Noname manuscript No.
(will be inserted by the editor)

Robust Return Risk Measures

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Received: date / Accepted: date

Abstract In this paper we provide an axiomatic foundation to Orlicz risk measures in terms of properties of their acceptance sets, by exploiting their natural correspondence with shortfall risk ([17]), thus paralleling the characterization in [40]. From a financial point of view, Orlicz risk measures assess the stochastic nature of *returns*, in contrast to the common use of risk measures to assess the stochastic nature of a position's monetary *value*. The correspondence with shortfall risk leads to several robustified versions of Orlicz risk measures, and of their optimized translation invariant extensions ([35], [20]), arising from an ambiguity averse approach as in [19], [30], [9], or from a multiplicity of Young functions. We study the properties of these *robust* Orlicz risk measures, derive their dual representations, and provide some examples and applications.

Keywords Orlicz premium · Shortfall risk · Robustness · Ambiguity averse preferences · Orlicz norms and spaces · Convex risk measures · Positive homogeneity.

JEL Classification: D81, G10, G22.

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1 Introduction

Theories of risk measurement, dating back to [4], [12], [33], [38], and [37], often take random values designated in monetary units as basic objects. These monetary values specify the final wealth levels in *absolute* terms. With uncertainty represented by a space of states of nature, numerical representations are then considered over state-contingent final wealth. In the context of financial risk measurement and capital requirements, modern robust versions of such theories are provided by convex measures of risk ([16, 17], [18], [36], and [27]).

Already [31] argued that, rather than considering final wealth levels in absolute terms, risk measurement should instead be based on an assessment *relative* to a reference (or inflection) point.[1] Furthermore, there is a long-standing tradition of considering *relative* risk in measurement of risk aversion (see [32]).

In this paper, we develop a theory of risk measurement that takes log returns (in relative terms) rather than monetary values (in absolute terms) as basic objects.[2] Our theory may, in some sense, be viewed as the analog for returns of the theory of robust shortfall risk ([17], Chapter 4) for monetary values. The new classes of risk measures induced by our theory include some canonical special cases that are not contained in the classes of monetary ([17]) or convex measures of risk, and maintain the interpretation of measuring risk by means of acceptance sets. In the law-invariant case, our theory is connected to risk measurement based on Orlicz norms.

Orlicz premium principles were introduced in the actuarial literature in [21]. For a random nonnegative loss X, the Orlicz premium H_Φ is defined by

$$H_\Phi(X) := \inf\left\{k > 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \leq 1\right\},$$

where the Young function $\Phi\colon [0, +\infty) \to [0, +\infty)$ is convex and satisfies $\Phi(0) = 0$, $\Phi(1) = 1$ and $\Phi(+\infty) = +\infty$. By construction, Orlicz premia are positively homogeneous, monotone and subadditive. From a mathematical point of view, Orlicz premia are Luxemburg norms, and their natural domain is the Orlicz space

$$L^\Phi = \left\{X \in L^0(\Omega, \mathcal{F}, P) \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{|X|}{k}\right)\right] < +\infty \text{ for some } k > 0\right\}.$$

We refer the interested reader to [21], [34], [10, 11], [1, 2], [13], [28] and the references therein for further properties of Orlicz premia and Orlicz spaces and their use in risk measurement and portfolio choice. The economic motivation behind Orlicz premia is to provide a multiplicative version of the actuarial zero

[1] Indeed, [31] anticipated the development of reference-dependent theories of risk measurement, such as, perhaps most noticeably, the popular prospect theory of [24], and more recent extensions thereof.

[2] Contrary to reference-dependent theories, however, our theory does not require the specification of a reference level.

utility premium principle; indeed, when $X \in L^\infty_+$, the Orlicz premium $H_\Phi(X)$ can be equivalently defined as the unique solution to the equation

$$\mathbb{E}\left[\Phi\left(\frac{X}{H_\Phi(X)}\right)\right] = 1.$$

The first contribution of this paper is to formalize a one-to-one correspondence between Orlicz premia and measures of (utility-based) shortfall risk defined in [16] as

$$\rho_\ell(X) = \inf\left\{m \in \mathbb{R} \mid \mathbb{E}[\ell(X - m)] \le 0\right\},$$

where $\ell\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing and satisfies $\ell(-\infty) < 0 < \ell(+\infty)$. Measures of shortfall risk occur as special cases in the class of convex measures of risk.

The second contribution is to provide an axiomatic foundation to Orlicz premia, which can be considered as the appropriate analog of the well-known characterization of utility-based shortfall risk (see [40], [14]). From a financial point of view, we discover that Orlicz premia arise naturally when assessing the stochastic nature of *returns*, in contrast to the more common use of risk measures that are applied to the monetary *value* of financial positions.

Our third contribution is to exploit the correspondence between Orlicz premia and utility-based shortfall risk in order to define two families of *robust Orlicz premia*. In the first family, the decision-maker faces ambiguity with respect to the correct probabilistic model P. Ambiguity will be modeled in three economically different but mathematically unifiable ways: by means of multiple priors as in [19], by means of variational preferences as in [7], and by means of homothetic preferences as in [7] and [9]. These ambiguity averse theories belong to the wide family of uncertainty averse preferences of [7] and can be viewed as formalizations—and significant extensions—of the classical decision rule of [39] (see also [22]). In the second family of robust Orlicz premia, the decision-maker considers a multiplicity of Young functions. Both families lead to risk measures that are suprema of Luxemburg norms, related to different probability measures on a suitable rearrangement-invariant Banach space, or related to different Young functions. The most important difference between the two families of robustifications is that law invariance is not automatically inherited by the first family, while it is by the second family. The properties of the different types of robust Orlicz premia are studied in detail. We also provide an axiomatic foundation to the notion of robust shortfall risk (introduced by [17]) and the related axiomatization of robust Orlicz premia.

Our fourth contribution is the development of optimized translation invariant extensions of robust Orlicz premia, which we refer to as *robust Haezendonck-Goovaerts risk measures* in the spirit of [20]. Robust Orlicz premia are positively homogeneous, monotone and subadditive, but not translation invariant. By extending the well-known Rockafellar-Uryasev [35] construction (see also [3], [20], and [1, 2]), we introduce robust Haezendonck-Goovaerts risk measures, analyze their properties, and provide their dual representation as coherent measures of risk. We conclude with a few applications of our results to Pareto optimal allocations and optimal risk sharing.

The outline of this paper is as follows. Section 2 introduces some preliminaries. In Section 3, we establish a connection between risk measurement based on returns and risk measurement based on monetary values, leading to an axiomatic characterization of Orlicz premia which parallels the well-known characterization of utility-based shortfall risk in [40]. In Section 4, we introduce robust Orlicz risk measures, study their properties and general representations. We also characterize those robust Orlicz risk measures that are translation invariant and discuss the axiomatic foundation of robust Orlicz risk measures. In Section 5, we introduce and analyze robust Haezendonck-Goovaerts risk measures, and provide dual representation results. Finally, in Section 6 we provide some examples and applications.

2 Basic definitions and notation

We assume that there is an underlying probability space $(\Omega, \mathcal{F})$ with a fixed reference measure P. The probability space $(\Omega, \mathcal{F}, P)$ is nonatomic. We denote by $\mathcal{Q}$ the set of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P. Without further mentioning we will often identify $\mathcal{Q}$ with the set of Radon-Nikodym densities

$$\mathcal{D} = \{\phi \in L^1(\Omega, \mathcal{F}, P),\ \phi \geq 0\ P\text{-a.s.},\ \mathbb{E}_P[\phi] = 1\}.$$

In this paper we will consider finite-valued risk measures and premium principles defined on $L^\infty(\Omega, \mathcal{F}, P)$ or on the smaller domains

$$L^\infty_+(\Omega, \mathcal{F}, P) := \{X \in L^\infty \mid X \geq 0\ P\text{-a.s.}\}$$
$$L^\infty_{++}(\Omega, \mathcal{F}, P) := \{X \in L^\infty \mid X > 0\ P\text{-a.s.}\}.$$

(Throughout, positive realizations of X represent losses while negative realizations represent gains.) For a risk measure $\rho\colon L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$, we say that[3]

- ρ is translation invariant if $\rho(X + h) = \rho(X) + h,\ \forall h \in \mathbb{R}, \forall X \in L^\infty$
- ρ is monotone if $X \leq Y$ P-a.s. $\Rightarrow \rho(X) \leq \rho(Y)$
- ρ is monetary if it is monotone, translation invariant and satisfies $\rho(0) = 0$
- ρ is convex if it is monetary and

$$\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y),\ \forall X, Y \in L^\infty, \forall \alpha \in [0,1]$$

- ρ is positively homogeneous if $\rho(\lambda X) = \lambda\rho(X),\ \forall \lambda \geq 0, \forall X \in L^\infty$
- ρ is coherent if it is convex and positively homogeneous.

[3] Different from most of the financial mathematics literature on risk measures and as a consequence of the adopted sign convention on profits and losses, risk measures are here assumed to be monotone increasing.

The acceptance set of a risk measure ρ (at the level of random variables) is

$$A_\rho := \{X \in L^\infty \mid \rho(X) \leq 0\}.$$

If ρ is translation invariant, it can be recovered from its acceptance set by

$$\rho(X) = \inf\{m \in \mathbb{R} \mid X - m \in A_\rho\},$$

so ρ can be interpreted as the minimal amount, i.e., the required capital, that has to be subtracted from the loss X in order to make it acceptable. If ρ is law invariant, we define its acceptance set at the level of distributions

$$\mathcal{A}_\rho := \{F \in \mathcal{M}_{1,c} \mid \rho(F) \leq 0\},$$

with $\mathcal{M}_{1,c}$ the set of probability measures with compact support in $\mathbb{R}$. We say that ρ has the Fatou property if

$$X_n \xrightarrow{P} X,\ \|X_n\|_\infty \leq k \Rightarrow \rho(X) \leq \liminf_{n\to+\infty} \rho(X_n).$$

The Fatou property is equivalent to continuity from above; see [17] or [13]. A convex risk measure with the Fatou property has the following dual representation:

$$\rho(X) = \sup_{Q\in\mathcal{Q}} \left\{\mathbb{E}_Q[X] - c(Q)\right\}, \tag{1}$$

where the penalty function $c\colon \mathcal{Q} \to [0, +\infty)$ is convex and lower semicontinuous with respect to the weak topology $\sigma(L^1, L^\infty)$. If moreover ρ satisfies the stronger Lebesgue property

$$X_n \xrightarrow{P} X,\ \|X_n\|_\infty \leq k \Rightarrow \rho(X_n) \to \rho(X),$$

which is equivalent to continuity from below, then the supremum in the dual representation (1) is always attained and the penalty function $c(Q)$ has the so-called WC property, which means that its lower level sets are weakly compact. We refer to [18], [17] and [13] for extensive treatments of convex duality theory for risk measures. If ρ is law-invariant in the sense that

$$X \overset{d}{=} Y \Rightarrow \rho(X) = \rho(Y),$$

then it can be seen as a functional on $\mathcal{M}_{1,c}$. Each probability measure will be identified with its distribution function $F(x) := \mu(-\infty, x]$. The operation of convex combination in $\mathcal{M}_{1,c}$ is the mixture of distribution functions and should not be confused with the state-wise convex combination of random variables in $L^\infty(\Omega, \mathcal{F}, P)$. We say that ρ is mixture continuous if the function $\lambda \mapsto \rho\left(\lambda F + (1-\lambda)\, G\right)$ is continuous for $\lambda \in [0, 1]$ and that ρ has Convex Level Sets (CxLS) if $\rho(F) = \rho(G) = \gamma \Rightarrow \rho(\lambda F + (1-\lambda)G) = \gamma$, $\forall\lambda \in (0, 1)$. On $\mathcal{M}_{1,c}$ we will consider the Ψ-weak topology $\sigma(\mathcal{M}_{1,c}, C_\Psi)$ where

$$C_\Psi = \{f\colon \mathbb{R} \to \mathbb{R}, f \text{ continuous with } |f| \leq C \cdot \Psi\},$$

for some gauge function $\Psi\colon \mathbb{R} \to [1, +\infty)$. We refer to [17] for the properties of the Ψ-weak topology. Convergence in the Ψ-weak topology will be denoted by $\xrightarrow{\Psi}$. We say that ρ is Ψ-weakly continuous if $F_n \xrightarrow{\Psi} F \Rightarrow \rho(F_n) \to \rho(F)$.

3 Monetary risk measures and return risk measures

In this section, we discuss risk measurement based on the *relative* assessment of a financial position. As will become apparent, the primitive object here is not the position's monetary *value*, but the position's log *return*. We provide the following definition:

Definition 1 A *return risk measure* $\tilde{\rho}\colon L^\infty_{++} \to (0,+\infty)$ is a positively homogeneous and monotone risk measure with $\tilde{\rho}(1) = 1$.

The following lemma shows that properties of $\tilde{\rho}$ translate to properties of its associated acceptance set and that, vice versa, properties of the acceptance set are inherited by $\tilde{\rho}$. Even more, $\tilde{\rho}$ can be recovered from its acceptance set.

Lemma 1 *Let $\tilde{\rho} : L^\infty_{++} \to (0,+\infty)$ be positively homogeneous with $\tilde{\rho}(1) = 1$, and let*

$$B_{\tilde{\rho}} := \left\{X \in L^\infty_{++} | \tilde{\rho}(X) \le 1\right\}$$

be its associated acceptance set.
a) We have that

$$\tilde{\rho}(X) = \min\left\{k > 0 \mid X/k \in B_{\tilde{\rho}}\right\}.$$

b) $\tilde{\rho}$ is subadditive if and only if $B_{\tilde{\rho}}$ is convex.
c) $\tilde{\rho}$ is monotone if and only if $B_{\tilde{\rho}}$ satisfies $X \in B_{\tilde{\rho}}$, $Y \le X \Rightarrow Y \in B_{\tilde{\rho}}$.

Proof a) Since $\tilde{\rho}$ is positively homogeneous, $\tilde{\rho}\left(\frac{X}{k}\right) = \frac{\tilde{\rho}(X)}{k}$, and obviously $\tilde{\rho}(X) = \min\left\{k > 0 \mid \frac{\tilde{\rho}(X)}{k} \le 1\right\}$.
b) If $\tilde{\rho}$ is subadditive, then, from positive homogeneity, it is also convex, hence the lower level set $B_{\tilde{\rho}}$ is convex. Conversely, let $X, Y \in L^\infty_{++}$ with $\tilde{\rho}(X) = \alpha$, $\tilde{\rho}(Y) = \beta$, with $\alpha, \beta > 0$. Then, from positive homogeneity, $\tilde{\rho}\left(\frac{X}{\alpha}\right) = \tilde{\rho}\left(\frac{Y}{\beta}\right) = 1$, hence, from the convexity of $B_{\tilde{\rho}}$, we get that, for each $\lambda \in [0,1]$,

$$\tilde{\rho}\left(\lambda \frac{X}{\alpha} + (1-\lambda)\frac{Y}{\beta}\right) \le 1.$$

Choosing $\lambda = \frac{\alpha}{\alpha+\beta}$, we obtain

$$\tilde{\rho}\left(X + Y\right) \le \alpha + \beta,$$

which shows the subadditivity of $\tilde{\rho}$.
c) Let $\tilde{\rho}$ be monotone. Then, $X \in B_{\tilde{\rho}}$, $Y \le X \Rightarrow \tilde{\rho}(Y) \le \tilde{\rho}(X) \le 1$. On the other hand, let $Y \le X$. Then,

$$\frac{Y}{\tilde{\rho}(X)} \le \frac{X}{\tilde{\rho}(X)} \in B_{\tilde{\rho}} \Rightarrow \frac{Y}{\tilde{\rho}(X)} \in B_{\tilde{\rho}} \Rightarrow \tilde{\rho}(Y) \le \tilde{\rho}(X),$$

from which monotonicity follows. □

A natural *one-to-one correspondence* between return risk measures and monetary risk measures is as follows:

- given a monetary risk measure $\rho\colon L^\infty \to \mathbb{R}$, the associated monetary risk measure $\tilde{\rho}\colon L^\infty_{++} \to (0,+\infty)$ is given by

$$\boxed{\tilde{\rho}(X) := \exp\left(\rho\left(\log(X)\right)\right);} \tag{2}$$

- given a return risk measure $\tilde{\rho}\colon L^\infty_{++} \to (0,+\infty)$, the associated monetary risk measure $\rho\colon L^\infty \to \mathbb{R}$ is given by

$$\boxed{\rho(Z) := \log\left(\tilde{\rho}\left(\exp(Z)\right)\right).} \tag{3}$$

The following is immediate:

Lemma 2 *Let $\rho\colon L^\infty \to \mathbb{R}$ and $\tilde{\rho}\colon L^\infty_{++} \to (0,+\infty)$ be as in (2) and (3). Then:*

a) $\rho(0) = 0 \iff \tilde{\rho}(1) = 1$
b) *ρ is translation invariant $\iff$ $\tilde{\rho}$ is positively homogeneous*
c) *ρ is monotone $\iff$ $\tilde{\rho}$ is monotone*
d) *ρ is subadditive $\iff$ $\tilde{\rho}(XY) \le \tilde{\rho}(X)\tilde{\rho}(Y)$, $\forall X, Y \in L^\infty_{++}$*
e) *ρ is positively homogeneous $\iff$ $\tilde{\rho}(X^\alpha) = \tilde{\rho}^\alpha(X)$, $\forall X \in L^\infty_{++}$, $\alpha > 0$*
f) *ρ is convex $\iff$ $\tilde{\rho}(X^\alpha Y^{1-\alpha}) \le \tilde{\rho}^\alpha(X)\tilde{\rho}^{1-\alpha}(Y)$, $\forall X, Y \in L^\infty_{++}$, $\alpha \in (0,1)$*
g) *ρ is law-invariant $\iff$ $\tilde{\rho}$ is law-invariant.*

When ρ and $\tilde{\rho}$ are law invariant, the correspondence given by (2) and (3) is also well-defined at the level of distributions. Let us denote by $\mathcal{M}_{1,c}(0,+\infty)$ the set of distribution functions with support in $(0,+\infty)$. If $F \in \mathcal{M}_{1,c}(0,+\infty)$, we have that

$$\tilde{\rho}(F) = \exp\left(\rho\left(F(e^t)\right)\right), \tag{4}$$

and, correspondingly, if $F \in \mathcal{M}_{1,c}(\mathbb{R})$, we have

$$\rho(F) = \log(\tilde{\rho}(F(\log t))). \tag{5}$$

Lemma 3 *Let $\Psi\colon \mathbb{R} \to [1,+\infty)$ be a gauge function, and let $\tilde{\Psi}\colon (0,+\infty) \to [1,+\infty)$ be given by $\tilde{\Psi}(t) = \Psi(\log t)$, for $t > 0$. Let $\tilde{F}_n(t) = F_n(\log t)$ and $\tilde{F}(t) = F(\log t)$. Then*

$$\tilde{F}_n \xrightarrow{\tilde{\Psi}} \tilde{F} \iff F_n \xrightarrow{\Psi} F.$$

Proof Let $\tilde{F}_n, \tilde{F} \in \mathcal{M}_{1,c}(0,+\infty)$ with $\tilde{F}_n \xrightarrow{\tilde{\Psi}} \tilde{F}$, that is, $\tilde{F}_n \to \tilde{F}$ weakly and $\int \tilde{\Psi} \mathrm{d}\tilde{F}_n \to \int \tilde{\Psi} \mathrm{d}\tilde{F}$. Since $F(t) = \tilde{F}(e^t)$, it follows that $F_n \to F$ weakly, and

$$\int_0^{+\infty} \tilde{\Psi} \mathrm{d}\tilde{F}_n = \int_0^{+\infty} \Psi(\log t) \mathrm{d}F_n(\log t) = \int_{-\infty}^{+\infty} \Psi(t) \mathrm{d}F_n(t),$$

and similarly

$$\int_0^{+\infty} \tilde{\Psi} \mathrm{d}\tilde{F} = \int_0^{+\infty} \Psi(\log t) \mathrm{d}F(\log t) = \int_{-\infty}^{+\infty} \Psi(t) \mathrm{d}F(t),$$

which implies that

$$\int_{-\infty}^{+\infty} \Psi(t) \mathrm{d}F_n(t) \to \int_{-\infty}^{+\infty} \Psi(t) \mathrm{d}F(t),$$

which gives the thesis. The converse implication can be proved similarly. □

Lemma 4 *Let* $\rho \colon \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ *and* $\tilde{\rho} \colon \mathcal{M}_{1,c}(0, +\infty) \to (0, +\infty)$ *be associated as in* (4) *and* (5). *Let* $\mathcal{B}_{\tilde{\rho}} := \{F \in \mathcal{M}_{1,c} \mid \tilde{\rho}(F) \leq 1\}$ *and let* $\tilde{\Psi}(t) = \Psi(\log t)$. *Then:*

a) ρ *is mixture continuous* $\iff$ $\tilde{\rho}$ *is mixture continuous*
b) ρ *has CxLS* $\iff$ $\tilde{\rho}$ *has CxLS*
c) ρ *is* Ψ*-weakly continuous* $\iff$ $\tilde{\rho}$ *is* $\tilde{\Psi}$*-weakly continuous.*
d) $\mathcal{A}_\rho$ *is convex with respect to mixtures* $\iff$ $\mathcal{B}_{\tilde{\rho}}$ *is convex with respect to mixtures*
e) $\mathcal{A}_\rho$ *is* Ψ*-weakly closed* $\iff$ $\mathcal{B}_{\tilde{\rho}}$ *is* $\tilde{\Psi}$*-weakly closed.*

Proof a) From (4),

$$\tilde{\rho}(\lambda F + (1 - \lambda)G) = \exp(\rho(\lambda F(e^t) + (1 - \lambda)G(e^t))),$$

hence mixture continuity of $\tilde{\rho}$ is equivalent to mixture continuity of ρ.
b) Let $\tilde{\rho}(F) = \tilde{\rho}(G)$. From (4) it follows that $\rho(F(e^t)) = \rho(G(e^t))$. If ρ has the CxLS property, then

$$\rho(\lambda F(e^t) + (1 - \lambda)G(e^t)) = \rho(F(e^t)) = \rho(G(e^t)),$$

hence

$$\tilde{\rho}(\lambda F + (1 - \lambda)G) = \tilde{\rho}(F) = \tilde{\rho}(G),$$

which implies that also $\tilde{\rho}$ has the CxLS property. The converse is similar.
c) Let ρ be Ψ-weakly continuous. Let $\tilde{F}_n, \tilde{F} \in \mathcal{M}_{1,c}(0, +\infty)$, with $\tilde{F}_n \xrightarrow{\tilde{\Psi}} \tilde{F}$. Let

$$F_n(t) := \tilde{F}_n(e^t), \quad F(t) := \tilde{F}(e^t).$$

From Lemma 3 it follows that $F_n \xrightarrow{\Psi} F$, which implies that $\rho(F_n) \to \rho(F)$. Since

$$\tilde{\rho}(\tilde{F}_n) = \exp\left(\rho\left(F_n\right)\right), \quad \tilde{\rho}(\tilde{F}) = \exp\left(\rho\left(F\right)\right),$$

it follows that $\tilde{\rho}(\tilde{F}_n) \to \tilde{\rho}(\tilde{F})$, which yields the thesis.
d) Let $F, G \in \mathcal{B}_{\tilde{\rho}}$, that is, $\tilde{\rho}(F) \leq 1, \tilde{\rho}(G) \leq 1$. Hence, $\rho(F(e^t)) \leq 0, \rho(G(e^t)) \leq 0$, that is,

$$F(e^t),\, G(e^t) \in \mathcal{A}_\rho.$$

From the convexity of $\mathcal{A}_\rho$, for each $\lambda \in (0,1)$,

$$\lambda F(e^t) + (1-\lambda)G(e^t) \in \mathcal{A}_\rho,$$

that is, $\rho(\lambda F(e^t)+(1-\lambda)G(e^t)) \leq 0$, which is equivalent to $\tilde{\rho}(\lambda F+(1-\lambda)G) \leq 1$. The converse is similar.
e) We prove that if $\mathcal{N}_\rho$ is Ψ-weakly closed, then $\mathcal{B}_{\tilde{\rho}}$ is $\tilde{\Psi}$-weakly closed. The converse is similar. Let $\tilde{F}_n \in \mathcal{B}_{\tilde{\rho}}$, with $\tilde{F}_n \xrightarrow{\tilde{\Psi}} \tilde{F}$. As before, let

$$F_n(t) := \tilde{F}_n(e^t), \quad F(t) := \tilde{F}(e^t).$$

From Lemma 3 it follows that $F_n \xrightarrow{\Psi} F$, which implies that $F \in \mathcal{N}_\rho$. As a consequence,

$$\tilde{\rho}(\tilde{F}) = \exp(\rho(F)) \leq 1,$$

that is, $\tilde{F} \in \mathcal{B}_{\tilde{\rho}}$. □

A return risk measure can be given the following interpretation. Consider a risk manager with initial capital $x_0 > 0$. Suppose he assesses the risk of a financial loss $X \in L^\infty_{++}$ (recall that X is a loss r.v.) by considering its log *return* with respect to initial capital x_0 given by $\log\left(\frac{X}{x_0}\right)$ rather than by considering the loss X itself. In particular, he asks the question of whether this log return is acceptable: does $\log\left(\frac{X}{x_0}\right) \in A_\rho$ hold?

If the log return $\log\left(\frac{X}{x_0}\right)$ is not acceptable, then he finds the smallest amount of capital $k > 0$ such that, when translated into a log return with respect to initial capital x_0, i.e., $\log\left(\frac{k}{x_0}\right)$, and subtracted from the log return $\log\left(\frac{X}{x_0}\right)$, the resulting log return given by

$$\log\left(\frac{X}{x_0}\right) - \log\left(\frac{k}{x_0}\right) = \log\left(\frac{X}{k}\right)$$

becomes acceptable, that is,

$$\log\left(\frac{X}{k}\right) \in A_\rho. \tag{6}$$

In other words, the risk manager determines $\inf\left\{k > 0 | \log\left(\frac{X}{k}\right) \in A_\rho\right\}$. Eq. (6) tells us that if k were the initial capital instead of x_0, then the log return with respect to initial capital, $\log\left(\frac{X}{k}\right)$, would have been acceptable.

Now if the resulting log return $\log\left(\frac{X}{k}\right)$ is acceptable, then, by the definition of acceptability,

$$\rho\left(\log\left(\frac{X}{k}\right)\right) \leq 0.$$

Clearly, this is equivalent to $\exp\left(\rho\left(\log\left(\frac{X}{k}\right)\right)\right) \leq 1$. Using the one-to-one correspondence (2) and (3) this, in turn, means that

$$\tilde{\rho}\left(\frac{X}{k}\right) \leq 1,$$

hence

$$\frac{X}{k} \in B_{\tilde{\rho}}. \tag{7}$$

Thus, while monetary risk measures conventionally assess acceptability at the level of monetary values, return risk measures can, from a conventional risk measures perspective, be interpreted to assess acceptability at the level of *returns*, as (6) reveals. Relatedly, one may directly interpret k as the amount of initial required capital *relative* to which the loss X becomes acceptable, as (7) stipulates. This interpretation naturally induces the positive homogeneity and monotonicity properties that return risk measures satisfy.

In the special case of the Value-at-Risk measure of risk, risk assessment at the level of monetary values is equivalent to risk assessment at the level of returns, as Example 1 below makes explicit. However, as soon as we depart from Value-at-Risk-based risk assessment, this equivalence is no longer valid in general, as Examples 2 and 3 illustrate.

Note that the amount of initial capital x_0 is irrelevant for determining k. This means in particular that our theory does not require the specification of a "reference level", contrary to reference-dependent theories.

3.1 Orlicz premia and shortfall risk

A particular case of the correspondence in (2) and (3) arises when ρ is a utility-based shortfall risk and $\tilde{\rho}$ is an Orlicz premium. Let $X \in L^\infty_+$. Given a nondecreasing Young function $\Phi\colon [0, +\infty) \to [0 +\infty)$ with $\Phi(0) < 1 < \Phi(+\infty)$, we define

$$H_\Phi(X) = \inf\left\{k > 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \leq 1\right\}.$$

Notice that convexity of Φ is not required and that we allow the possibility that $\Phi(0) > 0$ or $\Phi(+\infty) < +\infty$. Similarly, given a nondecreasing loss function $\ell\colon \mathbb{R} \to \mathbb{R}$ with $\ell(-\infty) < 0 < \ell(+\infty)$, we define

$$\rho_\ell(X) = \inf\{m \in \mathbb{R} \mid \mathbb{E}[\ell(X - m)] \leq 0\}.$$

Proposition 1 *Let ρ and $\tilde{\rho}$ be associated as in (2) and (3). A monetary risk measure ρ is a utility-based shortfall risk with loss function ℓ if and only if the corresponding return risk measure $\tilde{\rho}$ is an Orlicz premium with Young function Φ, with $\Phi(x) = 1 + \ell(\log x)$.*

Proof By definition $\tilde{\rho}_\ell(X) = \exp(\rho_\ell(\log X))$. We compute

$$\begin{aligned}\tilde{\rho}_\ell(X) &= \exp\left(\inf\{m \in \mathbb{R} \mid \mathbb{E}[\ell(\log X - m)] \leq 0\}\right)\\ &= \inf\{k > 0 \mid \mathbb{E}[\ell(\log X - \log k)] \leq 0\}\\ &= \inf\left\{k > 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \leq 1\right\} = H_\Phi(X).\end{aligned}$$

The reverse implication can be proved similarly. □

Remark 1 When ℓ is convex, the continuity properties of ρ_ℓ with respect to the Ψ-weak topology have been established in [26], under a Δ_2 condition.

Example 1 (Value-at-Risk $\longleftrightarrow$ Value-at-Risk) Let $\alpha \in (0,1)$ and

$$\ell(x) = \begin{cases}\alpha - 1, & \text{if } x \leq 0,\\ \alpha, & \text{if } x > 0.\end{cases}$$

Then,

$$\begin{aligned}\rho_\ell(X) &= \inf\{m \mid \alpha P(X > m) + (\alpha - 1)P(X \leq m) \leq 0\} = \inf\{m \mid F_X(m) \geq \alpha\}\\ &= q_\alpha(X).\end{aligned}$$

Correspondingly,

$$\Phi(x) = 1 + \ell(\log x) = \begin{cases}\alpha, & \text{if } 0 \leq x \leq 1,\\ 1 + \alpha, & \text{if } x > 1,\end{cases}$$

and

$$\begin{aligned}H_\Phi(X) &= \inf\left\{k > 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \leq 1\right\}\\ &= \inf\left\{k > 0 \;\middle|\; (1+\alpha)P\left(\frac{X}{k} > 1\right) + \alpha P\left(\frac{X}{k} \leq 1\right) \leq 1\right\}\\ &= \inf\left\{k > 0 \;\middle|\; (1+\alpha)P\left(X > k\right) + \alpha P\left(X \leq k\right) \leq 1\right\} = q_\alpha(X).\end{aligned}$$

So, when ρ is an α-quantile, then also the corresponding $\tilde{\rho}$ is the same α-quantile. This is not surprising from a financial point of view: the use of the Value-at-Risk does not require a distinction between measuring the risk of a financial position at the level of returns or at the level of monetary values. Notice also that in this example Φ is not convex and satisfies $\Phi(0) > 0$ and $\Phi(1) < 1$, which motivates the need for the slightly generalized version of the definition of Orlicz premia given at the beginning of this subsection.

Example 2 (Mean $\longleftrightarrow$ logarithmic certainty equivalent) Let $\ell(x) = x$. Then, $\rho_\ell(X) = \mathbb{E}[X]$, $\Phi(x) = 1 + \log x$, and

$$\begin{aligned} H_\Phi(X) &= \inf \left\{ k > 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \le 1 \right\} \\ &= \inf \left\{ k > 0 \;\middle|\; \mathbb{E}\left[1 + \log\left(\frac{X}{k}\right)\right] \le 1 \right\} \\ &= \inf \left\{ k > 0 \;\middle|\; \mathbb{E}\left[\log X - \log k\right] \le 0 \right\} = \exp\left(\mathbb{E}\left[\log X\right]\right). \end{aligned}$$

This example shows that the convexity of ρ_ℓ does not imply the convexity of H_Φ. Indeed, as we saw in Lemma 2 item f), the convexity of ρ is equivalent to a multiplicative convexity of $\tilde{\rho}$.

Example 3 (Entropic risk measure $\longleftrightarrow$ p-norm) Let $\ell(x) = \exp(\gamma x) - 1$, with $\gamma > 0$. Then, ρ_ℓ is the entropic risk measure, also known as the exponential premium, given explicitly by

$$\rho_\ell(X) = \frac{1}{\gamma} \log\left(\mathbb{E}[\exp(\gamma X)]\right).$$

Correspondingly, $\Phi(x) = 1 + \ell(\log x) = x^\gamma$ and $H_\Phi(X) = \|X\|_\gamma$.

One of the most widely used families of utility functions is the power (CRRA) family. The corresponding certainty equivalent under expected utility is a p-norm. It is perhaps the most widely adopted measure of risk in economics. This example shows that the class of return risk measures encompasses p-norms, contrary to the classes of monetary or convex measures of risk.

Under expected utility preferences (or homothetic preferences as considered in Section 4), the certainty equivalent coincides with the utility-based shortfall risk if and only if the utility function is exponential. Also, the only certainty equivalent under expected utility (or homothetic) preferences that gives rise to a convex measure of risk is the one with exponential utility (see [27]). Now suppose we use an exponential utility function, but not to measure the risk of monetary values (in absolute terms) but to measure risk in relative terms, that is, to measure the risk of log returns. Under exponential utility, the certainty equivalent agrees with the utility-based shortfall risk under expected utility (or homothetic) preferences, and induces a (β-discounted, in the case of homothetic preferences) p-norm for risk measured in absolute terms, as illustrated in this example. Thus, it gives rise to one of the most widely adopted measures of risk.

3.2 Axiomatization of Orlicz premia

An axiomatic characterization of utility-based shortfall risk has been provided in [40]. We recall the main result:

Theorem 1 (Weber (2006)) *Let $\rho\colon \mathcal{M}_{1,c} \to \mathbb{R}$ be a law invariant monetary risk measure. If the following conditions hold:*

a) the acceptance set $\mathcal{A}_\rho$ *and its complement* $\mathcal{A}_\rho^c$ *are convex with respect to mixtures*
b) the acceptance set $\mathcal{A}_\rho$ *is* Ψ*-weakly closed for some gauge function* Ψ
c) for each $x < 0$, $y > 0$, *there exists* $\alpha \in (0, 1)$ *such that* $\alpha\delta_x + (1-\alpha)\delta_y \in \mathcal{A}_\rho$

then ρ *is a utility-based shortfall risk.*

[14] proved a version of this result in the special case of law invariant *convex* risk measures, in which the assumptions b) and c) are not needed. Here we translate Weber's Theorem into a characterization of those return risk measures that are Orlicz premia.

Theorem 2 *Let* $\tilde{\rho}\colon \mathcal{M}_{1,c}(0, +\infty) \to \mathbb{R}$ *be a law invariant return risk measure and let*

$$\mathcal{B}_{\tilde{\rho}} := \{F \in \mathcal{M}_{1,c}(0, +\infty) \mid \tilde{\rho}(F) \leq 1\}.$$

Assume that

a) $\mathcal{B}_{\tilde{\rho}}$ *and* $\mathcal{B}_{\tilde{\rho}}^c$ *are convex with respect to mixtures*
b) $\mathcal{B}_{\tilde{\rho}}$ *is* $\tilde{\Psi}$*-weakly closed for some gauge function* $\tilde{\Psi}$
c) for each $0 < \tilde{x} < 1$ *and* $\tilde{y} > 1$ *there exists* $\alpha \in (0, 1)$ *such that*

$$\alpha\delta_{\tilde{x}} + (1-\alpha)\delta_{\tilde{y}} \in \mathcal{B}_{\tilde{\rho}}.$$

Then there exists $\Phi\colon [0, +\infty) \to [0, +\infty)$, *with* $0 \leq \Phi(0) < 1 < \Phi(+\infty)$, *such that*

$$\tilde{\rho}(F) = \inf\left\{k > 0 \mid \int \Phi\left(x/k\right)\, \mathrm{d}F(x) \leq 1\right\}.$$

Thus, $\tilde{\rho}$ *is an Orlicz premium.*

Proof Let us consider the risk measure $\rho\colon L^\infty \to \mathbb{R}$ associated to $\tilde{\rho}$ by (3). From Lemma 2 it follows that ρ is monetary with $\rho(0) = 0$. From item a) and Lemma 4, item d) it follows that $\mathcal{A}_\rho$ and $\mathcal{A}_\rho^c$ are convex with respect to mixtures. From item b) and Lemma 4, item d) it follows that $\mathcal{A}_\rho$ is Ψ-weakly closed. Let now $x < 0 < y$ and $\tilde{x} = e^x$, $\tilde{y} = e^y$. Since $\tilde{x} < 1 < \tilde{y}$, from item c) there exists $\alpha \in (0, 1)$ such that

$$\tilde{\rho}(\alpha\delta_{\tilde{x}} + (1-\alpha)\delta_{\tilde{y}}) \leq 1,$$

which implies

$$\rho(\alpha\delta_x + (1-\alpha)\delta_y) \leq 0,$$

that is, $\alpha\delta_x + (1-\alpha)\delta_y \in \mathcal{A}_\rho$. Hence, all the hypotheses of Weber's Theorem are satisfied, and we can conclude that ρ is a utility-based shortfall risk corresponding to some loss function ℓ. From Proposition 1 it then follows that $\tilde{\rho}(X) = H_\Phi(X)$, with $\Phi(x) = 1 + \ell(\log x)$. □

4 Robustification of Orlicz premia

So far, no uncertainty on the choice of P and of Φ has been considered. What happens if we consider uncertainty with respect to the probabilistic model P (i.e., ambiguity) or with respect to the Young function Φ? In the first case, the decision-maker faces a set $\mathcal{S} \subset \mathcal{Q}$ of probability measures, while in the second case he faces a set $\mathfrak{F}$ of Young functions. Our goal in the present section is to extend and robustify Orlicz premia, by adopting ambiguity averse preferences or by means of a worst-case approach over a multiplicity of Young functions.

4.1 Ambiguity over Q

When $X \in L^\infty_+$, Orlicz premia are implicitly defined by

$$L\left(\frac{X}{H_\Phi(X)}\right) = 1,$$

where $L(X) = \mathbb{E}[\Phi(X)]$. Robust versions of Orlicz premia are obtained by replacing the expected utility loss with the following "robust" versions of expected utility:

- Multiple priors ([19]):

$$L(X) = \sup_{Q \in \mathcal{S}} \mathbb{E}_Q[\ell(X)],$$

  with $\mathcal{S} \subset \mathcal{Q}$.
- Variational preferences ([30]):

$$L(X) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[\ell(X)] - c(Q)\},$$

  with $c\colon \mathcal{Q} \to [0, +\infty]$.
- Homothetic preferences ([7] and [9]):

$$L(X) = \sup_{Q \in \mathcal{Q}} \{\beta(Q)\, \mathbb{E}_Q[\ell(X)]\},$$

  where $\beta\colon \mathcal{Q} \to [0, 1]$, with $\sup_{Q \in \mathcal{Q}} \beta(Q) = 1$.

More precisely, we introduce the following definitions:

Definition 2 (Robust Orlicz premia - multiple Q) Let $X \in L^\infty_+$, $X \neq 0$, and let $\Phi\colon [0, +\infty) \to [0, +\infty)$ be convex, with $\Phi(0) = 0$, $\Phi(1) = 1$ and $\Phi(+\infty) = +\infty$.
Let $\mathcal{S} \subset \mathcal{Q}$. We define

$$H_{\Phi,\mathcal{S}}(X) := \inf \left\{ k > 0 \;\middle|\; \sup_{Q \in \mathcal{S}} \mathbb{E}_Q \left[\Phi\left(\frac{X}{k}\right)\right] \leq 1 \right\}. \tag{8}$$

Let $c\colon \mathcal{Q} \to [0, +\infty]$ be convex and lower semicontinuous, with $\inf_{Q\in\mathcal{Q}} c(Q) = 0$. We define

$$H_{\Phi,c}(X) := \inf \left\{ k > 0 \;\middle|\; \sup_{Q\in\mathcal{Q}} \left\{ \mathbb{E}_Q \left[\Phi\left(\frac{X}{k}\right)\right] - c(Q) \right\} \le 1 \right\}. \tag{9}$$

Finally, let $\beta\colon \mathcal{Q} \to [0,1]$ satisfying $\sup_{Q\in\mathcal{Q}} \beta(Q) = 1$. We define

$$H_{\Phi,\beta}(X) := \inf \left\{ k > 0 \;\middle|\; \sup_{Q\in\mathcal{Q}} \left\{ \beta(Q)\, \mathbb{E}_Q \left[\Phi\left(\frac{X}{k}\right)\right] \right\} \le 1 \right\}. \tag{10}$$

If $X = 0$ P-a.s., we set by definition $H_{\Phi,\mathcal{S}}(X) = H_{\Phi,c}(X) = H_{\Phi,\beta}(X) = 0$.

Clearly, the definition of $H_{\Phi,\mathcal{S}}(X)$ in (8) is a special case of both (9) and (10), corresponding to

$$c(Q) = \begin{cases} 0, & \text{if } Q \in \mathcal{S}, \\ +\infty, & \text{if } Q \notin \mathcal{S}, \end{cases}$$

or

$$\beta(Q) = \begin{cases} 1, & \text{if } Q \in \mathcal{S}, \\ 0, & \text{if } Q \notin \mathcal{S}. \end{cases}$$

Notice that multiple priors, variational preferences and homothetic preferences are special cases of the more general uncertainty averse preferences introduced in [7]. As suggested by an anonymous referee, it is possible to further generalize the definition of Robust Orlicz premia to a quasiconvex setting by invoking uncertainty averse preferences.

When $\mathcal{S} = \{P\}$, the robust Orlicz premium coincides with the usual Orlicz premium. The following lemma shows that on L^∞_+ robust Orlicz premia can always be expressed as the solution to an equation, just like the Orlicz premium itself.

Lemma 5 *Let $X \in L^\infty_+$, $X \neq 0$, and let $\Phi\colon [0,+\infty) \to [0,+\infty)$ be convex, with $\Phi(0) = 0$, $\Phi(1) = 1$ and $\Phi(+\infty) = +\infty$. Furthermore, let $H_{\Phi,\mathcal{S}}$, $H_{\Phi,c}$ and $H_{\Phi,\beta}$ be as in Definition 2. Then, $H_{\Phi,\mathcal{S}}$, $H_{\Phi,c}$ and $H_{\Phi,\beta}$ are the unique solutions to the following equations:*

$$\sup_{Q\in\mathcal{S}} \mathbb{E}_Q \left[\Phi\left(\frac{X}{H_{\Phi,\mathcal{S}}(X)}\right)\right] = 1, \tag{11}$$

$$\sup_{Q\in\mathcal{Q}} \left\{ \mathbb{E}_Q \left[\Phi\left(\frac{X}{H_{\Phi,c}(X)}\right)\right] - c(Q) \right\} = 1, \tag{12}$$

$$\sup_{Q\in\mathcal{Q}} \left\{ \beta(Q)\, \mathbb{E}_Q \left[\Phi\left(\frac{X}{H_{\Phi,\beta}(X)}\right)\right] \right\} = 1. \tag{13}$$

Proof We consider only $H_{\Phi,c}$ and $H_{\Phi,\beta}$, since $H_{\Phi,\mathcal{S}}$ in (11) can be regarded as a special case of (12) and (13). For $k \in [0,+\infty)$, let

$$g(k) := \sup_{Q\in\mathcal{Q}} \{\mathbb{E}_Q[\Phi(kX)] - c(Q)\}.$$

Then, $g(0) = -\inf_{Q\in\mathcal{Q}} c(Q) = 0$, and g is convex and nondecreasing, since $X \in L^\infty_+$. From

$$\mathbb{E}_Q[\Phi(kX)] - c(Q) \leq \mathbb{E}_Q[\Phi(k\|X\|_\infty)] \leq \Phi(k\|X\|_\infty),$$

we get that g is finite and hence continuous and strictly increasing on $\{g > 0\}$. Moreover, from the monotone convergence theorem, $g(+\infty) = +\infty$. It follows that the equation $g(k) = 1$ has exactly one solution. For the case of $H_{\Phi,\beta}$, the same arguments show that the function

$$g(k) := \sup_{Q\in\mathcal{Q}} \{\beta(Q)\,\mathbb{E}_Q[\Phi(kX)]\}$$

satisfies $g(0) = 0$, $g(+\infty) = +\infty$ and is strictly increasing on $\{g > 0\}$, from which the thesis follows as before. □

Remark 2 It is possible to extend the domain of the robust Orlicz premia beyond L^∞_+. For example, it is easy to see that $H_{\Phi,\mathcal{S}}$ is finite on

$$\mathbb{L}^\Phi_+ = \bigcap_{Q\in\mathcal{S}} L^\Phi_+(Q),$$

although Lemma 5 does not necessarily hold on this larger domain. In this paper we stick to the L^∞_+ case, leaving these extensions for further research.

In order to present the various robustifications of the Orlicz premium in a unified framework, we introduce the following definition.

Definition 3 (Non-normalized Orlicz premia) Let $Q \in \mathcal{Q}$, $\Phi\colon [0, +\infty) \to [0, +\infty)$ convex with $\Phi(0) = 0$, $\Phi(1) = 1$ and $\Phi(+\infty) = +\infty$, $X \in L^\infty_+$ and $c \geq 0$. We define $H_{Q,\Phi,c}(X)$ as the unique solution to the equation

$$\mathbb{E}_Q\left[\Phi\left(\frac{X}{H_{Q,\Phi,c}(X)}\right)\right] = 1 + c. \tag{14}$$

If $X = 0$ P-a.s. or if $c = +\infty$, we set by definition $H_{\Phi,Q,c}(X) = 0$.

Clearly, if $c = 0$, we have that $H_{\Phi,Q,c}$ is the usual Orlicz premium, and moreover

$$c_1 \leq c_2 \Rightarrow H_{Q,\Phi,c_1}(X) \geq H_{Q,\Phi,c_2}(X).$$

The next proposition shows that the three robust Orlicz premia $H_{\Phi,\mathcal{S}}$, $H_{\Phi,c}(X)$ and $H_{\Phi,\beta}$ can be expressed as (non-penalized) suprema of non-normalized Orlicz premia $H_{Q,\Phi,c}$.

Proposition 2 *Let $X \in L^\infty_+$ and $\Phi\colon [0, +\infty) \to [0, +\infty)$ be convex, with $\Phi(0) = 0$, $\Phi(1) = 1$ and $\Phi(+\infty) = +\infty$. Let $H_{\Phi,\mathcal{S}}$, $H_{\Phi,c}$, $H_{\Phi,\beta}$ be as in Definition 2 and let $H_{Q,\Phi,c}(X)$ be as in (3). Then,*

$$H_{\Phi,\mathcal{S}}(X) = \sup_{Q\in\mathcal{S}} H_{Q,\Phi,0}(X),$$

$$H_{\Phi,c}(X) = \sup_{Q\in\mathcal{Q}} H_{Q,\Phi,c(Q)}(X),$$

$$H_{\Phi,\beta}(X) = \sup_{Q\in\mathcal{Q}} H_{Q,\Phi,\frac{1-\beta(Q)}{\beta(Q)}}(X).$$

Proof From the proof of Lemma 5 we know that the function

$$g(k) = \sup_{Q \in \mathcal{Q}} \{\mathbb{E}_Q [\varPhi(kX)] - c(Q)\}$$

is finite, convex, continuous and strictly increasing on $\{g > 0\}$. For any $\overline{Q} \in \mathcal{Q}$,

$$\sup_{Q \in \mathcal{Q}} \left\{ \mathbb{E}_Q \left[\varPhi \left(\frac{X}{H_{\overline{Q},\varPhi,c(\overline{Q})}(X)} \right) \right] - c(Q) \right\} \geq \mathbb{E}_{\overline{Q}} \left[\varPhi \left(\frac{X}{H_{\overline{Q},\varPhi,c(\overline{Q})}(X)} \right) \right] - c(\overline{Q}) = 1,$$

which implies $H_{\overline{Q},\varPhi,c(\overline{Q})}(X) \leq H_{\varPhi,c}(X)$, so $\sup_{\overline{Q} \in \mathcal{Q}} H_{\overline{Q},\varPhi,c(\overline{Q})}(X) \leq H_{\varPhi,c}(X)$. On the other hand,

$$\sup_{Q \in \mathcal{Q}} \left\{ \mathbb{E}_Q \left[\varPhi \left(\frac{X}{\sup_{\overline{Q} \in \mathcal{Q}} H_{\overline{Q},\varPhi,c(\overline{Q})}(X)} \right) \right] - c(Q) \right\} \leq \sup_{Q \in \mathcal{Q}} \left\{ \mathbb{E}_Q \left[\varPhi \left(\frac{X}{H_{Q,\varPhi,c(Q)}(X)} \right) \right] - c(Q) \right\} = 1,$$

which gives $\sup_{\overline{Q} \in \mathcal{Q}} H_{\overline{Q},\varPhi,c}(X) \geq H_{\varPhi,c(\overline{Q})}(X)$, from which the thesis follows. Now for $\beta > 0$ define $H^{\beta}_{Q,\varPhi}(X)$ as the unique solution to the equation

$$\mathbb{E}_Q \left[\varPhi \left(\frac{X}{H^{\beta}_{Q,\varPhi}(X)} \right) \right] = \frac{1}{\beta}.$$

Notice that $H^{\beta}_{Q,\varPhi}(X) = H_{Q,\varPhi,\frac{1-\beta}{\beta}}(X)$. A similar argument as before shows that

$$H_{\varPhi,\beta}(X) = \sup_{Q \in \mathcal{Q}} H^{\beta(Q)}_{Q,\varPhi}(X) = \sup_{Q \in \mathcal{Q}} H_{Q,\varPhi,\frac{1-\beta(Q)}{\beta(Q)}}(X).$$

□

Let $c \colon \mathcal{Q} \to [0, +\infty]$ and let $\bar{H}_{\varPhi,c} \colon L^{\infty} \times \mathcal{Q} \to [0, +\infty)$ be defined as

$$\bar{H}_{\varPhi,c}(X, Q) := H_{Q,\varPhi,c(Q)}(X), \tag{15}$$

so that

$$H_{\varPhi,c}(X) = \sup_{Q \in \mathcal{Q}} \bar{H}_{\varPhi,c}(X, Q).$$

Lemma 6 *Let $c \colon \mathcal{Q} \to [0, +\infty]$ be convex and lower semicontinuous with respect to the $\sigma(L^1, L^{\infty})$ topology, with $\inf_{Q \in \mathcal{Q}} c(Q) = 0$. Let $\varPhi \colon [0, +\infty) \to [0, +\infty)$ be convex, with $\varPhi(0) = 0$, $\varPhi(1) = 1$ and $\varPhi(+\infty) = +\infty$. Let $\bar{H}_{\varPhi,c} \colon L^{\infty} \times \mathcal{Q} \to [0, +\infty)$ be as in (15). Then:*

a) $\bar{H}_{\varPhi,c}(X, Q)$ is positively homogeneous and subadditive with respect to X

b) $\|X_n\|_\infty \leq k,\ X_n \xrightarrow{P} X \Rightarrow \bar{H}_{\Phi,c}(X_n, Q) \to \bar{H}_{\Phi,c}(X, Q)$
c) $\bar{H}_{\Phi,c}(X, Q)$ *is quasiconcave with respect to* Q
d) $\bar{H}_{\Phi,c}(X, Q)$ *is upper semicontinuous with respect to* Q *in the* $\sigma(L^1, L^\infty)$ *topology*
e) if $c\colon \mathcal{Q} \to [0, +\infty]$ *has the WC property, then the upper level sets*

$$U_\gamma := \{Q \in \mathcal{Q} \mid \bar{H}_{\Phi,c}(X, Q) \geq \gamma\}$$

are compact in the $\sigma(L^1, L^\infty)$ *topology.*

Proof a) follows in the same way as the corresponding properties of the Orlicz premium. Positive homogeneity is immediate, while

$$\mathbb{E}_Q\left[\Phi\left(\frac{X+Y}{\bar{H}_{\Phi,c}(X,Q)+\bar{H}_{\Phi,c}(Y,Q)}\right)\right] \leq \mathbb{E}_Q\left[\frac{\bar{H}_{\Phi,c}(X,Q)}{\bar{H}_{\Phi,c}(X,Q)+\bar{H}_{\Phi,c}(Y,Q)}\Phi\left(\frac{X}{\bar{H}_{\Phi,c}(X,Q)}\right)\right]$$
$$+\,\mathbb{E}_Q\left[\frac{\bar{H}_{\Phi,c}(Y,Q)}{\bar{H}_{\Phi,c}(X,Q)+\bar{H}_{\Phi,c}(Y,Q)}\Phi\left(\frac{Y}{\bar{H}_{\Phi,c}(Y,Q)}\right)\right] = 1 + c(Q),$$

hence

$$\bar{H}_{\Phi,c}(X, Q) + \bar{H}_{\Phi,c}(Y, Q) \geq \bar{H}_{\Phi,c}(X + Y, Q).$$

b) For a fixed Q, the non-normalized Orlicz premium $H_{Q,\Phi,c}$ has the Lebesgue property; see for example [21].
c) Let $Q_1, Q_2 \in \mathcal{Q}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. We have that

$$\mathbb{E}_{\alpha Q_1+\beta Q_2}\left[\Phi\left(\frac{X}{\min(\bar{H}_{\Phi,c}(X,Q_1),\bar{H}_{\Phi,c}(X,Q_2))}\right)\right]$$
$$= \alpha\,\mathbb{E}_{Q_1}\left[\Phi\left(\frac{X}{\min(\bar{H}_{\Phi,c}(X,Q_1),\bar{H}_{\Phi,c}(X,Q_2))}\right)\right] + \beta\,\mathbb{E}_{Q_2}\left[\Phi\left(\frac{X}{\min(\bar{H}_{\Phi,c}(X,Q_1),\bar{H}_{\Phi,c}(X,Q_2))}\right)\right]$$
$$\geq \alpha\,\mathbb{E}_{Q_1}\left[\Phi\left(\frac{X}{\bar{H}_{\Phi,c}(X,Q_1)}\right)\right] + \beta\,\mathbb{E}_{Q_2}\left[\Phi\left(\frac{X}{\bar{H}_{\Phi,c}(X,Q_2)}\right)\right]$$
$$= \alpha(1 + c(Q_1)) + \beta(1 + c(Q_2)) \geq 1 + c\left(\alpha Q_1 + \beta Q_2\right),$$

hence

$$\min(\bar{H}_{\Phi,c}(X, Q_1), \bar{H}_{\Phi,c}(X, Q_2)) \leq \bar{H}_{\Phi,c}(X, \alpha Q_1 + \beta Q_2),$$

that is, $\bar{H}_{\Phi,c}(X, Q)$ is quasiconcave with respect to Q.
d) For a fixed $X \in L^\infty$, let

$$U_\gamma = \left\{Q \in \mathcal{Q} \mid \bar{H}_{\Phi,c}(X, Q) \geq \gamma\right\},$$

and let $Q_n \in U_\gamma$. Then

$$\mathbb{E}_{Q_n}\left[\Phi\left(\frac{X}{\gamma}\right)\right] \geq 1 + c(Q_n). \tag{16}$$

Let $Q_n \to Q$ weakly. Since $X \in L^\infty$, Φ is monotone and finite-valued and c is weakly lower semicontinuous, we have

$$0 \leq \limsup_{n\to+\infty}\left\{\mathbb{E}_{Q_n}\left[\Phi\left(\frac{X}{\gamma}\right)\right] - 1 - c(Q_n)\right\}$$
$$= \mathbb{E}_Q\left[\Phi\left(\frac{X}{\gamma}\right)\right] - 1 - \limsup_{n\to+\infty} c(Q_n) \leq \mathbb{E}_Q\left[\Phi\left(\frac{X}{\gamma}\right)\right] - 1 - c(Q),$$

hence $\bar{H}_{\Phi,c}(X,Q) \geq \gamma$.
e) Let again $Q_n \in U_\gamma$. From (16), we get

$$c(Q_n) \leq \mathbb{E}_{Q_n}\left[\Phi\left(\frac{X}{\gamma}\right)\right] - 1 \leq \Phi\left(\frac{\operatorname{ess\,sup} X}{\gamma}\right) - 1.$$

From the WC property, there exists a subsequence $Q_{n_k} \to \bar{Q}$. Since from item d) U_γ is weakly closed, it follows that $\bar{Q} \in U_\gamma$, which proves the thesis. □

Theorem 3 *Let $c\colon \mathcal{Q} \to [0,+\infty]$ be convex and lower semicontinuous with respect to the $\sigma(L^1, L^\infty)$ topology, with $\inf_{Q\in\mathcal{Q}} c(Q) = 0$. Let $\Phi\colon [0,+\infty) \to [0,+\infty)$ be convex, with $\Phi(0) = 0$, $\Phi(1) = 1$ and $\Phi(+\infty) = +\infty$. Let $\bar{H}_{\Phi,c}\colon L^\infty \times \mathcal{Q} \to [0,+\infty)$ be as in (15) and let as before*

$$H_{\Phi,c}(X) = \sup_{Q\in\mathcal{Q}} \bar{H}_{\Phi,c}(X,Q).$$

Then, $H_{\Phi,c}$ is monotone, positively homogeneous and subadditive. Moreover, for each $X \in L^\infty_+$ and $k > 0$, we have:

a) $H_{\Phi,c}(k) = k$
b) $H_{\Phi,c}(X+k) \leq H_{\Phi,c}(X) + k$
c) $c(Q^*) = 0 \Rightarrow H_{\Phi,c}(X) \geq \mathbb{E}_{Q^*}(X)$
d) $\inf_{Q\in\mathcal{Q}} H_{Q,\Phi,0}(X) \leq H_{\Phi,c}(X) \leq \sup_{Q\in\mathcal{Q}} H_{Q,\Phi,0}(X)$
e) $\|X_n\|_\infty \leq k$, $X_n \xrightarrow{P} X \Rightarrow H_{\Phi,c}(X) \leq \liminf H_{\Phi,c}(X_n)$*, that is $H_{\Phi,c}$ has the Fatou property*
f) if c has the WC property, then there exists $\bar{Q} \in \mathcal{Q}$ such that

$$H_{\Phi,c}(X) = \bar{H}_{\Phi,c}(X,\bar{Q})$$

Proof Monotonicity, positive homogeneity and subadditivity follow immediately from Proposition 2. If $X = k$ P-a.s., then

$$\sup_{Q\in\mathcal{Q}}\left\{\mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] - c(Q)\right\} = \Phi(1) - \sup_{Q\in\mathcal{Q}} c(Q) = 1,$$

from which a) follows. b) follows from subadditivity and a). From Proposition 2,

$$H_{\Phi,c}(X) = \sup_{Q\in\mathcal{Q}} H_{Q,\Phi,c(Q)}(X) \geq H_{Q^*,\Phi,0}(X) \geq \mathbb{E}_{Q^*}(X),$$

that is c). Similarly, d) follows from

$$H_{\Phi,c}(X) = \sup_{Q\in\mathcal{Q}} H_{Q,\Phi,c(Q)}(X) \leq \sup_{Q\in\mathcal{Q}} H_{Q,\Phi,0}(X)$$

and the observation that

$$\sup_{Q\in\mathcal{Q}}\left\{\mathbb{E}_Q\left[\Phi\left(\frac{X}{\inf\limits_{Q\in\mathcal{Q}} H_{Q,\Phi,0}(X)}\right)\right] - c(Q)\right\}$$
$$\geq \sup_{Q\in\mathcal{Q}}\left\{\mathbb{E}_Q\left[\Phi\left(\frac{X}{H_{Q,\Phi,0}(X)}\right)\right] - c(Q)\right\} = 1,$$

which shows that $\inf_{Q\in\mathcal{Q}} H_{Q,\Phi,0}(X) \leq H_{\Phi,c}(X)$.
e) Let $\|X_n\|_\infty \leq k$, $X_n \to X$ in probability. Then,

$$H_{\Phi,c}(X_n) \leq \gamma \Rightarrow H_{Q,\Phi,c(Q)}(X_n) \leq \gamma \Rightarrow H_{Q,\Phi,c(Q)}(X) \leq \gamma \Rightarrow H_{\Phi,c}(X) \leq \gamma,$$

from which the thesis follows.
f) From Lemma 6 we have that, if c has the WC property, then the upper level sets of $\bar{H}_{\Phi,c}(X,Q)$ are weakly compact. Moreover, $\bar{H}_{\Phi,c}(X,Q)$ is weakly upper semicontinuous, hence the maximum is attained. □

4.2 Translation invariance of robust Orlicz premia

Orlicz premia H_Φ are not in general translation invariant, with the exception of the case $\Phi(x) = x$, as was shown by [21]. If $\Phi(x) = x$, also robust Orlicz premia of the form $H_{\Phi,\mathcal{S}}(X)$ are translation invariant. The following theorem gives a partial converse:

Theorem 4 *Let $\Phi\colon [0,+\infty) \to [0,+\infty)$ be convex and twice differentiable, with $\Phi(0) = 0$ and $\Phi(1) = 1$. Let $\mathcal{S} \subset \mathcal{Q}$ be weakly compact, and let $H_{\Phi,\mathcal{S}}$ be defined as in (8). If $H_{\Phi,\mathcal{S}}$ is translation invariant, then $\Phi(x) = x$.*

Proof Let us consider a dyadic random variable X which can take the values $a > 0$ and $b > 0$, and let $p_Q := Q(X = a)$. The Orlicz premium $H_{Q,\Phi,0}(X)$ is the unique solution to the equation

$$p_Q\Phi\left(\frac{a}{H_{Q,\Phi,0}(X)}\right) + (1-p_Q)\Phi\left(\frac{b}{H_{Q,\Phi,0}(X)}\right) = 1,$$

and the robust Orlicz premium $H_{\Phi,\mathcal{S}}(X)$ is given by

$$H_{\Phi,\mathcal{S}}(X) = \sup_{Q\in\mathcal{S}} H_{Q,\Phi,0}(X).$$

Since $\mathcal{S}$ is weakly compact the supremum is attained, so $\exists\bar{Q} \in \mathcal{S}$ such that

$$p_{\bar{Q}}\Phi\left(\frac{a}{H_{\Phi,\mathcal{S}}(X)}\right) + (1-p_{\bar{Q}})\Phi\left(\frac{b}{H_{\Phi,\mathcal{S}}(X)}\right) = 1.$$

Notice that since $H_{Q,\Phi,0}(X)$ is monotonic with respect to first-order stochastic dominance, it holds that

$$p_{\bar{Q}} = \min_{Q\in\mathcal{S}} Q(X = a) = \min_{Q\in\mathcal{S}} p_Q.$$

Assuming translation invariance, for each $h > 0$,

$$\sup_{Q\in\mathcal{S}}\left\{p_Q\Phi\left(\frac{a+h}{H_{\Phi,\mathcal{S}}(X)+h}\right) + (1-p_Q)\Phi\left(\frac{b+h}{H_{\Phi,\mathcal{S}}(X)+h}\right)\right\} = 1,$$

and also

$$p_{\bar{Q}}\Phi\left(\frac{a+h}{H_{\Phi,\mathcal{S}}(X)+h}\right)+(1-p_{\bar{Q}})\Phi\left(\frac{b+h}{H_{\Phi,\mathcal{S}}(X)+h}\right)=1, \tag{17}$$

since trivially

$$\min_{Q\in\mathcal{S}} Q(X+h=a+h)=\min_{Q\in\mathcal{S}} Q(X=a)=p_{\bar{Q}}.$$

Summing up, for each $a>0$ and $b>0$, there exists $p_{\bar{Q}}$ such that for each $h>0$ equation (17) holds. Differentiating twice with respect to h we get

$$p_{\bar{Q}}\Phi''\left(\frac{a+h}{H_{\Phi,\mathcal{S}}(X)+h}\right)(H_{\Phi,\mathcal{S}}(X)-a)^2+(1-p_{\bar{Q}})\Phi''\left(\frac{b+h}{H_{\Phi,\mathcal{S}}(X)+h}\right)(H_{\Phi,\mathcal{S}}(X)-b)^2=0,$$

and from the generality of a,b and h we get $\Phi''(x)=0$ for each $x>0$. □

The theorem above shows basically that in order to have translation invariance of $H_{\Phi,\mathcal{S}}(X)$, it is necessary to stick to the case $\Phi(x)=x$. And indeed, under this hypothesis, $H_{\Phi,\mathcal{S}}$ is translation invariant. A further natural question is under which hypotheses on $c\colon \mathcal{Q}\to\mathbb{R}$ (or, alternatively, on $\beta:\mathcal{Q}\to[0,1]$) the robust Orlicz premium $H_{\Phi,c}$ (or $H_{\Phi,\beta}$) with $\Phi(x)=x$ is translation invariant. As shown in the following example, linearity of Φ does not guarantee translation invariance of $H_{\Phi,c}$ and $H_{\Phi,\beta}$. Notice that, for $\Phi(x)=x$, the robust Orlicz premium $H_{\Phi,\beta}$ has the explicit formulation

$$H_{\Phi,\beta}(X)=\sup_{Q\in\mathcal{Q}}\{\beta(Q)\,\mathbb{E}_Q[X]\}. \tag{18}$$

(18) can be interpreted as a worst-case β-discounted expectation.

Example 4 Consider $\Omega=\{\omega_1;\omega_2\}$, $\mathcal{Q}=\{Q_1;Q_2\}$, with $Q_1(\omega_1)=Q_1(\omega_2)=1/2$, $Q_2(\omega_1)=1/4, Q_2(\omega_2)=3/4$ and

$$c(Q_1)=0,\quad c(Q_2)=\frac{1}{4},$$
$$\beta(Q_1)=1,\quad \beta(Q_2)=\frac{5}{6},$$

and assume $\Phi(x)=x$.

Taking $k=3$ and $X=\begin{cases}0, & \text{on } \omega_1,\\ 1, & \text{on } \omega_2,\end{cases}$ it is easy to verify that $H_{\Phi,c}(X+k)=\frac{7}{2}<H_{\Phi,c}(X)+k=\frac{18}{5}$ and $H_{\Phi,\beta}(X+k)=\frac{7}{2}<H_{\Phi,\beta}(X)+k=\frac{29}{8}$. Hence, translation invariance fails for both $H_{\Phi,c}$ and $H_{\Phi,\beta}$ (although Φ is linear).

One may conjecture that, under linearity of Φ, translation invariance of $H_{\Phi,\beta}$ (resp. $H_{\Phi,c}$) is equivalent to $\beta\equiv 1$ (resp. $c\equiv 0$). This is essentially true when one considers only "relevant" probability measures. In order to clarify this point, we provide the following example.

Example 5 Consider $\Omega = \{\omega_1; \omega_2\}$, $\mathcal{Q} = \{Q_1; Q_2\}$, with $Q_1(\omega_1) = Q_1(\omega_2) = 1/2$, $Q_2(\omega_1) = 1/4, Q_2(\omega_2) = 3/4$ and

$$\beta(Q_1) = 1, \quad \beta(Q_2) = \frac{1}{2},$$

and assume $\Phi(x) = x$.

For any $X \in L^\infty_+$ and $k \geq 0$, it holds that $H_{\Phi,\beta}(X + k) = \mathbb{E}_{Q_1}(X) + k = H_{\Phi,\beta}(X) + k$. In other words, translation invariance holds for any $X \in L^\infty_+$ and $k \geq 0$ even if $\beta(Q_2) < 1$. The underlying reason is that here

$$\mathbb{E}_{Q_1}(X) \geq \beta(Q_2)\, \mathbb{E}_{Q_2}(X), \qquad \text{for any } X \in L^\infty_+,$$

that is, the probability Q_2 is "irrelevant" since the maximum is always attained in Q_1.

Assume $\Phi(x) = x$. The following result gives a *necessary* condition for $H_{\Phi,c}$ (respectively $H_{\Phi,\beta}$) to be translation invariant. This condition is the one we have conjectured and essentially requires that probability measures having $c(\cdot) > 0$ (resp. $\beta(\cdot) < 1$) can be dropped since they are always "irrelevant". The condition $c(Q) = 0$ (resp. $\beta(Q) = 1$) for any $Q \in \mathcal{Q}$ is always *sufficient* for the translation invariance of $H_{\Phi,c}$ (respectively $H_{\Phi,\beta}$).

Proposition 3 *Assume that $\Phi(x) = x$ and that the supremum is always attained in (12) (resp. (13)).*

If translation invariance holds for $H_{\Phi,c}$ (respectively $H_{\Phi,\beta}$), then there does not exist any $\bar{Q} \in \mathcal{Q}$ such that $c(\bar{Q}) > 0$ (resp. $\beta(\bar{Q}) < 1$) and such that for some $X \in L^\infty_+$ the maximum in (12) (resp. (13)) is realized only in such $\bar{Q}$.

Proof Let us focus first on the case of $H_{\Phi,c}$ and assume by contradiction that there exists a $\bar{Q} \in \mathcal{Q}$ such that $c(\bar{Q}) > 0$ and such that for some $\bar{X} \in L^\infty_+$ the maximum in (12) is realized only in such $\bar{Q}$. Hence,

$$H_{\Phi,c}(\bar{X}) = \frac{\mathbb{E}_{\bar{Q}}[\bar{X}]}{1 + c(\bar{Q})} > \frac{\mathbb{E}_{Q}[\bar{X}]}{1 + c(Q)},$$

for any $Q \in \mathcal{Q} \setminus \{\overline{Q}\}$.

We are now going to prove that the following two cases cannot occur when translation invariance holds: (a) $H_{\Phi,c}(\bar{X} + k) = \frac{\mathbb{E}_{\widetilde{Q}_k}[\bar{X}+k]}{1+c(\widetilde{Q}_k)}$ with $c(\widetilde{Q}_k) = 0$ for some $k > 0$; (b) $H_{\Phi,c}(\bar{X} + k) = \frac{\mathbb{E}_{\widetilde{Q}_k}[\bar{X}+k]}{1+c(\widetilde{Q}_k)}$ with $c(\widetilde{Q}_k) > 0$ for any $k > 0$.

(a) If $H_{\Phi,c}(\bar{X} + k) = \frac{\mathbb{E}_{\widetilde{Q}_k}[\bar{X}+k]}{1+c(\widetilde{Q}_k)}$ with $c(\widetilde{Q}_k) = 0$ for some $k > 0$, then, for such k,

$$H_{\Phi,c}(\bar{X} + k) - k = \frac{\mathbb{E}_{\widetilde{Q}_k}[\bar{X} + k]}{1 + c(\widetilde{Q}_k)} - k = \mathbb{E}_{\widetilde{Q}_k}[\bar{X} + k] - k = \mathbb{E}_{\widetilde{Q}_k}[\bar{X}] < H_{\Phi,c}(\bar{X}).$$

(b) If $H_{\Phi,c}(\bar{X}+k) = \frac{\mathbb{E}_{\widetilde{Q}_k}[\bar{X}+k]}{1+c(\widetilde{Q}_k)}$ with $c(\widetilde{Q}_k) > 0$ for any $k > 0$, then, for any k,

$$H_{\Phi,c}(\bar{X}+k)-k = \frac{\mathbb{E}_{\widetilde{Q}_k}[\bar{X}+k]}{1+c(\widetilde{Q}_k)}-k = \frac{\mathbb{E}_{\widetilde{Q}_k}[\bar{X}]-kc(\widetilde{Q}_k)}{1+c(\widetilde{Q}_k)} < \frac{\mathbb{E}_{\widetilde{Q}_k}[\bar{X}]}{1+c(\widetilde{Q}_k)} < H_{\Phi,c}(\bar{X}).$$

In both cases, translation invariance fails. This proves the thesis. The case of $H_{\Phi,\beta}$ can be proved similarly. □

4.3 Axiomatization of robust Orlicz premia and robust shortfall risk

The aim of this subsection is to discuss the axiomatic foundation of robust Orlicz premia introduced in Definition 2. In view of the correspondence between Orlicz premia and utility-based shortfall risk studied in Subsection 3.1, we focus on the axiomatization of robust shortfall risk, introduced in [17] by means of the acceptance set

$$A_{\rho_\mathcal{S}} := \{X \in L^\infty \mid \mathbb{E}_Q[\ell(X)] \le 0, \forall Q \in \mathcal{S}\},$$

for a suitable $\mathcal{S} \subset \mathcal{Q}$. A related characterization result has been provided by [27] for the particular case of robust entropic risk measures (called entropy coherent risk measures in [27]), which are of the form

$$\rho(X) := \sup_{Q\in\mathcal{S}} e_{Q,\bar{\gamma}}(X),$$

where

$$e_{Q,\bar{\gamma}}(X) = \bar{\gamma}\log\left(\mathbb{E}_Q\left[\exp\left(\frac{X}{\bar{\gamma}}\right)\right]\right), \qquad \bar{\gamma} > 0.$$

The Laeven-Stadje Theorem axiomatizes robust entropic risk measures in the wider class of robust monetary risk measures, which are defined by

$$\rho(X) = \sup_{Q\in\mathcal{S}} \rho_Q(X),$$

where each ρ_Q is translation invariant, Q-monotone in the sense that

$$X \le Y \ Q\text{-a.s.} \Rightarrow \rho_Q(X) \le \rho_Q(Y),$$

and Q-law invariant, in the sense that

$$F_X^Q = F_Y^Q \Rightarrow \rho_Q(X) = \rho_Q(Y),$$

where for each $X \in L^\infty$ and $Q \in \mathcal{S}$ we denote by F_X^Q the distribution function of X under the probabilistic model Q. In the axiomatization of [27], a crucial role is played by the axiom of acceptance neutrality:

$$\text{if } F_X^{Q_1} = F_Y^{Q_2} \text{ then } X \in A_{\rho_{Q_1}} \Leftrightarrow Y \in A_{\rho_{Q_2}}, \tag{19}$$

which is much stronger than Q-law invariance of each ρ_Q. Indeed, acceptance neutrality implies that the acceptance sets at the level of distributions are the same: for each $Q_1, Q_2 \in \mathcal{S}$,

$$\{F \in \mathcal{M}_{1,c} \mid \rho_{Q_1}(F) \leq 0\} = \{F \in \mathcal{M}_{1,c} \mid \rho_{Q_2}(F) \leq 0\}.$$

In words, we may say that a random variable X is acceptable under a model Q if and only if its distribution F_X^Q belongs to a common set of acceptable distributions. Under the acceptance neutrality axiom, it is immediate to provide an axiomatization of robust shortfall risk.

Theorem 5 *Let $X \in L^\infty$ and let*

$$\rho(X) = \sup_{Q \in \mathcal{S}} \rho_Q(X),$$

where for each $Q \in \mathcal{S}$
i) ρ_Q is translation invariant with $\rho_Q(X) = 0$ if $X = 0 \quad Q$-a.s.
ii) ρ_Q is Q-monotone
iii) $F_X^Q = F_Y^Q \Rightarrow \rho_Q(X) = \rho_Q(Y)$.
Assume moreover that acceptance neutrality (19) holds, and that there exists $\overline{Q} \in \mathcal{S}$ such that $\mathcal{A}_{\rho_{\overline{Q}}}$ satisfies the hypotheses of Weber's Theorem (Theorem 1). Then,

$$\rho(X) = \sup_{Q \in \mathcal{S}} \inf \{m \in \mathbb{R} | \, \mathbb{E}_Q[\ell(X - m)] \leq 0\}.$$

Thus, ρ is a robust shortfall risk.

Proof From Weber's Theorem it follows that there exists a nondecreasing $\ell \colon \mathbb{R} \to \mathbb{R}$ with $\ell(-\infty) < 0 < \ell(+\infty)$ such that

$$\rho_{\overline{Q}}(X) = \inf\{m \in \mathbb{R} \mid \mathbb{E}_{\overline{Q}}[\ell(X - m)] \leq 0\},$$

that is,

$$\mathcal{A}_{\rho_{\overline{Q}}} = \left\{F \in \mathcal{M}_{1,c} \mid \int \ell(x) \mathrm{d}F(x) \leq 0\right\}.$$

From acceptance neutrality, for each $Q \in \mathcal{S}$, it holds that $\mathcal{A}_{\rho_Q} = \mathcal{A}_{\rho_{\overline{Q}}}$, so

$$\rho_Q(X) = \inf\{m \in \mathbb{R} \mid \mathbb{E}_Q[\ell(X - m)] \leq 0\},$$

from which the thesis follows. □

4.4 The case of multiple Φ

In the previous subsections we considered robustified versions of Orlicz premia that originated from ambiguity on the true probabilistic model Q, modeled by a worst-case approach with an additive or multiplicative penalty function. In this subsection we consider the situation in which there is only one probabilistic model P, but the decision-maker is uncertain about his Young function Φ. Indeed, he has a multiplicity of possible Young functions, and takes a worst-case approach. This setting is analogous to a worst-case approach under a multiplicity of utility functions as considered by [29] and [8].

Definition 4 (Robust Orlicz premium - multiple Φ) Let $X \in L^\infty_+$ and $\mathfrak{F} = \{\Phi_\alpha\}_{\alpha\in I}$, where each Φ_α is convex and satisfies the usual assumptions $\Phi_\alpha(0) = 0$, $\Phi_\alpha(1) = 1$ and $\Phi_\alpha(+\infty) = +\infty$. Let $\sup_{\Phi\in\mathfrak{F}} \Phi(x) < +\infty$, for each $x > 0$. We define

$$H_{\mathfrak{F}}(X) := \inf\left\{k > 0 \,\middle|\, \sup_{\Phi\in\mathfrak{F}} \mathbb{E}_P\left[\Phi\left(\frac{X}{k}\right)\right] \leq 1\right\}.$$

We state the following result:

Proposition 4 *Let $X \in L^\infty_+$ and let $H_{\mathfrak{F}}(X)$ be as in Definition 4. Then:*

a) $H_{\mathfrak{F}}(X)$ is the unique solution to the equation

$$\sup_{\Phi\in\mathfrak{F}} \mathbb{E}_P\left[\Phi\left(\frac{X}{H_{\mathfrak{F}}(X)}\right)\right] = 1$$

b) It holds that

$$H_{\mathfrak{F}}(X) = \sup_{\Phi\in\mathfrak{F}} H_\Phi(X)$$

c) $H_{\mathfrak{F}}(X)$ is law invariant, monotone, positively homogeneous and subadditive
d) $H_{\mathfrak{F}}(k) = k$, for each $k \in \mathbb{R}$
e) $H_{\mathfrak{F}}(X) \geq \mathbb{E}_P[X]$
f) $\|X_n\|_\infty \leq k$, $X_n \xrightarrow{P} X \Rightarrow H_{\mathfrak{F}}(X) \leq \liminf H_{\mathfrak{F}}(X_n)$, that is, $H_{\mathfrak{F}}$ has the Fatou property.

Proof a) Let

$$g(k) = \sup_{\phi\in\mathfrak{F}} \mathbb{E}_P\left[\Phi(kX)\right].$$

Then, from the assumption $\sup_{\Phi\in\mathfrak{F}} \Phi(x) < +\infty$, we obtain that g is finite and convex, and hence continuous, from which the thesis follows.
b) Let $\bar{\Phi} \in \mathfrak{F}$. Since

$$\mathbb{E}_P\left[\bar{\Phi}\left(\frac{X}{H_{\bar{\Phi}}(X)}\right)\right] = 1,$$

clearly

$$\sup_{\Phi\in\mathfrak{F}} \mathbb{E}_P\left[\Phi\left(\frac{X}{H_{\bar{\Phi}}(X)}\right)\right] \geq 1,$$

which implies $H_{\bar{\Phi}}(X) \leq H_{\mathfrak{F}}(X)$, thus

$$\sup_{\bar{\Phi}\in\mathfrak{F}} H_{\bar{\Phi}}(X) \leq H_{\mathfrak{F}}(X).$$

On the other hand,

$$\sup_{\Phi\in\mathfrak{F}} \mathbb{E}_P\left[\Phi\left(\frac{X}{\sup_{\bar{\Phi}\in\mathfrak{F}} H_{\bar{\Phi}}(X)}\right)\right] \leq \sup_{\Phi\in\mathfrak{F}} \mathbb{E}_P\left[\Phi\left(\frac{X}{H_{\Phi}(X)}\right)\right] = 1,$$

from which we obtain the reverse inequality.
c), d), e), f) follow from b), as in the proof of Theorem 3. □

5 Robustification of Haezendonck-Goovaerts risk measures

Let us now recall the definition of the Haezendonck-Goovaerts risk measure:

$$\Pi_{Q,\Phi,c}(X) := \inf_{x\in\mathbb{R}} \left\{x + H_{Q,\Phi,c}\left((X-x)^+\right)\right\}$$

(see [20] or [1, 2]). Notice that the parameter $\alpha \in (0,1)$ in the original definition has been replaced here by $c \in [0,+\infty)$, in order to be consistent with (14) in Definition 3. Haezendonck-Goovaerts risk measures are coherent risk measures, they provide a natural extension to the Rockafellar-Uryasev [35] construction of Conditional Value-at-Risk, and their properties are well-known. In particular, their dual representation is given by

$$\Pi_{Q,\Phi,c}(X) = \max_{R\in\mathcal{R}_Q} \mathbb{E}_R[X],$$

where

$$\mathcal{R}_Q = \{R \ll P : \mathbb{E}_R[X] \leq H_{Q,\Phi,c}(X),\ \forall X \in L^\infty_+\}. \tag{20}$$

It is natural to suggest a similar construction in the robust case, so we define the robust Haezendonck-Goovaerts risk measure as

$$\Pi_{\Phi,c}(X) := \inf_{x\in\mathbb{R}} \left\{x + H_{\Phi,c}\left((X-x)^+\right)\right\}. \tag{21}$$

Proposition 5 *Let $X \in L^\infty(\Omega,\mathcal{F},P)$. Furthermore, let $c\colon \mathcal{Q} \to [0,+\infty]$ be convex and lower semicontinuous with respect to the $\sigma(L^1,L^\infty)$ topology, with $\min_{Q\in\mathcal{Q}} c(Q) = c(Q^*) = 0$, and let $\Phi\colon [0,+\infty) \to [0,+\infty)$ be convex, with $\Phi(0) = 0$, $\Phi(1) = 1$ and $\Phi(+\infty) = +\infty$. Finally, let $H_{\Phi,c}$ be as in (9) and $\Pi_{\Phi,c}$ as in (21). Then:*

a) $\Pi_{\Phi,c}\colon L^\infty(\Omega,\mathcal{F},P) \to \mathbb{R}$ is a coherent risk measure. Moreover, $\Pi_{\Phi,c}(X) \geq \mathbb{E}_{Q^*}[X]$.
b) $\Pi_{\Phi,c}\colon L^\infty \to \mathbb{R}$ has the Fatou property

c) For each $X \in L^\infty$, it holds that

$$\Pi_{\Phi,c}(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}_Q[X],$$

where

$$\mathcal{R} = \left\{ Q \ll P : \mathbb{E}_Q[X] \leq H_{\Phi,c}(X) \text{ for any } X \in L^\infty_+ \right\}. \tag{22}$$

d) If c satisfies the WC property, then we have the minimax identity

$$\Pi_{\Phi,c}(X) = \sup_{Q \in \mathcal{Q}} \Pi_{Q,\Phi,c(Q)}(X).$$

Proof a) Monotonicity, positive homogeneity, convexity and translation invariance (hence, coherence) of $\Pi_{\Phi,c}$ are straightforward. By Theorem 3 it follows that

$$x + H_{\Phi,c}\left((X-x)^+\right) \geq x + \mathbb{E}_{Q^*}\left[(X-x)^+\right] \geq x + \mathbb{E}_{Q^*}[X-x] = \mathbb{E}_{Q^*}[X],$$

hence $\Pi_{\Phi,c}(X) \geq \mathbb{E}_{Q^*}[X]$.
b) Let $\|X_n\|_\infty \leq k$, $X_n \xrightarrow{P} X$. Then there exists a subsequence (n_k) such that $X_{n_k} \searrow X$. Hence, by monotonicity of $\Pi_{\Phi,c}$,

$$\begin{aligned} \Pi_{\Phi,c}(X) &= \inf_{x \in \mathbb{R}} \left\{ x + H_{\Phi,c}\left((X-x)^+\right) \right\} \leq \inf_{x \in \mathbb{R}} \left\{ x + H_{\Phi,c}\left((X_{n_k}-x)^+\right) \right\} \\ &= \Pi_{\Phi,c}(X_{n_k}) \leq c, \end{aligned}$$

for any n_k. The thesis then follows immediately.
c) Following [1], we notice that $\Pi_{\Phi,c}$ can be seen as the inf-convolution of the proper convex functionals f and g defined by

$$\begin{aligned} f(X) &= H_{\Phi,c}\left(X^+\right) \\ g(X) &= \begin{cases} x, & \text{if } X = x, \ P\text{-a.s.}, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Indeed,

$$\Pi_{\Phi,c}(X) = \inf_{x \in \mathbb{R}} \left\{ x + H_{\Phi,c}\left((X-x)^+\right) \right\} = \inf_{Y \in L^\infty} \left\{ f(X-Y) + g(Y) \right\}.$$

We get

$$(\Pi_{\Phi,c})^*(\varphi) = f^*(\varphi) + g^*(\varphi) = \begin{cases} f^*(\varphi), & \text{if } \mathbb{E}[\varphi] = 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

for $\varphi \in L^1$, where f^* and g^* denote the convex conjugates of f and g. Furthermore,

$$\begin{aligned} f^*(\varphi) &= \sup_{X\in L^\infty}\{\mathbb{E}[\varphi X] - f(X)\} = \sup_{X\in L^\infty}\left\{\mathbb{E}[\varphi X] - H_{\Phi,c}\left(X^+\right)\right\} \\ &= \sup_{\lambda\geq 0}\left\{\lambda \sup_{X\in L^\infty:\|X\|_\infty\leq 1}\left\{E[\varphi X] - H_{\Phi,c}\left(X^+\right)\right\}\right\} \qquad (23) \\ &= \begin{cases} 0, & \text{if } H_{\Phi,c}(X^+) \geq \mathbb{E}[\varphi X] \text{ for any } X\in L^\infty \text{ s.t. } \|X\|_\infty \leq 1, \\ +\infty, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 0, & \text{if } H_{\Phi,c}(X) \geq \mathbb{E}[\varphi X] \text{ for any } X\in L^\infty_+, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned}$$

where equality (23) is due to positive homogeneity of $H_{\Phi,c}$. The thesis follows.
d) If c satisfies the WC property, it follows from Theorem 3 that there exists $\bar{Q}$ such that

$$H_{\Phi,c}(X) = H_{\bar{Q},\Phi,c(\bar{Q})}(X).$$

Hence,

$$\Pi_{\Phi,c}(X) = \inf_{x\in\mathbb{R}}\left\{x + H_{\bar{Q},\Phi,c(\bar{Q})}\left((X-x)^+\right)\right\} \leq \sup_{Q\in\mathcal{Q}} \Pi_{Q,\Phi,c(Q)}(X).$$

On the other hand, examining the sets of generalized scenarios in the dual representations (20) and (22), it is easy to see that

$$\bigcup_{Q\in\mathcal{Q}} \mathcal{R}_Q \subseteq \mathcal{R},$$

from which it follows that

$$\Pi_{\Phi,c}(X) \geq \sup_{Q\in\mathcal{Q}} \Pi_{Q,\Phi,c(Q)}(X).$$

□

5.1 The case of multiple Φ

We can also robustify the Haezendonck-Goovaerts risk measure in the case of multiple Φ. Just like in Section 4, an important difference with the case of ambiguity over Q is that we will now obtain a coherent risk measure that is *law invariant*. We define:

$$\Pi_{\mathfrak{F}}(X) := \inf_{x\in\mathbb{R}}\left\{x + H_{\mathfrak{F}}\left((X-x)^+\right)\right\}, \qquad (24)$$

with $H_{\mathfrak{F}}[X]$ as defined before.

Proposition 6 *Let $X \in L^\infty_+$ and $\mathfrak{F} = \{\Phi_\alpha\}_{\alpha\in I}$, where each Φ_α is convex and satisfies the usual assumptions $\Phi_\alpha(0) = 0$, $\Phi_\alpha(1) = 1$ and $\Phi_\alpha(+\infty) = +\infty$. Let $\sup_{\Phi\in\mathfrak{F}} \Phi(x) < +\infty$, for each $x > 0$. Let $\Pi_{\mathfrak{F}}(X)$ be as in (24). Then:*

a) $\Pi_{\mathfrak{F}}\colon L^\infty \to \mathbb{R}$ *is a coherent and law invariant risk measure. Moreover,* $\Pi_{\mathfrak{F}}(X) \geq E_P[X]$.
b) $\Pi_{\mathfrak{F}}\colon L^\infty \to \mathbb{R}$ *has the Fatou property*
c) For each $X \in L^\infty$*, it holds that*

$$\Pi_{\mathfrak{F}}(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}_Q(X),$$

where

$$\mathcal{R} = \left\{ Q \ll P : \mathbb{E}_Q[X] \leq H_{\mathfrak{F}}(X) \text{ for any } X \in L^\infty_+ \right\}.$$

Proof Similar to Proposition 5.

6 Applications to Pareto optimal allocations and optimal risk sharing

The aim of this section is to study and compare Pareto optimal allocations and optimal risk sharing under classical Orlicz premia and Haezendonck-Goovaerts risk measures and under the corresponding robustified premia/risk measures.

We recall that, given two premia/risk measures π_1 and π_2, these problems essentially consist in solving

$$\inf_{X_1, X_2 : X_1 + X_2 = X} \{\pi_1(X_1) + \pi_2(X_2)\}, \tag{25}$$

also known as the *optimal risk decomposition* or *inf-convolution*. Indeed, as shown by [15] and [23] (see also [5] and [6]), for convex functionals π_1 and π_2, an optimal risk decomposition is a Pareto optimal allocation and vice versa, and from an optimal risk decomposition one can build an optimal risk sharing rule.

In the following, any pair $(X_1; X_2)$ satisfying $X_1 + X_2 = X$ will be called an *admissible allocation* and, with slight abuse of notation, problem (25) will be called *optimal risk sharing* and a solution $(X_1^*; X_2^*)$ to it an *optimal risk decomposition*.

By [15] (see also [5] and [6]), given two Gateaux differentiable convex functionals[4] $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ an admissible pair $(X_1^*; X_2^*) \in L^\infty \times L^\infty$ is optimal for problem (25) if and only if

$$\pi_1'(X_1^*) = \pi_2'(X_2^*),$$

where π_i' stands for the Gateaux derivative of π_i. We recall that $\pi_i'(X)$ is such that $h'(0) = \mathbb{E}[\pi_i'(X) \cdot V]$, where $h(t) = \pi_i(X + tV)$ and $t \in \mathbb{R}$.

The following example illustrates optimal risk decomposition under Orlicz premia in a classical and robust framework. To this end, we recall that H_Φ is

[4] We note that Theorem 10 of [15] holds also without assuming that π_1 and π_2 satisfy constancy (that is $\pi(c) = c$ for any $c \in \mathbb{R}$) and translation invariance.

Table 1 Optimal risk decomposition in Example 6

	$\mathcal{Q}$	H	$(X_1^*; X_2^*)$	$H_1(X_1^*) + H_2(X_2^*)$
non-robust	$\{P\}$	H_1, H_2	$(X-c; c)$ for $c \in [0,1]$	$3/2$
robust	$\{P, Q\}$	$H_{1,\beta_1}, H_{2,\beta_2}$	$\begin{cases} (1-\frac{2}{\sqrt{7}}; \frac{2}{\sqrt{7}}); & \text{on } \omega_1 \\ (0; 2); & \text{on } \omega_2 \end{cases}$	$\frac{1}{4} + \frac{\sqrt{7}}{2}$

law invariant and, for a differentiable Young function Φ, the Gateaux derivative of $H_\Phi(X)$ is given by

$$H'_\Phi(X) = \frac{\Phi'\left(\frac{X}{H_\Phi(X)}\right)}{\mathbb{E}\left[\frac{X}{H_\Phi(X)}\Phi'\left(\frac{X}{H_\Phi(X)}\right)\right]}$$

(see [25] and [2]).

Example 6 Take $\Omega = \{\omega_1; \omega_2\}$, $P(\omega_1) = 1/2$ and $Q(\omega_1) = 1/4$.

For $i = 1, 2$, take $\Phi_i(x) = x^i$ and

$$H_i(X) = H_{\Phi_i}(X) \text{ (classical Orlicz premium);}$$
$$H_{i,\beta_i}(X) = H_{\Phi_i,\beta_i}(X) \text{ (robust Orlicz premium);}$$

where, in the second case, $\mathcal{Q} = \{P, Q\}$ and

$$\beta_1(R) = \begin{cases} \frac{1}{2}, \ R = P, \\ 1, \ R = Q, \end{cases} \qquad \beta_2(R) = \begin{cases} 1, \ R = P, \\ \frac{1}{3}, \ R = Q. \end{cases}$$

Then,

$$H_{1,\beta_1}(X) = \sup_{R \in \mathcal{Q}} \{\beta_1(R)\, \mathbb{E}_R[X]\}$$
$$H_{2,\beta_2}(X) = \sup_{R \in \mathcal{Q}} \left\{(\beta_2(R))^{1/2} \left(\mathbb{E}_R\left[X^2\right]\right)^{1/2}\right\}$$

are in general no longer law invariant.

It is easy to verify that any pair $(X_1^*; X_2^*) \in L_+^\infty \times L_+^\infty$ satisfying

$$1 = \frac{X_2^*}{\|X_2^*\|_2}, \qquad X_1^* = X - X_2^*,$$

is an optimal risk decomposition of X. We deduce, therefore, that any pair $(X - c; c)$, with $c \in [0; \operatorname{ess\,inf} X]$, is an optimal risk decomposition of X in the classical case.

Take now

$$X(\omega) = \begin{cases} 1, \ \omega = \omega_1, \\ 2, \ \omega = \omega_2. \end{cases}$$

We summarize in Table 1 the explicit optimal risk decomposition. It is worth to notice that, as one might expect, the optimal risk decomposition in the robust case is different from the one in the non-robust case. Furthermore, the minimal total premium to be paid for the robust case would be $\frac{1}{4} + \frac{\sqrt{7}}{2}$, while in the non-robust case it would be $\frac{3}{2}$ (i.e., lower). This result can be viewed as a consequence of the fact that in the robust case, model uncertainty is taken into account.

Let us now consider Haezendonck-Goovaerts risk measures. We recall that both classical and robust Haezendonck-Goovaerts risk measures defined on L^∞ are coherent and lower semicontinuous. Moreover, both in the classical and in the robust case, Haezendonck-Goovaerts risk measures defined on L^∞ can be represented as

$$\Pi_{\Phi,c}(X) = \sup_{R \in \mathcal{R}} \mathbb{E}_R\left[X\right],$$

where

$$\mathcal{R} = \left\{ R \ll P : E_R\left[X\right] \leq H_{\Phi,c}(X) \text{ for any } X \in L^\infty_+ \right\}$$

(see Proposition 17 of [1] and Proposition 5 in the present paper). By Theorem 3.1 of [23], it then follows that an admissible pair $(X_1^*; X_2^*)$ is a Pareto optimal allocation if and only if

$$\Pi_{1,c}(X_1^*) = \mathbb{E}_{Q^*}\left[X_1^*\right] \quad \text{and} \quad \Pi_{2,c}(X_2^*) = \mathbb{E}_{Q^*}\left[X_2^*\right],$$

for some $Q^* \in \mathcal{R}_1 \cap \mathcal{R}_2$.

Acknowledgements We are very grateful to Freddy Delbaen, Saul Jacka, Bernt Øksendal, and Stefan Weber for discussions, their comments and suggestions and to a Guest Editor and two anonymous referees for comments that have improved the paper. We are also grateful to conference participants at the 2nd EAJ Conference in Vienna (September 2014), the 2nd Conference on Stochastics of Environmental and Financial Economics in Oslo (April 2015), the ZIF Research Group Workshop on Mathematics and Financial Economics in Bielefeld (May 2015), the 19th International IME Congress in Liverpool (June 2015), the OR2015 International Conference on Operations Research in Vienna (September 2015), the 39th AMASES Meeting in Padova (September 2015), the International Conference on Probability Theory and Statistics in Tbilisi (September 2015), and the Workshop on Dependence and Risk Measures in Milano (November 2015). This research was funded in part by the Netherlands Organization for Scientific Research under grant NWO VIDI 2009 (Laeven).

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