -module M in the $\mathcal{O}G$ -block B , $\text{df}(B) = {}^G D$, the *height* of M is defined as the integer $\text{ht}(M)$ such that

$$\dim(M)_p |D| = p^{\text{ht}(M)} |P|,$$

where $\dim(M)_p$ denote the p -part of the dimension of the module M . Let $\mathbf{R}_{\mathbb{K}}^0(B)$ denote the abelian group being freely generated by the set $\text{Irr}_0(B)$ of isomorphism types of irreducible $B_{\mathbb{K}}$ -modules of height zero. By Theorem A, the map $-\mathbb{K}$ induces a canonical isomorphism $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(b) \longrightarrow \mathbf{R}_{\mathbb{K}}^0(b)$, and thus, by Green correspondence, a canonical isomorphism $\mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \simeq \mathbf{R}_{\mathbb{K}}^0(b)$. Hence, in order to relate the Grothendieck group $\mathbf{R}_{\mathbb{K}}^0(b)$ to $\mathbf{R}_{\mathbb{K}}^0(B)$ - as it is predicted by the Alperin-McKay conjecture (see [Alp76, Conjecture 3]) - one has to

find a suitable section $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ for the canonical projection $\pi_B: \mathbf{L}_{\mathcal{O}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B)$,

$$\begin{array}{ccc} & \mathbf{L}_{\mathcal{O}}(B) & \xleftarrow{\sigma_B} \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \\ & \swarrow \text{--}\mathbb{K} \quad \searrow \pi_B & \\ \mathbf{R}_{\mathbb{K}}(B) & & \end{array} \quad (4.1.6)$$

The Grothendieck group $\mathbf{L}_{\mathcal{O}}(B)$ comes equipped with the two symmetric bilinear forms defined in § 3.1. The first

$$\langle -, - \rangle: \mathbf{L}_{\mathcal{O}}(B) \times \mathbf{L}_{\mathcal{O}}(B) \longrightarrow \mathbb{Z}, \quad (4.1.7)$$

induced by the group homomorphism $\text{--}\mathbb{K}: \mathbf{L}_{\mathcal{O}}(B) \rightarrow \mathbf{R}_{\mathbb{K}}(B)$, and the second

$$\langle\langle -, - \rangle\rangle: \mathbf{L}_{\mathcal{O}}(B) \times \mathbf{L}_{\mathcal{O}}(B) \longrightarrow \mathbb{Z}, \quad (4.1.8)$$

induced by the projection $\pi_B: \mathbf{L}_{\mathcal{O}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B)$. As shown in § 3.1 one can define a third bilinear form by

$$(-, -)_B = \langle -, - \rangle - \langle\langle -, - \rangle\rangle: \mathbf{L}_{\mathcal{O}}(B) \times \mathbf{L}_{\mathcal{O}}(B) \longrightarrow \mathbb{Z} \quad (4.1.9)$$

which can be used to formulate two conjectures concerning Diagram 4.1.6.

Conjecture 1. For an $\mathcal{O}G$ -block B of a finite group G there exists a π_B -section $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ such that $\text{im}(\sigma_B)$ is totally isotropic with respect to $(-, -)_B$.

Suppose that σ_B is a π_B -section as described in Conjecture 1 for the $\mathcal{O}G$ -block B . Let $L \in \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(B)$ be an indecomposable linear source B -lattice with maximal vertex, i.e., $\mathbf{v}(L) = \text{df}(B) = {}^G D$. Then

$$\sigma_B(L)_{\mathbb{K}} \in \pm \text{Irr}_0(B) \subseteq \mathbf{R}_{\mathbb{K}}(B), \quad (4.1.10)$$

and thus σ_B induces an injective map $\tilde{\sigma}_B: \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \text{Irr}_0(B)$ defined as the composition $\text{--}\mathbb{K} \circ \sigma_B$ (see the proof of Theorem 4.3.4). In particular, if b denotes the $\mathcal{O}N_G(D)$ -block in Brauer correspondence to B , one obtains the inequality

$$|\text{Irr}_0(b)| \leq |\text{Irr}_0(B)|, \quad (4.1.11)$$

and thus “one half” of the Alperin-McKay conjecture.

Hence one may interpret the following conjecture as a reformulation of the classical Alperin-McKay conjecture (under the hypothesis that Conjecture 1 holds).

Conjecture 2. Let σ_B be a π_B -section for the $\mathcal{O}G$ -block B of the finite group G . Then $\tilde{\sigma}_B: \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \text{Irr}_0(B)$ is bijective.

Remark 4.1.1. It is known that the Alperin-McKay conjecture has positive answer in some cases, as for example p -solvable groups (see [OW80] and [Dad80]), the symmetric and alternating groups (see [Ols76]) and also for blocks B splendid derived equivalent to their Brauer correspondent b .

So, Conjecture 1 and Conjecture 2 can be seen as a reformulation of the Alperin-McKay conjecture in terms of linear source lattices, but they have also the following impact on a conjecture formulated by I. M. Isaacs and G. Navarro in [IN02] generalizing the Alperin-McKay conjecture (see Theorem 4.3.4).

Theorem B. Let B be an $\mathcal{O}G$ -block of the finite group G . If Conjecture 1 and 2 hold for B , then Conjecture B in [IN02] holds for B .

It is important to observe that Theorem B ensures that Conjecture 1 and the Alperin-McKay conjecture together imply Conjecture B in [IN02].

At this point one will ask the following question:

- Where do π_B -sections $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ for an $\mathcal{O}G$ -block B of a finite group G come from?

One source is described in the following theorems (see Theorem 4.4.5). and Theorem 4.5.1).

Theorem C. Let G be a finite group, and let B be an $\mathcal{O}G$ -block with defect groups $\text{df}(B) = {}^G D$. Let b be the $\mathcal{O}N_G(D)$ -block in Brauer correspondence to B , and suppose that B and b are splendid derived equivalent. Then Conjectures 1 holds for B , and thus Conjecture B in [IN02] holds for B .

Remark 4.1.2. Already some time ago M. Broué conjectured in [Bro90] that an $\mathcal{O}G$ -block B with abelian defect groups $\text{df}(B) = {}^G D$ is (splendid) derived equivalent to the $\mathcal{O}N_G(D)$ -block b in Brauer correspondence to B . Nowadays, this statement is known as *Broué's (splendid) abelian defect group conjecture* (see [Mal17, § 6]). Thus, by Theorem C, Broué's splendid abelian defect group conjecture implies Conjecture B from [IN02]. It is well known that Broué's splendid abelian defect group conjecture holds for blocks with cyclic defect (cf. [Rou98]).

4.2 Linear source lattices

One major tool in this context is the *Green correspondence* (see [Ben98, Theorem 3.12.2]), i.e., if L is an indecomposable linear source $\mathcal{O}G$ -lattice with vertex set ${}^G V = \{ {}^g V \mid g \in G \}$ and $N = N_G(V)$, then $\mathbf{f}(L)$ will denote the unique indecomposable summand of $\text{res}_N^G(L)$ with vertex set ${}^N V$. On the other hand, if M is an indecomposable linear source $\mathcal{O}N$ -lattice with vertex set ${}^N V$, $\mathbf{g}(M)$ will denote the unique indecomposable summand of $\text{ind}_N^G(M)$ with vertex set containing V .

4.2.1 Structure theorem

The result presented in this section has already been discussed in § 1.5.2; it is repeated here for convenience of the reader.

For a p -subgroup V of G we denote by $\hat{V} = \text{Hom}_{\text{grp}}(V, \mathcal{O}^*)$ the set of $\mathcal{O}$ -linear characters of V . In particular, $\hat{V}$ is a left $N_G(V)$ -set, where ${}^g\varphi$ is given by ${}^g\varphi(x) = \varphi(g^{-1}xg)$ for $g \in N_G(V)$, $\varphi \in \hat{V}$ and $x \in V$.

The stabilizer $I(\varphi) = \{g \in N_G(V) \mid {}^g\varphi = \varphi\}$ of φ coincides with the *inertia* subgroup of the $\mathcal{O}V$ -lattice $\mathcal{O}_\varphi$, where $\mathcal{O}_\varphi$ is - considered as $\mathcal{O}$ -module - isomorphic to $\mathcal{O}$ with left V -action given by φ . Moreover, for $\varphi \in \hat{V}$, $V \cdot C_G(V) \subseteq I(\varphi) \subseteq N_G(V)$. By construction $\text{IL}_{\mathcal{O}}^{\text{mx}}(V) = \{[\mathcal{O}_\varphi] \mid \varphi \in \hat{V}\}$. Let $\text{ind}_V^{I(\varphi)}(\mathcal{O}_\varphi) = L_1 \oplus \cdots \oplus L_r$, where L_i are indecomposable $\mathcal{O}I(\varphi)$ -lattices. As $\ker(\varphi)$ is a normal subgroup of $I(\varphi)$, $\text{ind}_V^{I(\varphi)}(\mathcal{O}_\varphi)$ is an $\mathcal{O}I(\varphi)$ -lattice which is inflated from $\bar{I}(\varphi) = I(\varphi)/\ker(\varphi)$. Hence L_i are indecomposable $\mathcal{O}I(\varphi)$ -lattices which are inflated from $\bar{I}(\varphi)$. Let $\bar{V} = V/\ker(\varphi)$. Then $\bar{V}$ is a cyclic p -group contained in $Z(\bar{I}(\varphi))$, the center of $\bar{I}(\varphi)$. Let

$$\mathcal{O}\bar{I}(\varphi) = \text{ind}_{\ker(\varphi)}^{I(\varphi)}(\mathcal{O}) = Q_1 \oplus \cdots \oplus Q_s, \quad (4.2.1)$$

where Q_i are projective indecomposable $\mathcal{O}\bar{I}(\varphi)$ -lattices. For an $\mathcal{O}\bar{I}(\varphi)$ -lattice Q for which $\text{res}_{\bar{V}}^{\bar{I}(\varphi)}(Q)$ is a projective $\mathcal{O}\bar{V}$ -lattice we put

$$Q(\varphi) = Q / \langle (c - \varphi(c) \cdot \text{id}_Q) \cdot q \mid c \in \bar{V}, q \in Q \rangle, \quad (4.2.2)$$

e.g., $\text{ind}_V^{I(\varphi)}(\mathcal{O}_\varphi) = \text{ind}_{\ker(\varphi)}^{I(\varphi)}(\mathcal{O})(\varphi)$. As $\bar{V}$ is a normal p -subgroup of $\bar{I}(\varphi)$, $\bar{V}$ acts trivially on any simple $\mathbb{F}\bar{I}(\varphi)$ -module. Hence $\text{hd}(Q_j(\varphi)) = \text{hd}(Q_j)$, and thus $Q_j(\varphi)$ is an indecomposable $\mathcal{O}\bar{I}(\varphi)$ -module. This implies that $r = s$, and after renumbering we may also assume $L_j = Q_j(\varphi)$. By construction, $\tilde{Q}_j(\varphi) = \inf_{\bar{I}(\varphi)}^{I(\varphi)}(Q_j(\varphi))$ is an indecomposable linear source $\mathcal{O}I(\varphi)$ -lattice with vertex set $\{V\}$. The following theorem gives a complete description of indecomposable linear source lattices modulo Green correspondence.

Theorem 4.2.1. *Let V be a p -subgroup of G , and let M be an indecomposable linear source $\mathcal{O}G$ -lattice with vertex set ${}^G V$.*

- (a) *There exist an element $\varphi \in \hat{V}$ and a projective indecomposable $\mathcal{O}\bar{I}(\varphi)$ -lattice Q such that $\mathbf{f}(M) \simeq \text{ind}_{I(\varphi)}^{N_G(V)}(\tilde{Q}(\varphi))$.*
- (b) *If $\psi \in \hat{V}$ and Q_o is a projective indecomposable $\mathcal{O}\bar{I}(\psi)$ -lattice such that $\mathbf{f}(M) \simeq \text{ind}_{I(\psi)}^{N_G(V)}(\tilde{Q}_o(\psi))$, then there exists $g \in N_G(V)$ such that $\psi = {}^g\varphi$ and $Q_o \simeq {}^gQ$.*
- (c) *Let $\varphi \in \hat{V}$, and let Q be a projective indecomposable $\mathcal{O}\bar{I}(\varphi)$ -lattice. Then $L = \text{ind}_{I(\varphi)}^{N_G(V)}(\tilde{Q}(\varphi))$ is an indecomposable linear source $\mathcal{O}N_G(V)$ -lattice, and $\mathbf{g}(L)$ is an indecomposable linear source $\mathcal{O}G$ -lattice.*

- Proof.* (a) Let $N = N_G(V)$, and let $L = \mathbf{f}(M)$ be its Green correspondent. Then L is an indecomposable linear source $\mathcal{O}N$ -lattice with vertex set $\{V\}$. Hence there exists $\varphi \in \hat{V}$ such that $\mathcal{O}_\varphi$ is a direct summand of $\text{res}_V^N(L)$. As V is normal in N , Clifford theory applies, i.e., there exists a direct summand $\tilde{Q}$ of $\text{ind}_V^{I(\varphi)}(\mathcal{O}_\varphi)$ such that $L \simeq \text{ind}_{I(\varphi)}^N(\tilde{Q})$ (see [Ben98, 3.13.2]). By the previously mentioned argument, one has $\tilde{Q} \simeq \tilde{Q}(\varphi)$ for some projective indecomposable $\mathcal{O}\bar{I}(\varphi)$ -lattice Q .
- (b) is a direct consequence of the fact that the source is uniquely determined modulo $N_G(V)$ -conjugation and the final remark of [Ben98, 3.13.2].
- (c) follows from [Ben98, 3.13.2] and Green correspondence. $\square$

4.2.2 Proof of Theorem A

Let G be a finite group, and let L be an indecomposable linear source $\mathcal{O}G$ -lattice in the block $B = \mathcal{O}G \cdot e_B$ with defect groups $\text{df}(B) = {}^G D$. Then $\mathbf{v}(L) \preceq \text{df}(B)$ (see [Ben06, Proposition 2.7.4]), and L is said to have *maximal vertex* if $\mathbf{v}(L) = \text{df}(B) = {}^G D$. The symbol ${}^G V \preceq {}^G U$ is used for p -subgroups U and V of G to express that there exists $g \in G$ such that $V \subseteq {}^g U$.

Remark 4.2.2. Let B be a block of $\mathcal{O}G$ with defect groups ${}^G D$. For H subgroup of G such that $D \cdot C_G(D) \subseteq H \subseteq N_G(D)$ and for $\chi \in \text{Irr}_0(H)$ in the Brauer correspondent $b = \mathcal{O}H \cdot f_b$ of B , then

$$(\text{ind}_H^G(\chi)(1))_p = \frac{|G|_p}{|H|_p} \chi(1)_p = \frac{|G|_p}{|H|_p} \cdot \frac{|H|_p}{|D|}. \quad (4.2.3)$$

Then, if $\text{ind}_H^G(\chi)$ is irreducible it has height zero.

Moreover, if B has normal defect group D and $\psi \in \text{Irr}_{\mathbb{K}}(B)$, then $\text{ht}(\psi)$ is equal to the p -value of the degree of every irreducible constituent of $\text{res}_D^G(\chi)$ (see [Rey63, Theorem 8]). So, in particular if ψ has height zero, then the irreducible constituents of $\text{res}_D^G(\chi)$ are linear characters.

Theorem 4.2.3. *Let G be a finite group, if a block B of G has normal defect group D , the irreducible representations of G/D in B are all modularly irreducible and of height zero. Taken modularly, they all remain distinct, and they yield all the irreducible $\mathbb{F}G$ -modules of B .*

Proof. See [Rey63, Theorem 10]. $\square$

Remark 4.2.4. Let B be a block of G with normal defect D . Then the block B in G/D can split in different components of defect 0. In particular, Theorem 4.2.3 ensures that irreducible $\mathbb{F}G$ -modules of B lie in different defect 0 blocks of G/D .

Theorem 4.2.5 (Theorem A). *Let G be a finite group, and let $B = \mathcal{O}G \cdot e_B$ be an $\mathcal{O}G$ -block of G with normal defect group, i.e., $\text{df}(B) = \{D\}$ and D is normal in G . Then for every indecomposable linear source B -lattice L with vertex set $\mathbf{v}(L) = \text{df}(B)$ the $\mathbb{K}G$ -module $L_{\mathbb{K}}$ is irreducible and has height zero. Moreover, the canonical map*

$$-\mathbb{K}: \text{IL}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \text{Irr}_0(B) \quad (4.2.4)$$

between the set of isomorphism types of linear source B -lattices of maximal vertex and the set of irreducible $B_{\mathbb{K}}$ -modules of height zero is a bijection.

Proof. Let $\varphi \in \hat{D}$ such that ${}^{N_G(D)}\mathcal{O}_{\varphi} = \mathbf{s}(L)$ and let $I = I(\varphi)$ be the inertia group of φ , i.e., $I(\varphi) = \{g \in N_G(D) \mid {}^g\varphi = \varphi\}$. Let Q_{φ} be the indecomposable linear source $\mathcal{O}I(\varphi)$ -lattice with vertex set $\{D\}$, such that $L \simeq \text{ind}_I^G(Q_{\varphi})$. Then by construction $(Q_{\varphi})_{\mathbb{F}}$ is a projective indecomposable $\mathbb{F}I(\varphi)/D$ -module. Then, the indecomposability of $(Q_{\varphi})_{\mathbb{F}}$ and Remark 4.2.4 imply that $(Q_{\varphi})_{\mathbb{F}}$ belongs to a 0-defect block of $I(\varphi)/D$. So one may conclude that $(Q_{\varphi})_{\mathbb{F}}$ is an irreducible $\mathbb{F}I(\varphi)/D$ -module. In particular, $(Q_{\varphi})_{\mathbb{K}}$ is an irreducible $\mathbb{K}I(\varphi)$ -module. Let M be an irreducible constituent of the $B_{\mathbb{K}}$ -module $L_{\mathbb{K}}$ and let $\mathbb{K}_{\varphi} = \mathcal{O}_{\varphi} \otimes_{\mathcal{O}} \mathbb{K}$. Then $I(\mathbb{K}_{\varphi})$ is the inertia subgroup of the irreducible $\mathbb{K}G$ -module $\mathbb{K}_{\varphi}$. Let H be the $\mathbb{K}_{\varphi}$ -homogeneous component of M . Then, by construction $Q_{\varphi} \otimes_{\mathcal{O}} \mathbb{K} \subseteq H$. By Clifford theory,

$$M \simeq \text{ind}_I^G(H) = \text{ind}_I^G(H) \quad (4.2.5)$$

and then

$$M \supseteq \text{ind}_I^G(Q_{\varphi})_{\mathbb{K}} \simeq L_{\mathbb{K}}. \quad (4.2.6)$$

Hence $L_{\mathbb{K}} = M$ is an irreducible $\mathbb{K}G$ -module.

By [Rey63, Theorem 10] there exists an height zero $\mathbb{K}I_{\varphi}/D$ -module that yields $(Q_{\varphi})_{\mathbb{K}}$, then also $(Q_{\varphi})_{\mathbb{K}}$ has height zero. So, thanks to Remark 4.2.2 it is possible to conclude that $L_{\mathbb{K}}$ has height zero.

Let L_1, L_2 be indecomposable linear source B -lattices with vertex set $\{D\}$ such that $(L_1)_{\mathbb{K}} = (L_2)_{\mathbb{K}} = \chi \in \text{Irr}_0(G)$. Since

$$\begin{aligned} \mathbf{s}(L_1) &= \{\varphi \in \hat{D} \mid \mathbb{K}_{\varphi} | \text{res}_D^G(L_1)_{\mathbb{K}}\} \\ &= \{\varphi \in \hat{D} \mid \mathbb{K}_{\varphi} | \text{res}_D^G(L_2)_{\mathbb{K}}\} = \mathbf{s}(L_2), \end{aligned} \quad (4.2.7)$$

L_1 and L_2 have same vertex and same source ${}^{N_G(D)}\varphi$. Then, by Theorem 1.5.10, $L_1 = \text{ind}_I^G(Q_{\varphi})$ and $L_2 = \text{ind}_I^G(P_{\varphi})$. Moreover,

$$P_{\mathbb{K}} \simeq Q_{\mathbb{K}} \text{ and } P_{\mathbb{F}} \simeq Q_{\mathbb{F}}. \quad (4.2.8)$$

Thus $P \simeq Q$ and Theorem 1.5.10 implies $L_1 \simeq L_2$.

Let $\chi \in \text{Irr}_0(B)$ and let M be the corresponding irreducible $\mathbb{K}G$ -module, then by Clifford theory

$$\text{res}_D^G(M) \simeq (\oplus_i {}^{g_i}J)^{(e)}, \quad (4.2.9)$$

where J is a simple $\mathbb{K}D$ -module contained in M and such that $J = (\mathcal{O}_\varphi)_\mathbb{K}$ for some $\varphi \in \hat{D}$ (see Remark 4.2.2). Let H be the $\mathbb{K}D$ -homogeneous component of $\text{res}_D^G(M)$ containing J , then $M \simeq \text{ind}_{\mathbb{K}I}^G(H)$, where $\mathbb{K}I$ is the stabilizer of H . In particular $\mathbb{K}I = (I_\varphi)_\mathbb{K}$ and $H = (Q_\varphi)_\mathbb{K}$. So, $M = (\text{ind}_{I_\varphi}^G(Q_\varphi))_\mathbb{K}$. Then Theorem 4.2.5 implies the surjectivity of the map defined in (4.2.4). $\square$

Remark 4.2.6. If G is a finite group, B a block with normal defect D and $T \in \text{ITr}_\mathcal{O}^{\text{mx}}(B)$, then $T_\mathbb{F}$ is irreducible.

4.2.3 Rigidity

In this subsection and in the following another characterization of $\mathcal{O}G$ -lattices with maximal vertex in a block B with normal or cyclic defect is given.

Let G be a finite group, and let L be an indecomposable $\mathcal{O}G$ -lattice with vertex set $\mathbf{v}(L) = {}^G V$. Assume that

$$L_\mathbb{F} = Y_1 \oplus \cdots \oplus Y_n, \quad (4.2.10)$$

where Y_i are indecomposable $\mathbb{F}G$ -modules. Then, by construction, for all $i \in \{1, \dots, n\}$, $\mathbf{v}(Y_i) \preceq {}^G V$.

Definition 4.2.7. An $\mathcal{O}G$ -lattice L with vertex set $\mathbf{v}(L) = {}^G V$ is said to be *rigid*, if $\mathbf{v}(Y_i) = {}^G V$ for all $i \in \{1, \dots, n\}$.

The following fact will turn out to be useful for our purpose.

Fact 4.2.8. Let G be a finite group, and let L be an indecomposable $\mathcal{O}G$ -lattice. Let $L_\mathbb{F} = Y_1 \oplus \cdots \oplus Y_n$, where Y_i are indecomposable $\mathbb{F}G$ -modules, and assume that $n \geq 2$. Then Y_i is not projective.

Proof. Suppose that Y_j , $j \in \{1, \dots, n\}$, is a projective $\mathbb{F}G$ -module, and let $\tilde{Q}$ be the projective $\mathcal{O}G$ -lattice satisfying $\tilde{Q}_\mathbb{F} \simeq Y_j$. Then, by hypothesis, there exists a homomorphism of $\mathcal{O}G$ -lattices $\pi_B: \tilde{Q} \rightarrow L$ which is injective, and $\text{im}(\pi_B)$ is a saturated $\mathcal{O}$ -lattice, i.e., if $y \in L$ and $r \in \mathcal{O}$ are satisfying $r \cdot y \in \text{im}(\pi_B)$, then $y \in \text{im}(\pi_B)$. Hence the short exact sequence

$$(\dagger): 0 \rightarrow \tilde{Q} \rightarrow L \rightarrow L/\text{im}(\pi_B) \rightarrow 0 \quad (4.2.11)$$

splits as a sequence of $\mathcal{O}$ -lattices. Since $\tilde{Q}$ is relative injective (with respect to $\mathcal{O}$ -split injections), $(\dagger)$ splits also as a short exact sequence of $\mathcal{O}G$ -lattices contradicting the indecomposability of L . $\square$

Remark 4.2.9. Let G be a finite group. If T is an indecomposable trivial source $\mathcal{O}G$ -lattice, then it is well known that $T_\mathbb{F}$ is an indecomposable $\mathbb{F}G$ -module and $\mathbf{v}(T) = \mathbf{v}(T_\mathbb{F})$ (cf. [Ben06, Remark after Corollary 2.6.3]). Hence T is rigid. Indeed, using a similar argument, one can show that every indecomposable endo-(trivial source) $\mathcal{O}G$ -lattice E has the property that $E_\mathbb{F}$ is an indecomposable endo-(trivial source) $\mathbb{F}G$ -module and $\mathbf{v}(E) = \mathbf{v}(E_\mathbb{F})$.

The situation for linear source lattices is more complicated.

Proposition 4.2.10. *Let G be a finite group, and let $B = \mathcal{O}G \cdot e_B$ be an $\mathcal{O}G$ -block with normal defect groups, i.e., $\text{df}(B) = \{D\}$. Then every indecomposable linear source B -lattice L with maximal vertex is rigid.*

Proof. Let L be an indecomposable linear source B -lattice with vertex D , then $L_{\mathbb{F}} = \sum \varphi_i$, where $\varphi_i \in \text{IBr}(B)$. In particular following the proof of Theorem 4.2.3 implies that φ_i are irreducible $\mathbb{F}G/D$ -modules. Then thanks to [Bro85, (3.6)] we can conclude that φ_i are trivial source $\mathbb{F}G$ -modules with vertex set $\{D\}$. $\square$

4.2.4 Linearly stable Brauer correspondence

Let $B = \mathcal{O}G \cdot e_B$ be an $\mathcal{O}G$ -block with defect groups $\text{df}(B) = {}^G D$, let H be a subgroup of G containing $N_G(D)$, and let $b = \mathcal{O}H \cdot e_b$ denote the Brauer correspondent of B . Then $e_B \mathcal{O}G e_b$ is a trivial source $B \otimes b^{\text{op}}$ -lattice containing a unique direct summand $\mathcal{M}$ with vertex $\mathbf{v}(\mathcal{M}) = {}^{G \times H} \Delta(D)$, where $\Delta(D) = \{(g, g) \in G \times H \mid g \in D\}$. We say that the Brauer correspondence (B, b) is *linearly stable*, if

$$\mathcal{M} \otimes_C L = \mathbf{g}(L) + \text{projective} \quad (4.2.12)$$

for every indecomposable linear source b -lattice L of maximal vertex.

Remark 4.2.11. If $\text{df}(B) = {}^G D$ has the property that $D \cap {}^g D = \mathbf{triv}$ for all $g \in G \setminus H$, then, by Green correspondence (see [Ben98, Theorem 3.12.2]), the Brauer correspondence (B, b) is linearly stable.

Remark 4.2.12. Let B be an $\mathcal{O}G$ -block with cyclic defect groups $\text{df}(B) = {}^G D$ such that $O_p(G) \neq 1$. Put $N = N_G(D)$, and let b denote the Brauer correspondent of B in $\mathcal{O}N$. Then either $\mathcal{M}$ or $\tilde{\Omega}(\mathcal{M})$ yields a Morita equivalence between B and b , where $\tilde{\Omega}(-)$ denotes the Heller operator in the category of $B \otimes b^{\text{op}}$ -lattices (cf. [Rou98, §10.4.2]). In the first case one concludes that $\mathcal{M} \otimes_b L \simeq \mathbf{g}(L)$ for every indecomposable linear source b -lattice L of maximal vertex. Hence in this case (B, b) is linearly stable.

In the second case one has $\tilde{\Omega}(\mathcal{M}) \otimes_b \Omega_b(L) \simeq \mathbf{g}(\Omega_b(L)) \simeq \Omega_B(\mathbf{g}(L))$ for every indecomposable linear source b -lattice L of maximal vertex. As $\tilde{\Omega}(\mathcal{M})$ is a projective right b -lattice, this yields a short exact sequence

$$0 \longrightarrow \Omega_B(\mathbf{g}(L)) \longrightarrow \mathcal{M} \otimes_b P_L \longrightarrow \mathcal{M} \otimes_b L \longrightarrow 0, \quad (4.2.13)$$

where $P_L \rightarrow L$ denotes the minimal projective cover of the indecomposable linear source b -lattice L . Hence $\mathcal{M} \otimes_b L \simeq \mathbf{g}(L) \oplus Q$ for some projective B -lattice Q , and (B, b) is linearly stable in this case as well.

Lemma 4.2.13. *Let G be a finite group, let B be an $\mathcal{O}G$ -block with defect groups $\text{df}(B) = {}^G D$, and let H be a subgroup of G containing $N_G(D)$ such*

that (B, b) is linearly stable, where b is the $\mathcal{O}H$ -block in Brauer correspondence to B . Then, if the indecomposable linear source b -lattice L of maximal vertex is rigid, then $\mathbf{g}(L)$ is also rigid.

Proof. For any linear source b -lattice L , $L_{\mathbb{F}}$ is a trivial source $b_{\mathbb{F}}$ -module. Thus, if L is a rigid indecomposable b -lattice with $\mathbf{v}(L) = {}^G D$, then one has $L_{\mathbb{F}} = Y_1 \oplus \cdots \oplus Y_n$, where Y_i are indecomposable trivial source $b_{\mathbb{F}}$ -modules satisfying $\mathbf{v}(Y_i) = {}^G D$. By hypothesis, there exists a projective B -lattice Q and projective $B_{\mathbb{F}}$ -modules P_i , $1 \leq i \leq n$, such that $\mathcal{M} \otimes_b L \simeq \mathbf{g}(L) \oplus Q$ and $\mathcal{M} \otimes_b Y_i \simeq \mathbf{g}(Y_i) \oplus P_i$. In particular,

$$\mathbf{g}(L)_{\mathbb{F}} \oplus Q_{\mathbb{F}} \simeq \bigoplus_{1 \leq i \leq n} \mathbf{g}(Y_i) \oplus \bigoplus_{1 \leq i \leq n} (P_i)_{\mathbb{F}}. \quad (4.2.14)$$

Thus from the Krull-Schmidt theorem and Fact 4.2.8 one concludes that $\mathbf{g}(L)_{\mathbb{F}}$ is isomorphic to $\bigoplus_{1 \leq i \leq n} \mathbf{g}(Y_i)$, and hence $\mathbf{g}(L)$ is rigid. $\square$

Proposition 4.2.14. *Let G be a finite group and let $B = \mathcal{O}G \cdot e_B$ be an $\mathcal{O}G$ -block with cyclic defect groups $\text{df}(B) = {}^G D$. Then every linear source B -lattice L of maximal vertex is rigid.*

Proof. Put $N = N_G(D)$ and $N_1 = N_G(D_1)$, where $D_1 \subseteq D$, $|D_1| = p$. Let b be the $\mathcal{O}N$ -block in Brauer correspondence to B , let b_1 be the $\mathcal{O}N_1$ -block in Brauer correspondence to B , let L'' denote the $\mathcal{O}N$ -Green correspondent of L , and let L' denote the $\mathcal{O}N_1$ -Green correspondent of L . By Proposition 4.2.10, L'' is a rigid linear source b -lattice of maximal vertex. In Remark 4.2.12 we have seen that the Brauer correspondence (b_1, b) is linearly stable. Hence Lemma 4.2.13 implies that $L' \simeq \mathbf{g}(L'')$ is a rigid linear source b_1 -lattice of maximal vertex. Moreover, for $g \in G \setminus N_1$ one has that $D \cap {}^g D = \mathbf{triv}$. Hence the Brauer correspondence (B, b_1) is linearly stable (cf. Remark 4.2.11), and again, by Lemma 4.2.13, one concludes that $L = \mathbf{g}(L')$ is rigid. $\square$

4.3 The Alperin-McKay conjecture

The main goal of this section is the proof of Theorem B, as proposed in § 4.1. In 1972 in [McK72], J. McKay conjectured that for a finite simple group G the number of irreducible characters of odd degree is equal to the number of irreducible characters of odd degree of the normalizer of a Sylow 2-subgroup of G . In [Isa73] M.I. Isaacs proved the conjecture for all groups of odd order. In [Alp76] J. Alperin generalized the statement of the conjecture to the now known as the *McKay conjecture* considering any finite group G and any prime p .

Conjecture (McKay conjecture). Let G be a finite group, p a prime and P a Sylow p -subgroup of G , then

$$|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(N_G(P))|.$$

Moreover, he stated a stronger refinement, considering the decomposition in blocks, known as the *Alperin-McKay conjecture*.

Conjecture (Alperin-McKay conjecture). Let G be a finite group, p a prime and B a p -block of G with defect group D ; then

$$|\text{Irr}_0(B)| = |\text{Irr}_0(b)|,$$

where b is the Brauer correspondent of B in $N_G(D)$.

These local-global conjectures have been deeply investigated and generalized (cf. [Mal17] for details) in different directions with the purpose to find a theory explaining the relation between the representation theory of a finite group G and the representation theory of its local subgroups. In this framework I. M. Isaacs and G. Navarro in [IN02] presented Conjecture A and Conjecture B, which add to the McKay and the Alperin-McKay conjecture informations about the p' -part of the degree of the irreducible height zero characters.

Let k be an integer not divisible by p and let $M_k(G)$ be the number of irreducible p' -characters having degree congruent modulo p to $\pm k$ and let $M_k(B)$ be the number of irreducible height zero characters of B for which the p' -part of the degree is congruent modulo p to $\pm k$.

Conjecture (Conjecture A). Let G be a finite group and let N be the normalizer of a Sylow p -subgroup of G . Then

$$M_k(G) = M_k(N),$$

for every integer k not divisible by p .

Conjecture (Conjecture B). Let B be a p -block of an arbitrary finite group G and suppose that b is the Brauer correspondent of B with respect to some defect group D . Then for each integer k not divisible by p , we have

$$M_{ck}(B) = M_k(b),$$

where $c = |G : N_G(D)|_{p'}$.

It is clear that Conjecture B implies Conjecture A and they are respectively a refinement of the Alperin-McKay and the McKay conjecture.

4.3.1 Proof of Theorem B

A crucial role in Conjecture B is played by the constant $c = |N_G(D) : G|_{p'}$. The following results show how this constant appears in Theorem B.

Proposition 4.3.1. *Let M be an indecomposable $\mathcal{O}G$ -lattice and let D be a vertex of M . Let P be a Sylow p -subgroup of G containing D , then the rank of M is divisible by the index $|P : D|$.*

Proof. See [CR90, Theorem 19.26] $\square$

Proposition 4.3.2. *Let B be an $\mathcal{O}G$ -block and let D be a defect group of B . Let M be an indecomposable linear source B -lattice with maximal vertex. Then $\text{rk}(M)_p = p^{n-d}$, where $d = |D|$ and n is the order of a Sylow p -subgroup of G .*

Proof. Let $N = N_G(D)$ and $S \in \text{Syl}_p(N)$. Let $m = |S|$ and $\mathbf{f}(M)$ the Green correspondent of L in the $\mathcal{O}N$ -block b . Theorem 4.2.5 implies $\text{rk}(\mathbf{f}(M))_p = p^{m-d}$. By Green correspondence, $\text{ind}_N^G(\mathbf{f}(M)) = M + \sum_i L_i$, where $L_i \in \text{IL}_{\mathcal{O}}^{\prec}(B)$. Then,

$$\frac{|G|}{|N|} \text{rk}(\mathbf{f}(M)) = \text{rk}(M + \sum_i L_i) \quad (4.3.1)$$

and in particular,

$$p^{n-m} \cdot p^{m-d} = \text{rk}(M + \sum_i L_i)_p. \quad (4.3.2)$$

By Proposition 4.3.1, p^{n-d+1} divides $\text{rk}(L_i)$. Then $\text{rk}(M)_p = p^{n-d}$. $\square$

Proposition 4.3.3. *Let G be a finite group, B an $\mathcal{O}G$ -block with defect groups $\text{df}(B) = {}^G D$ and b the $\mathcal{O}N_G(D)$ -block in Brauer correspondence to B . Let L be an indecomposable B -lattice with maximal vertex and $\mathbf{f}(L)$ its Green correspondent in b . Then $\text{rk}(L)_{p'}$ and $c \cdot \text{rk}(\mathbf{f}(L))_{p'}$ are congruent modulo p , where $\text{rk}(L)_{p'}$ and $\text{rk}(\mathbf{f}(L))_{p'}$ denote the p' -part of the rank of L and $\mathbf{f}(L)$ respectively and $c = |G : N_G(D)|_{p'}$.*

Proof. Let $\tau : \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ be the ring homomorphism given by the congruence modulo p . Let $\text{rk}(L)_{p'} = b_L$ and $\text{rk}(\mathbf{f}(L))_{p'} = b_{\mathbf{f}(L)}$. By the Green correspondence

$$\text{ind}_{N_G(D)}^G(\mathbf{f}(L)) = L + \sum_{L' \in \text{IL}_{\mathcal{O}}^{\prec}(B)} a_{L'} \cdot L', \quad (4.3.3)$$

where $a_{L'} \in \mathbb{N}_0$. Then,

$$\begin{aligned} \text{rk}(\text{ind}_{N_G(D)}^G(\mathbf{f}(L))) &= \text{rk}(L) + \sum_{L' \in \text{IL}_{\mathcal{O}}^{\prec}(B)} a_{L'} \cdot \text{rk}(L') \\ &= (b_L + \sum_{L' \in \text{IL}_{\mathcal{O}}^{\prec}(B)} a_{L'} \cdot b_{L'} \cdot p^{d-d_{L'}}) \cdot p^{n-d}, \end{aligned} \quad (4.3.4)$$

where $b_{L'} = \text{rk}(L')_{p'}$, p^n is the order of a Sylow p -subgroup of G , $p^d = |D|$ and $p^{n-d_{L'}} > p^{n-d}$ is equal to $\text{rk}(L')_p$.

Hence, $\text{rk}(\text{ind}_{N_G(D)}^G(\mathbf{f}(L)))_{p'} = b_L + \sum_{L' \in \text{IL}_{\mathcal{O}}^{\prec}(B)} a_{L'} \cdot b_{L'} \cdot p^{d-d_{L'}}$ and

$$\tau(\text{rk}(\text{ind}_{N_G(D)}^G(\mathbf{f}(L)))_{p'}) = \tau(b_L) = \tau(\text{rk}(L)_{p'}). \quad (4.3.5)$$

On the other hand, $\text{rk}(\text{ind}_{N_G(D)}^G(\mathbf{f}(L)))_{p'} = \frac{|G|_{p'}}{|N_G(D)|_{p'}} \cdot b_{\mathbf{f}(L)}$. Thus,

$$\tau(\text{rk}(\text{ind}_{N_G(D)}^G(\mathbf{f}(L)))_{p'}) = c \cdot \tau(b_{\mathbf{f}(L)}), \quad (4.3.6)$$

which concludes the proof. $\square$

Now a proof of Theorem B can be provided.

Theorem 4.3.4 (Theorem B). *Let B be an $\mathcal{O}G$ -block. If Conjecture 1 and Conjecture 2 hold for B , then Conjecture B in [IN02] holds for the $\mathcal{O}G$ -block B and its Brauer correspondent b .*

Proof. Theorem 4.2.5 and the Green correspondence imply that there exists a bijection between $\text{IL}_{\mathcal{O}}^{\text{mx}}(B)$ and $\text{Irr}_0(b)$.

The map σ_B is a π_B -section, then $\langle\langle\sigma_B(L), \sigma_B(L)\rangle\rangle_B = 1$ for every indecomposable linear source B -lattice L with maximal vertex. Moreover, thanks to Conjecture 1, $\text{im}(\sigma_B)$ is totally isotropic with respect to $(-, -)_B$ then $\langle\sigma_B(L), \sigma_B(L)\rangle_B = 1$, i.e.,

$$\tilde{\sigma}_B(L) \in \pm \text{Irr}_{\mathbb{K}}(B). \quad (4.3.7)$$

Let L_1, L_2 be two non-isomorphic indecomposable linear source B -lattices with maximal vertex, then $\langle\langle\sigma_B(L_1), \sigma_B(L_2)\rangle\rangle_B = 0$ and, since $\text{im}(\sigma_B)$ is totally isotropic, this implies $\tilde{\sigma}_B(L_1) \neq \tilde{\sigma}_B(L_2)$.

Moreover,

$$\begin{aligned} \tilde{\sigma}_B(L)(1) &= \varepsilon_L \cdot \text{rk}(L) + \sum_{L' \in \text{IL}_{\mathcal{O}}^{\prec}(B)} a_{L'} \cdot \text{rk}(L') \\ &= (\varepsilon_L \cdot b_L + \sum_{L' \in \text{IL}_{\mathcal{O}}^{\prec}(B)} a_{L'} \cdot b_{L'} \cdot p^{d-d_{L'}}) \cdot p^{n-d}, \end{aligned} \quad (4.3.8)$$

where $b_L = \text{rk}(L)_{p'}$, $b_{L'} = \text{rk}(L')_{p'}$, p^n is the order of a Sylow p -subgroup of G , $p^d = |D|$ and $p^{n-d_{L'}} > p^{n-d}$ is equal to $\text{rk}(L')_p$. Hence, $\tilde{\sigma}_B(L)$ has height zero. Then the Alperin-McKay conjecture holds and the 1-1 correspondence established between the irreducible height zero characters of B and b is given by $\varepsilon_L \cdot \sigma_B(L)_{\mathbb{K}}$ and $\mathbf{f}(L)_{\mathbb{K}}$.

As before, let $\tau: \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ be the ring homomorphism given by the congruence modulo p , then to prove Conjecture B it is enough to consider $\tau(\text{rk}(\mathbf{f}(L))_{p'})$ and $\tau(\text{rk}(\sigma_B(L))_{p'})$ for L indecomposable linear source B -lattice with maximal vertex. From (4.3.8) follows that

$$\tau(\text{rk}(\sigma_B(L))_{p'}) = \tau(\varepsilon_L \cdot \text{rk}(L)_{p'}). \quad (4.3.9)$$

Hence Proposition 4.3.3 implies the thesis. $\square$

Corollary 4.3.5. *Let B be an $\mathcal{O}G$ -block. If Conjecture 1 and the Alperin-McKay conjecture hold for B , then Conjecture B (cf. [IN02]) holds for B .*

Proof. It is clear that Conjecture 1 and the Alperin-McKay conjecture imply Conjecture 2, then Theorem 4.3.4 imply the thesis. $\square$

4.4 Splendid derived equivalence

The goal of this section is the proof of Theorem C. In the first part some possible equivalences between blocks are presented for convenience of the reader. For further details see for example [Rou98].

Let $B\text{-}\mathbf{mod}$ denote the stable category of B , whose objects are finitely generated B -modules. Let H be a finite group and let $\bar{B}$ be an $\mathcal{O}H$ -block. Let M be an $(B \otimes \bar{B}^{op})$ -module, projective as a B -module and also as a $\bar{B}^{op}$ -module, where $\bar{B}^{op}$ is the opposite of $\bar{B}$.

Definition 4.4.1. The module M induces a *stable equivalence of Morita type* between B and $\bar{B}$ if

$$M \otimes_{\bar{B}} M^* \simeq B \oplus \text{projective modules.} \quad (4.4.1)$$

In the case that M is a module inducing a stable equivalence between B and $\bar{B}$, then the functors $M \otimes_{\bar{B}} -$ and $M^* \otimes_B -$ induce inverse equivalences of triangulated categories between the stable categories $\bar{B}\text{-}\mathbf{mod}$ and $B\text{-}\mathbf{mod}$.

Let $C^\bullet$ be a bounded complex of $(B \otimes \bar{B}^{op})$ -modules, all of which are projective as B -modules and as $\bar{B}^{op}$ -modules.

Definition 4.4.2. The complex $C^\bullet$ induces a *Rickard equivalence* between B and $\bar{B}$ if

$$\begin{aligned} C^\bullet \otimes_{\bar{B}} (C^\bullet)^* &\simeq B \oplus \text{complex homotopy equivalent to } 0 \\ (C^\bullet)^* \otimes_B C^\bullet &\simeq \bar{B} \oplus \text{complex homotopy equivalent to } 0 \end{aligned} \quad (4.4.2)$$

In this case the complex $C^\bullet$ is called *Rickard complex*.

Assume that the blocks B and $\bar{B}$ have a common defect group D , when the modules occurring in the complex $C^\bullet$ are trivial source lattices induced from $\Delta(D) = \{(g, g^{-1}) | g \in D\} \subset G \times H^{op}$, then the complex C is called *splendid* and the equivalence of derived categories induced by $C^\bullet$ is a *splendid Rickard equivalence* or, shortly, a *splendid equivalence*. The suggestive name *splendid* is an acronym of *split-**e**ndomorphism two-sided tilting complex of direct summands of permutation modules Induced from **D**iagonal subgroups*.

In particular, if B has cyclic defect and b its Brauer correspondent, the following result holds.

Theorem 4.4.3. *The blocks B and b are splendidly Rickard equivalent and, if $O_p(G) \neq 1$, then B and b are Morita equivalent too.*

Proof. See [Rou98, Theorem 10.1]. □

For finite simple groups with blocks with cyclic defect group D of order p , the following local result gives an explicit description of a complex $C^\bullet$ inducing the splendid Rickard equivalence between B and b .

Theorem 4.4.4. *Let G be a finite group and $B = \mathcal{O}G \cdot e_B$ an $\mathcal{O}G$ -block with a non-normal cyclic defect group D . Let Q be the subgroup of D containing $R = O_p(G)$ as a subgroup of index p . Let $H = N_G(Q)$ and $\bar{B} = \mathcal{O}H \cdot e_{\bar{B}}$ the block of H corresponding to B . Let M be an indecomposable direct summand of the $(B \otimes \bar{B}^{op})$ -module $e_B \mathcal{O}G e_{\bar{B}}$ with vertex $\Delta(D)$. Then there is a direct summand N of the $(B \otimes \bar{B}^{op})$ -module $B \otimes_{\mathcal{O}R} \bar{B}^{op}$ such that the complex*

$$C^\bullet = 0 \rightarrow N \xrightarrow{m} M \rightarrow 0 \quad (4.4.3)$$

induces a splendid equivalence between B and $\bar{B}$. In this case m is the restriction of the multiplication map $B \otimes_{\mathcal{O}R} \bar{B}^{op} \rightarrow e_B \mathcal{O}G e_{\bar{B}}$. If $R \neq 1$, then $C^\bullet$ has homology only in one degree.

Proof. See [Rou98, Theorem 10.3]. □

4.4.1 Proof of Theorem C

Theorem 4.4.5. *Let G be a finite group, and let B be an $\mathcal{O}G$ -block with defect groups $\text{df}(B) = {}^G D$. Let b be the $\mathcal{O}N_G(D)$ -block in Brauer correspondence to B , and suppose that B and b are splendid derived equivalent. Then Conjecture 1 holds for B , and thus Conjecture B of [IN02] holds for B .*

Proof. The blocks B and b are splendid derived equivalent, then the Alperin-McKay conjecture holds for them. Let $C^\bullet$ be a “splendid” complex inducing the equivalence between the blocks B and b

$$C^\bullet = 0 \rightarrow C^{-r} \rightarrow \cdots \rightarrow C^{-2} \rightarrow C^{-1} \rightarrow C^0 \rightarrow 0. \quad (4.4.4)$$

For every $i \in \{0, \dots, r\}$, the $\mathcal{O}(B \otimes b^{op})$ -lattice C^i is a trivial source lattice and each indecomposable component of C^i , for $i \neq 0$, has vertex strictly contained in $\Delta(D)$ and the indecomposable components of C^0 have vertex $\Delta(D)$. Let

$$\gamma^j = C^j \otimes_b -: \text{IL}_{\mathcal{O}}^{\text{mx}}(b) \rightarrow \mathbf{L}_{\mathcal{O}}(B). \quad (4.4.5)$$

For a lattice $M \in \text{IL}_{\mathcal{O}}^{\text{mx}}(b)$, let

$$h^j(M) = H^j(\gamma^\bullet(M[0])) = H^j(C^\bullet \otimes_b M[0]), \quad (4.4.6)$$

where $H^j(-)$ denotes the j^{th} group of cohomology and $M[0]$ is the module M seen as a complex concentrated in 0.

In particular, $H^j(C^\bullet \otimes_b M[0])$ is torsion free as $\mathcal{O}$ -module and $\gamma^\bullet(M[0])_{\mathbb{K}}$ is (directly) indecomposable in the bounded derived category of $B_{\mathbb{K}}$ -modules. So there exists $k \in \mathbb{Z}$ such that

$$\gamma^\bullet(M[0])_{\mathbb{K}} = (C^\bullet \otimes_b M[0])_{\mathbb{K}} \simeq \bigoplus_{j \in \mathbb{Z}} h^j(M)[j]_{\mathbb{K}} = h^k(M)[k]_{\mathbb{K}}. \quad (4.4.7)$$

This implies that $h^k(M)_{\mathbb{K}}$ is irreducible and $h^m(M)_{\mathbb{K}} = 0$ for every $m \neq k$. Thus, for every $m \neq k$, $H^m(C^\bullet \otimes_b M)$ has torsion as $\mathcal{O}$ -module and this implies

$$H^k(C^\bullet \otimes_b M) \neq 0 \text{ and } H^m(C^\bullet \otimes_b M) = 0 \ \forall m \neq k. \quad (4.4.8)$$

For an indecomposable linear source B -lattice with maximal vertex L , we define

$$\sigma_B(L) = \sum_{n \in \mathbb{Z}} (-1)^n \gamma^n(\mathbf{f}(L)). \quad (4.4.9)$$

Thus σ_B is a section of the canonical projection $\pi_B : \mathbf{L}_{\mathcal{O}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B)$, in fact $\sum_{n \in \mathbb{Z}} (-1)^n \gamma^n(\mathbf{f}(L))$ is an algebraic sum of indecomposable linear source B -lattices where only L has maximal vertex (cf. [Rou98]). Moreover,

$$\sigma_B(L)_{\mathbb{K}} = (-1)^k h^k(\mathbf{f}(L))_{\mathbb{K}} \in \pm \text{Irr}(B). \quad (4.4.10)$$

Let $L, M \in \text{IL}_{\mathcal{O}}^{\text{mx}}(B)$, such that $[L] \neq [M]$. Then $\pi_B(\sigma(L)) \neq \pi_B(\sigma(M))$. Let $k, j \in \mathbb{Z}$ be such that $h^k(\mathbf{f}(L)) \neq 0$ and $h^j(\mathbf{f}(M)) \neq 0$, then $h^k(\mathbf{f}(L))_{\mathbb{K}} \neq h^j(\mathbf{f}(M))_{\mathbb{K}}$. Thus $\text{im}(\sigma_B)$ is totally isotropic and Conjecture 1 has been proved. Corollary 4.3.5 implies [IN02, Conjecture B]. $\square$

The previous theorem is a first non trivial case in which Conjecture 1 is positively verified. Theorem 4.4.3 ensures that blocks with cyclic defect groups satisfy the hypothesis of Theorem C, then Conjecture B holds, as already proved by I. M. Isaacs and G. Navarro (see [IN02, Theorem 2.1]).

Something more can be said considering the splendid form of Broué conjecture.

Conjecture (Broué splendid conjecture). Let B be a block of a finite group G with abelian defect group D and b its Brauer corresponding block of $N_G(D)$. Then the bounded derived module categories of B and of b are splendid equivalent.

A positive answer to the Broué splendid conjecture will imply not only the classical Alperin -McKay conjecture, but also Conjecture B.

4.5 McKay bijection and canonical section

In the study of local-global conjectures a particular focus is given to the natural bijections between characters of a group G and its local subgroups. In this context a *McKay bijection* is a (canonical) bijection between the sets $\text{Irr}_{p'}(G)$ and $\text{Irr}_{p'}(N_G(P))$, where $N_G(P)$ is the normalizer of a Sylow p -subgroup of G . Of course, if such a bijection exists, then the McKay conjecture is true in a very strong form; it is not only a statement on the equality of the cardinality of two sets of characters but reveals also a deeper

link between the representations of the groups G and $N_G(P)$.

Given a finite group G and a prime p , we will say that (G, p) has a *McKay bijection given by restriction* if for every $\chi \in \text{Irr}_{p'}(G)$, there exists a unique $\lambda \in \text{Irr}_{p'}(N_G(P))$ such that $\text{res}_{N_G(P)}^G(\chi) = \lambda + \Delta$ and Δ is the sum of characters in $\text{Irr}(N_G(P)) \setminus \text{Irr}_{p'}(N_G(P))$; and for each $\lambda \in \text{Irr}_{p'}(N_G(P))$ there exists a unique $\chi \in \text{Irr}_{p'}(G)$ such that $\text{res}_{N_G(P)}^G(\chi) = \lambda + \Delta$.

In this context there exists a section $\sigma: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}(G)$ satisfying Conjecture 1. It is constructed considering the canonical section

$$\mathfrak{l}_G: \mathbf{R}_{\mathbb{K}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}(G) \quad (4.5.1)$$

defined in Chapter 2 and introduced previously by R. Boltje (cf. [Bol98a]).

Theorem 4.5.1 (Theorem D). *Let G be a finite group, P a Sylow p -subgroup of G and $N = N_G(P)$ its normalizer in G . If (G, p) has a McKay bijection given by restriction, then there exists a section $\sigma: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) \rightarrow \mathbf{L}_{\mathcal{O}}(G)$ satisfying Conjecture 1, and thus Conjecture A of Isaacs and Navarro is verified (see [IN02, Conjecture A]).*

Proof. For $[L] \in \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(G)$, let $\mathbf{f}([L])$ be its Green correspondent in $\mathbf{IL}_{\mathcal{O}}^{\text{mx}}(N)$. Then there exist $\lambda \in \text{Irr}_{p'}(N)$ such that $\mathbf{f}([L])_{\mathbb{K}} = \lambda$ and $\chi \in \text{Irr}_{p'}(G)$ such that $\text{res}_{N_G(P)}^G(\chi) = \lambda + \Delta$, where Δ is the sum of characters whose degree is a multiple of p .

Let us define the map

$$\begin{aligned} \sigma: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(G) &\rightarrow \mathbf{L}_{\mathcal{O}}(G) \\ [L] &\mapsto \mathfrak{l}_G(\chi) \end{aligned} \quad (4.5.2)$$

By construction $\sigma([L]) = \sum L' \in \mathbf{IL}_{\mathcal{O}}(G)_{a_{L'}} L'$, for $a_{L'} \in \mathbb{Z}$. Let $\text{supp}(\sigma(L)) = \{L \in \mathbf{IL}_{\mathcal{O}}(G) | a_L \neq 0\}$. Since $\mathfrak{l}_G$ is a section of $-\mathbb{K}$,

$$\sigma([L])_{\mathbb{K}} = \mathfrak{l}_G(\chi)_{\mathbb{K}} = \chi \in \text{Irr}_{p'}(G). \quad (4.5.3)$$

Moreover,

$$\begin{aligned} \text{res}_{N_G(D)}^G(\sigma([L])) &= \text{res}_{N_G(D)}^G(\mathfrak{l}_G(\chi)) = \mathfrak{l}_{N_G(P)}(\text{res}_{N_G(P)}^G(\chi)) \\ &= \mathfrak{l}_{N_G(P)}(\lambda + \Delta) = \mathfrak{l}_{N_G(P)}(\lambda) + \mathfrak{l}_{N_G(P)}(\Delta). \end{aligned} \quad (4.5.4)$$

Proposition 2.8.3 implies $\mathfrak{l}_{N_G(D)}(\Delta) \in \mathbf{L}_{\mathcal{O}}^{\prec}(N)$ and Remark 2.8.6 implies that $\mathfrak{l}_{N_G(D)}(\lambda) = \mathbf{f}(L)$. So,

$$\text{res}_{N_G(D)}^G(\sigma([L])) = \mathbf{f}([L]). \quad (4.5.5)$$

Then $[L] \in \text{supp}(\sigma(L))$.

Let us suppose that there exists $[M] \neq [L] \in \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(G) \in \text{supp}(\sigma(L))$. Then by Green correspondence, $\mathbf{f}([M]) \in \text{supp}(\text{res}_{N_G(D)}(\sigma([L])))$ which contradicts (4.5.5). Thus, $(\sigma([L]), \sigma([L])) = 0$.

Moreover, if $[L] \neq [M] \in \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(G)$, $\sigma([L])_{\mathbb{K}} \neq \sigma([M])_{\mathbb{K}}$ and $\mathbf{f}([L]) \neq \mathbf{f}([M])$ and this implies $(\sigma([L]), \sigma([M])) = 0$. Therefore, the map σ is a section with $\text{im}(\sigma)$ totally isotropic with respect to $(-, -)$. $\square$

As already said, Theorem D can be applied in the case of the symmetric groups S_{2^n} with $p = 2$ (cf. [Gia17, Theorem 1.1]) and in the case of p -solvable groups with self-normalising Sylow p -subgroups (cf. [Isa73]). It is clear that Conjecture A is not adding any information in the case of the prime 2 and it is well known that it has been positively verified for p -solvable groups (see [IN02]). However it is interesting to see that the section conjectured exists.

4.6 Further potential developments

At this point a natural question is whether it is possible to find such a link between linear source lattices and other refinements of the Alperin-McKay conjecture.

In the last forty years the Alperin-McKay conjecture has been refined in many different directions. One of those is for example given by Conjecture D of [IN02] and Conjecture B of [Nav04].

Conjecture (Conjecture D). Let B be a p -block for a finite group G and suppose that b is the Brauer correspondent of B with respect to some defect group. Let μ be an automorphism of the cyclotomic field $\mathbb{Q}_{|G|}$ and assume that μ has p -power order and that it fixes all p' -roots of unity in $\mathbb{Q}_{|G|}$. Then μ fixes equal numbers of height zero characters in $\text{Irr}_{\mathbb{K}}(B)$ and $\text{Irr}_{\mathbb{K}}(b)$.

In particular, following the idea of Conjecture D one can ask if the following holds.

Conjecture 3. Let G be a finite group, B be an $\mathcal{O}G$ -block and b its Brauer correspondent with respect to some defect group. Let

$$\mathcal{G} = \text{Gal}(\mathbb{Q}[\chi(g) \mid \chi \in \text{Irr}(G), g \in G] / \mathbb{Q}[\xi \mid \xi \text{ a } p'\text{-root of unity}]). \quad (4.6.1)$$

Then $\mu \in \mathcal{G}$ fixes the same number of height zero characters in B as it does in b .

In this case the idea is to “translate” these conjectures in terms of species of the linear source lattices. To do that, a more complicated setting is necessary.

Let $(\mathbb{K}, \mathbb{K}^{\text{un}}, \mathcal{O}, \mathcal{O}^{\text{un}}, \mathbb{F})$ be a split p -modular bi-system for G (cf. § 4.6.3). Then every indecomposable linear source B -lattice L defines a field extension $\mathbb{K}(L)/\mathbb{K}^{\text{un}}$ (see Remark 4.6.7), where $\mathbb{K}^{\text{un}} \subseteq \mathbb{K}(L) \subseteq \mathbb{K}$. If σ_B is a π_B -section, then one has

$$\sigma_B([L]) = [L] + \sum_{[L'] \in \text{IL}_{\mathcal{O}}^{\prec}(B)} a_{[L']} \cdot [L'], \quad (4.6.2)$$

for integers $a_{L'} \in \mathbb{Z}$ and $\text{IL}_{\mathcal{O}}^{\prec}(B) = \text{IL}_{\mathcal{O}}(B) \setminus \text{IL}_{\mathcal{O}}^{\text{mx}}(B)$, for every indecomposable linear source B -lattice L with maximal vertex. For short we put

$$\sup_{\sigma_B}^{\prec}(L) = \{ [L'] \in \text{IL}_{\mathcal{O}}^{\prec}(B) \mid a_{L'} \neq 0 \}. \quad (4.6.3)$$

The section $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ is said to be *rational*, if $\mathbb{K}(L') \subsetneq \mathbb{K}(L)$ or $\mathbb{K}(L) = \mathbb{K}(L') = \mathbb{K}^{\text{un}}$ for every $[L'] \in \text{sup}_{\sigma_B}^{\prec}(L)$. Conjecture 1 may be strengthened in the following way.

Conjecture 1'. For an $\mathcal{O}G$ -block B of a finite group G there exists a rational π_B -section $\sigma_B: \mathbf{L}_{\mathcal{O}}^{\text{mx}}(B) \rightarrow \mathbf{L}_{\mathcal{O}}(B)$ such that $\text{im}(\sigma_B)$ is totally isotropic with respect to $(-, -)_B$.

It turns out that Conjecture 1' and Conjecture 2 imply Conjecture 3 under the following assumptions, where, as usual, D is a defect group of the $\mathcal{O}G$ -block B .

- (A1) Given $[L] \in \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(G)$ and its Green correspondent $\mathbf{f}(L) \in \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(N_G(D))$, the field extensions $\mathbb{K}(L)/\mathbb{K}^{\text{un}}$ and $\mathbb{K}(\mathbf{f}(L))/\mathbb{K}^{\text{un}}$ are the same
- (A2) Given $[L] \in \mathbf{IL}_{\mathcal{O}}^{\text{mx}}(G)$ the field extension $\mathbb{K}(L)/\mathbb{K}^{\text{un}}$ is equal to the extension $\mathbb{K}^{\text{un}}(s(L) \mid s = (h, \langle h_p \rangle))/\mathbb{K}^{\text{un}}$

In order to say something more in this direction, further investigations of the set of species of the linear source lattices are necessary. By the way, at the moment it is interesting to see how to construct the (hopefully) right setting. For our purpose Conjecture 3 can be reformulated as follows.

Conjecture 3'. Let G be a finite group, B be an $\mathcal{O}G$ -block and b its Brauer correspondent. Let

$$\mathcal{G} = \text{Gal}(\mathbb{K}[\chi(g) \mid \chi \in \text{Irr}(G), g \in G]/\mathbb{K}^{\text{un}}). \quad (4.6.4)$$

Then $\mu \in \mathcal{G}$ fixes the same number of height zero characters in B as it does in b .

4.6.1 Unramified complete discrete valuation domains

Let $p > 0$ be a prime number. A *complete discrete valuation domain* $\mathcal{O}$ of characteristic 0 and with residue field $\mathbb{F} = \text{res}(\mathcal{O}) = \mathcal{O}/\mathfrak{m}$ of characteristic p will be called a *0/p-c.d.v.d.* Such a domain $\mathcal{O}$ is said to be *unramified*, if $p\mathcal{O}$ is the unique maximal ideal of $\mathcal{O}$.

4.6.2 Unramified pseudo-split p -modular systems

Let G be a finite group, and let p be a prime number. Let $\mathcal{O}^{\text{un}}$ be an unramified $0/p$ -c.d.v.d. with residue field $\mathbb{F} = \text{res}(\mathcal{O}^{\text{un}})$ and quotient field $\mathbb{K}^{\text{un}} = \text{quot}(\mathcal{O}^{\text{un}})$.

Definition 4.6.1. A triple $(\mathbb{K}^{\text{un}}, \mathcal{O}^{\text{un}}, \mathbb{F})$ is said to be an *unramified pseudo-split p -modular system* for G , if

(PS₁) for every subgroup U of G any Wedderburn component of $\mathbb{F}U/\text{rad}(\mathbb{F}U)$ is isomorphic to a matrix algebra over $\mathbb{F}$, i.e., if

$$\mathbb{F}U/\text{rad}(\mathbb{F}U) \simeq \bigoplus_{1 \leq i \leq s} A_i, \quad (4.6.5)$$

where A_i , $1 \leq i \leq s$, are simple $\mathbb{F}$ -algebras, then there exists a positive integer m_i such that $A_i \simeq \text{Mat}_{m_i \times m_i}(\mathbb{F})$;

(PS₂) for every subgroup U of G any Wedderburn component of $\mathbb{K}^{\text{un}}U$ is isomorphic to a matrix algebra over a finite totally ramified extension of $\mathbb{K}^{\text{un}}$, i.e., if $\mathbb{K}^{\text{un}}U \simeq \bigoplus_{1 \leq j \leq t} B_j$, where B_j , $1 \leq j \leq t$, are simple $\mathbb{K}^{\text{un}}$ -algebras, then there exists a finite totally ramified extension $\mathbb{E}_j/\mathbb{K}^{\text{un}}$ and a positive integer n_j such that $B_j \simeq \text{Mat}_{n_j \times n_j}(\mathbb{E}_j)$.

Let $\mathbb{Q}_p$ denote the field of p -adic numbers and by $\mathbb{Z}_p \subset \mathbb{Q}_p$ the ring of p -adic integers. The following property can be used quite efficiently to show the existence of unramified pseudo-split p -modular systems.

Proposition 4.6.2. *Let $\mathbb{E}$ be a finite extension field of $\mathbb{Q}_p$, and let C be a finite-dimensional simple $\mathbb{E}$ -algebra. Then there exists a finite extension $\mathbb{E}_1/\mathbb{E}$ such that for the maximal unramified subextension $\mathbb{E}_1^{\text{un}}/\mathbb{E}$ of $\mathbb{E}_1/\mathbb{E}$ one has*

$$C \otimes_{\mathbb{E}} \mathbb{E}_1^{\text{un}} \simeq \text{res}_{\mathbb{E}_1^{\text{un}}}^{\mathbb{E}_1}(\text{Mat}_{n \times n}(\mathbb{E}_1)) \quad (4.6.6)$$

for some positive integer n .

Proof. By Wedderburn's theorem, one has $C \simeq \text{Mat}_{k \times k}(D)$ for some positive integer k and some finite-dimensional $\mathbb{E}$ -skew field D . Hence it suffices to prove the claim for D .

Let $\mathbb{E}_0 = Z(D)$ be the centre of D . Then $\mathbb{E}_0$ is a finite field extension of $\mathbb{E}$, and D is a central simple $\mathbb{E}_0$ -algebra. Moreover, there exists a finite unramified extension $\mathbb{E}_1/\mathbb{E}_0$ such that $D \otimes_{\mathbb{E}_0} \mathbb{E}_1 \simeq \text{Mat}_{n \times n}(\mathbb{E}_1)$ (cf. [Rei03, (31.10) Corollary]). Let $\mathbb{E}_0^{\text{un}}/\mathbb{E}$ denote the maximal unramified subextension of $\mathbb{E}_0/\mathbb{E}$.

Let $\mathbb{E}_1^{\text{un}}/\mathbb{E}$ denote the maximal unramified subextension of $\mathbb{E}_1/\mathbb{E}$. In particular, $\mathbb{E}_0^{\text{un}} \subseteq \mathbb{E}_1^{\text{un}}$, and $\mathbb{E}_0^{\text{un}} = \mathbb{E}_0 \cap \mathbb{E}_1^{\text{un}}$. As $f(\mathbb{E}_1 : \mathbb{E}_0^{\text{un}}) = |\mathbb{E}_1^{\text{un}} : \mathbb{E}_0^{\text{un}}|$ and $e(\mathbb{E}_1 : \mathbb{E}_0^{\text{un}}) = |\mathbb{E}_0 : \mathbb{E}_0^{\text{un}}|$, the canonical map $\mathbb{E}_0 \otimes_{\mathbb{E}_0^{\text{un}}} \mathbb{E}_1^{\text{un}} \rightarrow \mathbb{E}_1$ is an isomorphism of $\mathbb{E}_0^{\text{un}}$ -algebras and left/right $(\mathbb{E}_0, \mathbb{E}_1^{\text{un}})$ -bimodules. Hence one has isomorphisms of $\mathbb{E}_1^{\text{un}}$ -algebras

$$D \otimes_{\mathbb{E}_0^{\text{un}}} \mathbb{E}_1^{\text{un}} \simeq \text{res}_{\mathbb{E}_1^{\text{un}}}^{\mathbb{E}_1}(\mathbb{E}_1 \otimes_{\mathbb{E}_0} \mathbb{E}_1) \simeq \text{res}_{\mathbb{E}_1^{\text{un}}}^{\mathbb{E}_1}(\text{Mat}_{n \times n}(\mathbb{E}_1)) \quad (4.6.7)$$

which yields the claim. $\square$

Remark 4.6.3. Let G be a finite group, and let p be a prime number. Then there exists a finite unramified extension $\mathbb{E}/\mathbb{Q}_p$ such that for $\mathcal{O}_{\mathbb{E}} = \text{int}_{\mathbb{Z}_p}(\mathbb{E})$ its residue field $\mathbb{F}_{\mathbb{E}} = \mathcal{O}_{\mathbb{E}}/p\mathcal{O}_{\mathbb{E}}$ satisfies the hypothesis (PS₁).

By Proposition 4.6.2, there exists a finite unramified extension $\mathbb{K}^{\text{un}}/\mathbb{E}$ such that for every subgroup U of G the Wedderburn decomposition

$$\mathbb{K}^{\text{un}}U \simeq \bigoplus_{1 \leq j \leq t} B_j \quad (4.6.8)$$

satisfies $B_j \simeq \text{Mat}_{n_j \times n_j}(\mathbb{E}_j)$ for some finite totally ramified field extension $\mathbb{E}_j/\mathbb{K}^{\text{un}}$ and some positive integer n_j . Thus for

$$\mathcal{O}^{\text{un}} = \text{int}_{\mathbb{K}^{\text{un}}}(\mathbb{Z}_p) \text{ and } \mathbb{F} = \text{res}(\mathcal{O}^{\text{un}}) \quad (4.6.9)$$

the triple $(\mathbb{K}^{\text{un}}, \mathcal{O}^{\text{un}}, \mathbb{F})$ is an unramified pseudo-split p -modular system for the group G .

4.6.3 Split p -modular bi-systems

Let G be a finite group, and let p be a prime number.

Definition 4.6.4. A finite extension $\mathcal{O}/\mathcal{O}^{\text{un}}$ of $0/p$ -c.d.v.d.'s is said to be a *split p -modular bi-system* for G , if for $\mathbb{K} = \text{quot}(\mathcal{O})$, $\mathbb{K}^{\text{un}} = \text{quot}(\mathcal{O}^{\text{un}})$ and $\mathbb{F} = \text{res}(\mathcal{O}^{\text{un}})$ one has the following:

- (sPS₁) $\mathcal{O}^{\text{un}}$ is unramified and $(\mathbb{K}^{\text{un}}, \mathcal{O}^{\text{un}}, \mathbb{F})$ is pseudo-split;
- (sPS₂) $\mathbb{K}/\mathbb{K}^{\text{un}}$ is an abelian totally ramified Galois extension;
- (sPS₃) $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ is a split p -modular system for G .

By definition, a split p -modular bi-system for a finite group G simply consists of an extension $\mathcal{O}/\mathcal{O}^{\text{un}}$ of $0/p$ -c.d.v.d.'s, but also defines the two p -modular systems $(\mathbb{K}^{\text{un}}, \mathcal{O}^{\text{un}}, \mathbb{F})$ and $(\mathbb{K}, \mathcal{O}, \mathbb{F})$.

Remark 4.6.5. Let p be a prime number, and let G be a finite group of exponent $\exp(G) = m \cdot p^k$, $\gcd(m, p) = 1$. By Remark 4.6.3, there exists an unramified pseudo-split p -modular system $(\mathbb{K}^{\text{un}}, \mathcal{O}^{\text{un}}, \mathbb{F})$ for G . Moreover, we may assume that $\mathbb{K}^{\text{un}}$ contains a primitive m^{th} -root of unity. Let $\bar{\mathbb{K}}^{\text{un}}/\mathbb{K}^{\text{un}}$ be an algebraic closure of $\mathbb{K}^{\text{un}}$, and let ξ denote a primitive $(p^k)^{\text{th}}$ -root of unity. It is well known that $\mathbb{K} = \mathbb{K}^{\text{un}}[\xi]$ is a splitting field for G (cf. [Isa06, Corollary 9.15]), and that $\mathbb{K}/\mathbb{K}^{\text{un}}$ is a totally ramified abelian Galois extension. Thus for $\mathcal{O} = \text{int}_{\mathcal{O}^{\text{un}}}(\mathbb{K})$ the extension $\mathcal{O}/\mathcal{O}^{\text{un}}$ of $0/p$ -c.d.v.d.'s is a split p -modular bi-system for G .

The properties of irreducible $\mathbb{K}^{\text{un}}G$ -modules for a split p -modular bi-system $\mathcal{O}/\mathcal{O}^{\text{un}}$ for G can be summarized as follows.

Proposition 4.6.6. *Let $\mathcal{O}/\mathcal{O}^{\text{un}}$ be a split p -modular bi-system for the finite group G , and let M be an irreducible $\mathbb{K}^{\text{un}}G$ -module. Then one has the following.*

- (a) $\mathbb{E}_o = \text{End}_G(M)$ is a field isomorphic to a subfield $\mathbb{E}$ of $\mathbb{K}$ containing $\mathbb{K}^{\text{un}}$. In particular, $\mathbb{E}_o/\mathbb{K}^{\text{un}} \simeq \mathbb{E}/\mathbb{K}^{\text{un}}$ is a totally ramified Galois extension with abelian Galois group.
- (b) The field $\mathbb{E}$ is the minimal splitting field for the Wedderburn component C of $\mathbb{K}^{\text{un}}G$ associated with M which is contained in $\mathbb{K}$.
- (c) Let $\Gamma = \text{Gal}(\mathbb{E}/\mathbb{K}^{\text{un}})$, and let Q be an irreducible constituent of the $\mathbb{K}G$ -module $M_{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{K}^{\text{un}}} M$. Then $M_{\mathbb{K}} \simeq \bigoplus_{\gamma \in \Gamma} {}^{\gamma}Q$.
- (d) If $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, then ${}^{\gamma_1}Q \not\simeq {}^{\gamma_2}Q$.
- (e) $\mathbb{E} \simeq \mathbb{K}^{\text{un}}(\chi_Q) = \mathbb{K}^{\text{un}}[\chi_Q(g) \mid g \in G]$, where $\chi_Q: G \rightarrow \mathbb{K}$ is the $\mathbb{K}$ -character associated with Q .

Proof. (a) By Schur's lemma, $\mathbb{E}_o$ is a skew-field. For $n = \dim_{\mathbb{E}_o}(M)$ the Wedderburn component C of $\mathbb{K}^{\text{un}}G$ associated with M is isomorphic to $\text{Mat}_{n \times n}(\mathbb{E}_o^{\text{op}})$. Thus, by definition, $\mathbb{E}_o$ is a field. As $\mathbb{K}$ is a splitting field for C , one concludes that $Z(\mathbb{K} \otimes_{\mathbb{K}^{\text{un}}} C) = \mathbb{K} \otimes_{\mathbb{K}^{\text{un}}} \mathbb{E}_o \simeq \mathbb{K}^m$ for some positive integer m . Hence $\mathbb{E}_o$ is $\mathbb{K}^{\text{un}}$ -isomorphic to a subfield $\mathbb{E}$ of $\mathbb{K}$, and thus, by definition, $\mathbb{E}_o/\mathbb{K}^{\text{un}}$ is an abelian Galois extension.

- (b) From now on we identify $\mathbb{E}$ with $\mathbb{E}_o$. If $\mathbb{F}L$ is a splitting field for C satisfying $\mathbb{K}^{\text{un}} \subseteq \mathbb{F}L \subseteq \mathbb{K}$, one has $\mathbb{F}L \otimes_{\mathbb{K}^{\text{un}}} \mathbb{E} \simeq \mathbb{F}L^m$. Hence $\mathbb{E} \subseteq \mathbb{F}L$.
- (c) As $\mathbb{E}/\mathbb{K}^{\text{un}}$ is a finite Galois extension, one has

$$\mathbb{E} \otimes_{\mathbb{K}^{\text{un}}} \mathbb{E} = \bigoplus_{\alpha \in \Gamma} \mathbb{E}_{\alpha} \simeq \mathbb{E}^m, \quad (4.6.10)$$

where $m = |\mathbb{E} : \mathbb{K}^{\text{un}}|$ and $\mathbb{E}_{\alpha}$ is the $(\mathbb{E}, \mathbb{E})$ -bisubmodule of $\mathbb{E} \otimes_{\mathbb{K}^{\text{un}}} \mathbb{E}$ satisfying $x \cdot \omega \cdot y = \omega \cdot (\alpha(x) \cdot y)$ for $x, y \in \mathbb{E}$ and $\omega \in \mathbb{E}_{\alpha}$. The $\mathbb{K}^{\text{un}}$ -algebra $\mathbb{E}_{\alpha}$ contains a unique idempotent ω_{α} , and, by the double centralizer property, $M_{\alpha} = \omega_{\alpha} \cdot M$ are the irreducible constituents of the $\mathbb{E}G$ -module $M_{\mathbb{E}} = \mathbb{E} \otimes_{\mathbb{K}^{\text{un}}} M$. Moreover, ${}^{\gamma}M_{\alpha} \simeq M_{\alpha \circ \gamma}$. This yields (c).

- (d) is a direct consequence of the Wedderburn theorem.
- (e) Since $\mathbb{E}$ is a splitting field for C , $\mathbb{E}$ must contain

$$\mathbb{K}_Q = \mathbb{K}^{\text{un}}(\alpha(\chi_Q) \mid \alpha \in \Gamma) = \mathbb{K}^{\text{un}}[\chi_{\alpha Q}(g) \mid \alpha \in \Gamma, g \in G]. \quad (4.6.11)$$

As $\mathbb{K}_Q \otimes_{\mathbb{K}^{\text{un}}} C$ contains m pairwise orthogonal central idempotents, $\mathbb{K}_Q$ is a splitting field for C , and thus $\mathbb{E} = \mathbb{K}_Q$. The field $\mathbb{E}$ is generated by the $\text{Gal}(\mathbb{E}/\mathbb{K}^{\text{un}})$ -orbit of the field $\mathbb{K}^{\text{un}}(\chi_Q)$. Hence, as $\text{Gal}(\mathbb{E}/\mathbb{K}^{\text{un}})$ is abelian, from the main theorem in Galois theory one concludes that $\mathbb{E} = \mathbb{K}^{\text{un}}(\chi_Q)$.

□

Remark 4.6.7. Let $\text{spec}(\mathbf{L}_{\mathcal{O}}(G))$ be the set of the species of $\mathbf{L}_{\mathcal{O}}(G)$, i.e., the set of $\star$ -algebra homomorphisms $s: \mathbf{L}_{\mathcal{O}}(G)_{\mathbb{C}} \rightarrow \mathbb{K} \subset \mathbb{C}$ (cf. [Ben06, § 2.2]). Then if $\mathcal{O}/\mathcal{O}^{\text{un}}$ is a splitting p -modular bi-system, a lattice $L \in \text{IL}_{\mathcal{O}}(B)$ defines a field extension $\mathbb{K}(L)/\mathbb{K}^{\text{un}}$, where

$$\begin{aligned} \mathbb{K}(L) &= \mathbb{K}^{\text{un}}[s(L) \mid s \in \text{spec}(\mathbf{L}_{\mathcal{O}}(G))] \\ &= \mathbb{K}^{\text{un}}[L(g, V) \mid (g, V) \in \text{spec}(\mathbf{L}_{\mathcal{O}}(G))]. \end{aligned} \quad (4.6.12)$$

The purpose of this last section was to establish a possible link between linear source lattices and the action of a (suitable) Galois group on the set of height zero characters of an $\mathcal{O}G$ -block B and of its Brauer correspondent b with respect to some defect group D . The definition and the construction of a *split p -modular bi-system* seems to go in the right direction thanks to the following fact.

Fact 4.6.8. Let the assumptions (A1) and (A2) be positively verified. Let G be a finite group and B an $\mathcal{O}G$ -block satisfying Conjecture 1' and Conjecture 2, then Conjecture 3' holds.

Proof. Let b be the Brauer correspondent of B with respect to some defect group D . Let $[M]$ be an indecomposable linear source b -lattice with maximal vertex and let $\mu \in \mathcal{G}$ such that it fixes the height zero representation $\sigma(\mathbf{g}([M]))_{\mathbb{K}}$. Then (A2) implies that μ fixes $\sigma(\mathbf{g}([M]))$ and the rationality of σ ensures that μ fixes $[M]$ and thus $[M]_{\mathbb{K}}$.

On the other hand, if μ fixes the height zero representation $[M]_{\mathbb{K}}$, it also fixes $[M]$ and then (A1) implies that it fixes $\mathbf{g}([M])$. As σ is rational, μ fixes $\sigma(\mathbf{g}([M]))$ and thus $\sigma(\mathbf{g}([M]))_{\mathbb{K}}$. $\square$

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