The Compressible to Incompressible Limit of 1D Euler Equations: the Non Smooth Case

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June 23, 2015

Abstract

We prove a rigorous convergence result for the compressible to incompressible limit of weak entropy solutions to the isothermal 1D Euler equations.

Keywords: Incompressible limit, Compressible Euler Equations, Hyperbolic Conservation Laws

2010 MSC: 35L65, 35Q35

1 Introduction

The compressible to incompressible limit in Euler equations is widely studied in the literature, see for instance the well known results [6, 9, 10, 11, 12, 14], the more recent [15], the review [13] and the references therein. For the case of Navier–Stokes equations, we refer for instance to [7, 8]. The usual setting considers regular solutions, whose existence is proved only for a finite time, to the compressible equations in 2 or 3 space dimensions. As the Mach number vanishes, these solutions are proved to converge to the solutions of the incompressible Euler equations, while only a weak* convergence holds for the pressure.

Here, we recover the same convergence results, in a 1D setting, but within the framework of *weak entropy solutions* proved to exist for *all times*. In particular, $\mathbf{L^1}$-convergence is proved for all positive times and the space regularity of solutions is $\mathbf{L^1} \cap \mathbf{BV}$.

In the present 1D setting, a compressible to incompressible limit over all the real line is of no interest. One may then consider two compressible immiscible fluids, say a gas and a liquid, letting the liquid tend to become incompressible. The limiting procedure basically yields a boundary value problem for the gas, while the boundary between the fluids turns into a solid wall. Indeed, an incompressible fluid in 1D is essentially a solid.

Therefore, below we consider a droplet of a compressible inviscid fluid that fills the segment $[a, b]$ and is surrounded by another compressible inviscid fluid filling the rest of the real line. The fluids are assumed to be immiscible. For simplicity, we refer to the fluid forming the droplet as to a *liquid*, while its complement is labeled as *gas*. In the isentropic (or isothermal) approximation,

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this whole system can be described by

$$
\begin{cases}
\begin{cases}
\partial_t \rho_g + \partial_x (\rho_g\, v_g) = 0 \\
\partial_t (\rho_g\, v_g) + \partial_x \left(\rho_g\, {v_g}^2 + p_g(\rho_g)\right) = 0
\end{cases} & x \in \mathbb{R} \setminus [a(t), b(t)] \qquad \text{gas} \\[2ex]
\begin{cases}
\partial_t \rho_l + \partial_x (\rho_l\, v_l) = 0 \\
\partial_t (\rho_l\, v_l) + \partial_x \left(\rho_l\, {v_l}^2 + p_l(\rho_l)\right) = 0
\end{cases} & x \in [a(t), b(t)] \qquad \text{liquid} \\[2ex]
\begin{cases}
\dot a(t) = v_g\left(t, a(t)-\right) \\
\dot b(t) = v_g\left(t, b(t)+\right)
\end{cases} & \text{kinetic relations (immiscibility)} \\[2ex]
\begin{cases}
v_g\left(t, a(t)-\right) = v_l\left(t, a(t)+\right) \\
v_g\left(t, b(t)+\right) = v_l\left(t, b(t)-\right)
\end{cases} & \text{mass conservation} \\[2ex]
\begin{cases}
p_g\left(\rho_g(t, a(t)-)\right) = p_l\left(\rho_l(t, a(t)+)\right) \\
p_g\left(\rho_g(t, b(t)+)\right) = p_l\left(\rho_l(t, b(t)-)\right)
\end{cases} & \text{momentum conservation.}
\end{cases}
\tag{1.1}
$$

Here, ρ_l is the density of liquid in the droplet, v_l is its speed and $p_l = p_l(\rho_l)$ is the pressure law, while ρ_g, v_g and p_g denote the analogous quantities for the gas. The above conditions not only ensure the conservation of mass, but also prevent any exchange of matter between the two fluids. In particular, they do not mix but, clearly, there is exchange of information between the two fluids, thanks to the global conservation of momentum. Note also that these conditions obviously ensure the energy conservation at the interfaces, while shocks lead to the dissipation of energy in the interior of the 2 fluids. Indeed, as is well know [5], energy plays here the role of the mathematical entropy. The above kinetic relations provide a link between the fluid interfaces and the fluid speeds.

Passing to the incompressible limit in the liquid phase, we expect to obtain the system

$$
\begin{cases}
\begin{cases}
\partial_t \rho_g + \partial_x (\rho_g\, v_g) = 0 \\
\partial_t (\rho_g\, v_g) + \partial_x \left(\rho_g\, {v_g}^2 + p_g(\rho_g)\right) = 0
\end{cases} & x \in \mathbb{R} \setminus [a(t), b(t)] \qquad \text{gas;} \\[2ex]
\begin{cases}
\rho_l = \bar\rho_l \\
\dot v_l = \frac{p_g\left(t, a(t-)\right) - p_g\left(t, b(t+)\right)}{\left(b(t) - a(t)\right)\bar\rho_l}
\end{cases} & x \in [a(t), b(t)] \qquad \text{liquid;} \\[2ex]
\begin{cases}
v_g\left(t, a(t)-\right) = \dot a(t) \\
v_g\left(t, b(t)+\right) = \dot b(t) \\
v_l(t) = \dot a(t) = \dot b(t)
\end{cases} & \text{mass and momentum conservation,}
\end{cases}
\tag{1.2}
$$

consisting of a conservation law describing the compressible gas coupled with an ordinary differential equation for the incompressible droplet. We recall that (1.2) is known to be well posed, see [1, Proposition 3.1].

Since we assume the two phases immiscible, a natural choice is to pass to Lagrangian coordinates, so that the interfaces between the two phases become stationary.

In these coordinates, introducing the total mass m of the liquid, the above system (1.1) reads:

$$\begin{cases} \begin{cases} \partial_t \tau_g - \partial_z v_g = 0 \\ \partial_t v_g + \partial_z p_g(\tau_g) = 0 \end{cases} & z \in \mathbb{R} \setminus [0, m] & \text{gas} \\ \begin{cases} \partial_t \tau_l - \partial_z v_l = 0 \\ \partial_t v_l + \partial_z p_l(\tau_l) = 0 \end{cases} & z \in [0, m] & \text{liquid} \\ \begin{cases} v_g\,(t, 0-) &=& v_l\,(t, 0+) \\ v_g\,(t, m+) &=& v_l\,(t, m-) \end{cases} & & \begin{array}{l}\text{mass}\\ \text{conservation}\end{array} \\ \begin{cases} p_g\left(\tau_g(t, 0-)\right) &=& p_l\left(\tau_l(t, 0+)\right) \\ p_g\left(\tau_g(t, m+)\right) &=& p_l\left(\tau_l(t, m-)\right) \end{cases} & & \begin{array}{l}\text{momentum}\\ \text{conservation,}\end{array} \end{cases} \tag{1.3}$$

while its incompressible limit in the liquid phase, i.e., (1.2) in Lagrangian coordinates, becomes

$$\begin{cases} \begin{cases} \partial_t \tau_g - \partial_z v_g = 0 \\ \partial_t v_g + \partial_z p_g(\tau_g) = 0 \end{cases} & z \in \mathbb{R} \setminus [0, m] & \text{gas;} \\ \dot{v}_l = \frac{p_g(t,0-) - p_g(t,m+)}{m} & & \text{liquid;} \\ \begin{cases} v_g\,(t, 0-) &=& v_l(t) \\ v_g\,(t, m+) &=& v_l(t) \end{cases} & & \begin{array}{l}\text{mass and momentum}\\ \text{conservation.}\end{array} \end{cases} \tag{1.4}$$

Problems (1.1) and (1.3), respectively (1.2) and (1.4), are well posed in $\mathbf{L^1}$ globally in time, see [4, Theorem 2.5], respectively [2, Theorem 3.6].

The main result of this paper states the rigorous convergence of (1.1) to (1.2) or, equivalently, of (1.3) to (1.4). In this very singular limit, the sound speed $\bar{c}$ in the liquid phase tends to $+\infty$; the density converges to a fixed reference value $\bar{\rho}$; the graph of the pressure law becomes vertical and the eigenvectors of the Jacobian of the flow tend to coalesce. In [9, 10, 11, 12, 13], this behavior is reproduced requiring that $\|\nabla p\| \to +\infty$. Here, similarly, we fix the constant initial specific volume $\bar{\tau}_l$ of the liquid and locally describe the liquid phase in Lagrangian coordinates in a neighborhood of $\bar{\tau}$ through the pressure law

$$p^\eta(\tau) = \bar{p} - \eta^2(\tau - \bar{\tau}_l) \quad \text{so that} \quad \tau^\eta(p) = \bar{\tau}_l - \frac{1}{\eta^2}\,(p - \bar{p}) \quad \text{where} \quad \eta = \frac{\bar{c}}{\bar{\tau}_l}\,, \tag{1.5}$$

where, $\bar{c}$ is the Eulerian sound speed and η is the Lagrangian sound speed.

Informally, we present here the main result of the present paper. The rigorous statement is in Section 2. More precisely, points 1. and 2. in Theorem 1.1 are consequences of Theorem 2.3, while point 3. follows from Theorem 2.4.

Theorem 1.1. *Fix in the gas phase a pressure law satisfying* **(p)** *and the pressure law* (1.5) *in the liquid phase.*

1. *There exists a nontrivial set of initial data $\mathcal{D}$ such that the Cauchy problem consisting of* (1.1) *is well posed for all initial data in $\mathcal{D}$ and for all $\bar{c}$ sufficiently large.*
2. *The initial data in $\mathcal{D}$ in the liquid phase are incompressible, i.e., they have a constant density $\bar{\rho}_l$ and speed $\bar{v}_l$.*
3. *For any initial data in $\mathcal{D}$, the solution to the Cauchy problem* (1.1) *converges as $\bar{c} \to +\infty$ and the limit solves* (1.2)*. In particular, in the liquid phase:*
   (i) the density converges in $\mathbf{L^1_{loc}}$ to $\bar{\rho}_l$, which is constant in space and time;

(ii) *the velocity converges in* $\mathbf{L^1_{loc}}$ *to a function* $v_l = v_l(t)$ *which is constant in space and whose time evolution is specified in* (1.2)*;*

(iii) *the pressure weak* converges in* $\mathbf{L}^\infty$ *to the linear interpolation of the values of the gas pressure at the liquid boundaries, for almost every time* $t \geq 0$*.*

Remark that the convergences stated above extend the classical result [9] to the case of *non smooth*, albeit 1D, solutions.

The present setting comprises also the case of the compressible to incompressible limit on the free boundary value problem in which only the liquid is present and it is constrained in the moving strip $[a(t), b(t)]$, with prescribed pressure along the boundaries.

As usual in the study of non smooth solutions to 1D systems of conservation laws, see [3, 5], we exploit the wave front tracking techniques, so that the desired estimates follow from suitable bounds on the approximate solutions. A first key analytical difficulty in obtaining the present result lies in the need for bounds in the total variation of solutions that are uniform when the sound speed tends to $+\infty$. A very careful choice of the parametrization allows to control the interactions of waves against the phase boundaries.

Within the liquid phase, waves may well bounce back and forth with a diverging speed. This phenomenon requires an *ad hoc* procedure to bound the total number of interaction points in the ε-approximations. In turn, we are able to use the wave front tracking algorithm in [3], but without the need of nonphysical waves. A further consequence of these possible bounces is that the total variation of the pressure along lines $x = \bar{x}$ grows unboundedly, which is why only a weak* convergence of the pressure is possible. Nevertheless, we recover the Newton law for the incompressible liquid thanks to the bounds on the total variation and to the conservative form of (1.3) and (1.4).

The next Section presents the rigorous setting for (1.3) and (1.4). First, the well posedness of (1.3) proved in [4, Theorem 2.5] is improved to obtain uniform estimates that allow to pass to the incompressible limit. Then, Theorem 2.4 presents our main result. Section 3 is devoted to the analytical proofs. There, we highlight the key modifications to the standard procedure [3, 5], without repeating the now classical wave front tracking constructions.

2 The Compressible $\to$ Incompressible Limit

Throughout, we identify the state u of the fluid by (τ, v) or (p, v), once an invertible pressure law $p = p(\tau)$ is assigned. Given a function $u \colon \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$, we denote

$$(\tau_g, v_g) = u_g = u\, \chi_{\mathbb{R}\setminus[0,m]} \qquad \text{and} \qquad (\tau_l, v_l) = u_l = u\, \chi_{[0,m]}\,.$$

In Lagrangian coordinates, a standard assumption on the pressure law is

(p) $p \in \mathbf{C^4}(\mathbb{R}^+; \mathbb{R}^+)$ is such that $p'(\tau) < 0$ and $p''(\tau) > 0$ for all $\tau > 0$.

Recall the definition of solution or (1.3):

Definition 2.1. *Fix a gas state* $\bar{u}_g$ *and a liquid state* $\bar{u}_l$*. Denote* $\bar{u} = \bar{u}_g + (\bar{u}_l - \bar{u}_g)\, \chi_{[0,m]}$*. Let* $T > 0$ *be fixed. By solution to* (1.3) *we mean a map*

$$u \in \mathbf{C^0}\left([0,T]; \bar{u} + (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R})\right)$$

such that:

1. *it is a weak entropy solution to* $\begin{cases} \partial_t \tau_g - \partial_z v_g = 0 \\ \partial_t v_g + \partial_z p_g(\tau_g) = 0 \end{cases}$ *in* $[0,T] \times (\mathbb{R} \setminus [0,m])$*;*

2. *it is a weak entropy solution to* $\begin{cases} \partial_t \tau_l - \partial_z v_l = 0 \\ \partial_t v_l + \partial_z p_l(\tau_l) = 0 \end{cases}$ *in* $[0,T] \times [0,m]$*;*

3. *for a.e.* $t \in [0,T]$*, the conditions at the junction* $\begin{cases} p_g\left(\tau_g(t,0-)\right) &= p_l\left(\tau_l(t,0+)\right) \\ v_g\left(t,0-\right) &= v_l\left(t,0+\right) \end{cases}$ *and* $\begin{cases} p_g\left(\tau_g(t,m+)\right) &= p_l\left(\tau_l(t,m-)\right) \\ v_g\left(t,m-\right) &= v_l\left(t,m+\right) \end{cases}$ *are satisfied.*

Above, the continuity in $\mathbf{C^0}\left([0,T];\bar{u} + (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^+ \times \mathbb{R})\right)$ is understood with respect to the $\mathbf{L^1}$ norm.

In the case of the mixed model (1.4), we adapt [2, Definition 3.1] to the present situation.

Definition 2.2. *Fix a gas state* $\bar{u}_g$*. Let* $T > 0$ *be fixed. By solution to* (1.4) *we mean a pair of functions*

$$(u, v_l) \in \mathbf{C^0}\left([0,T];\bar{u}_g + (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R} \setminus [0,m];\mathbb{R}^+ \times \mathbb{R})\right) \times \mathbf{W^{1,\infty}}([0,T];\mathbb{R}) \tag{2.1}$$

such that:

1. u *is a weak entropy solution to* $\begin{cases} \partial_t \tau_g - \partial_z v_g = 0 \\ \partial_t v_g + \partial_z p_g(\tau_g) = 0 \end{cases}$ *in* $[0,T] \times (\mathbb{R} \setminus [0,m])$*;*

2. v_l *is a solution to* $\dot{v}_l = \frac{1}{m}\left(p_g\left(\tau_g(s,0-)\right) - p_g\left(\tau_g(s,m+)\right)\right)$ *on* $[0,T]$*;*

3. *the conditions at the interface* $\begin{cases} v_g\left(t,0-\right) &= v_l(t) \\ v_g\left(t,m+\right) &= v_l(t) \end{cases}$ *are satisfied for a.e.* $t \in [0,T]$*.*

The definitions of solutions to the Cauchy problems for (1.3) and (1.4) are an immediate adaptation of the definitions above.

Aiming at the rigorous compressible $\to$ incompressible limit, we need new estimates on the compressible problem (1.3), improving analogous results in [4, Theorem 2.5].

Theorem 2.3. *Fix a state* $\bar{u}_g$ *in the gas phase and a state* $\bar{u}_l$ *in the liquid phase. Let* p_g *satisfy* **(p)** *and* p^η *be as in* (1.5)*. Then, there exist positive* $\Delta, \delta_g, \eta_*, L$ *such that for all* $\eta > \eta_*$ *and for suitable positive* δ_l^η*,* L^η*, problem* (1.3)– (1.4) *generates a semigroup*

$$S^\eta \colon \mathbb{R}^+ \times \mathcal{D}^\eta \to \mathcal{D}^\eta$$

with the following properties:

1. $\mathcal{D}^\eta \supseteq \left\{u \in \bar{u} + (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};\mathbb{R}^+ \times \mathbb{R})\colon \mathrm{TV}(u_g) < \delta_g \text{ and } \mathrm{TV}(u_l) < \delta_l^\eta\right\}$*;*
2. S^η *is a semigroup:* $S_0^\eta = \mathbf{Id}$ *and* $S_{t_1}^\eta \circ S_{t_2}^\eta = S_{t_1+t_2}^\eta$*;*
3. S^η *is Lipschitz in u: for any* u^1, u^2 *in* $\mathcal{D}^\eta$ *and for all* $t \in \mathbb{R}^+$
$$\left\|S_t^\eta u^1 - S_t^\eta u^2\right\|_{\mathbf{L^1}} \leq L^\eta \cdot \left\|u^1 - u^2\right\|_{\mathbf{L^1}};$$
4. S^η *is Lipschitz in t: for any u in* $\mathcal{D}^\eta$ *and for all* $t_1, t_2 \in \mathbb{R}^+$*, setting* $(\tau^\eta, v^\eta)(t) = S_t^\eta u$
$$\begin{aligned} \left\|\tau_g^\eta(t_1) - \tau_g^\eta(t_2)\right\|_{\mathbf{L^1}(\mathbb{R}\setminus[0,m])} &\leq L\,|t_1 - t_2| \\ \left\|\tau_l^\eta(t_1) - \tau_l^\eta(t_2)\right\|_{\mathbf{L^1}([0,m])} &\leq \frac{1}{\eta}\,L\,|t_1 - t_2| \\ \left\|v^\eta(t_1) - v^\eta(t_2)\right\|_{\mathbf{L^1}(\mathbb{R})} &\leq L\,|t_1 - t_2|; \end{aligned}$$

5. *if $u \in \mathcal{D}^\eta$ is piecewise constant then, for t small, the map $S_t^\eta u$ locally coincides with the standard Lax solutions to the Riemann problems for* (1.3) *at the points of jump, at $x = 0$ and at $x = m$;*
6. *for all $u \in \mathcal{D}^\eta$, the map $t \to S_t^\eta u$ is an entropy solution to* (1.3) *with initial datum u in the sense of Definition 2.1;*
7. *if $u \in \mathcal{D}^\eta$, then for all $t \geq 0$ and $\eta \geq \eta_*$, calling $\left(\tau^\eta(t), v^\eta(t)\right) = S_t^\eta u$*

$$\begin{aligned} \mathrm{TV}\left(p^\eta\left(\tau_l^\eta(t,\cdot)\right)\right) + \mathrm{TV}\left(p_g\left(\tau_g^\eta(t,\cdot)\right)\right) &< \Delta \\ \eta\,\mathrm{TV}\left(v_l^\eta(t,\cdot)\right) + \eta^2\,\mathrm{TV}\left(\tau_l^\eta(t,\cdot)\right) &< \Delta\,; \end{aligned} \tag{2.2}$$

8. *if $u \in \mathcal{D}^\eta$, then for all $\eta \geq \eta_*$, calling $\left(\tau^\eta(t), v^\eta(t)\right) = S_t^\eta u$*

$$\mathrm{TV}\left(\tau_g^\eta(\cdot,x)\right) + \mathrm{TV}\left(v_g^\eta(\cdot,x)\right) + \mathrm{TV}\left(p_g\left(\tau_g^\eta(\cdot,x)\right)\right) \leq \Delta \qquad x \in \mathbb{R}\setminus[0,m]\,; \tag{2.3}$$

9. *for a.e. x_1, x_2 with either $x_1, x_2 < 0$ or $x_1, x_2 > m$*

$$\int_0^t \left\| u_g^\eta(s,x_2) - u_g^\eta(s,x_1)\right\| \mathrm{d}s \leq L\,|x_2 - x_1|\,. \tag{2.4}$$

The points 2., 3., 5. and 6. follow from [4, Theorem 2.5]. To obtain the *a priori* estimates in points 1., 7. and 8., we have to substantially improve the wave front tracking construction in [4], devising and exploiting a different parametrization of the Lax curves. These estimates allow to obtain the key Lipschitz type estimates at point 4. and 9., where the dependence of the Lipschitz constants on η, where present, is explicit.

Remark that point 8. above may not hold in the liquid phase, since the total variation of the pressure therein may well blow up. The proof of Theorem 2.3 is deferred to Section 3.

The bounds above have a key role in proving our main result below.

Theorem 2.4. *Fix a state $\bar u_g$ in the gas phase and a state $\bar u_l$ in the liquid phase. Let p satisfy* **(p)** *and p^η be as in* (1.5). *Then, with reference to the semigroup S^η and its domain $\mathcal{D}^\eta$ constructed in Theorem 2.3, there exist positive δ_* and η_* such that for all $u_g^o \in \bar u_g^o + \mathbf{L^1}(\mathbb{R}\setminus[0,m];\mathbb{R}^+\times\mathbb{R})$ with $\mathrm{TV}(u_g^o) < \delta_*$:*

1. *the function $u^o = u_g^o\,\chi_{\mathbb{R}\setminus[0,m]} + \bar u_l\,\chi_{[0,m]}$ is in $\mathcal{D}^\eta$ for all $\eta > \eta_*$;*
2. *the limit $(\tau_*, v_*) = \lim_{\eta\to+\infty} S_t^\eta u^o$ is well defined and in $\mathcal{D}^\eta$ for all $t \in \mathbb{R}^+$ and $\eta > \eta_*$;*
3. *$\tau_*(t,x) = \bar\tau_l$ and $v_*(t,x) = v_l(t)$ for all $(t,x) \in \mathbb{R}^+\times[0,m]$, with $v_l \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R})$;*
4. *for all $\bar x \in \mathbb{R}$, the traces converge in the sense that*

$$\lim_{\eta\to+\infty}\int_0^t \left\| u^\eta(s,\bar x\pm) - u_*(s,\bar x\pm)\right\| \mathrm{d}s = 0\,;$$

5. *let $u_g = u_*\,\chi_{\mathbb{R}\setminus[0,m]}$. Then, the map $t \to \left(u_g(t), v_l(t)\right)$ solves* (1.4) *in the sense of Definition 2.2;*
6. *as $\eta \to +\infty$, the map $p^\eta \circ \tau_l^\eta$ weak* converges in $\mathbf{L^\infty}$ to a function $p_l \in \mathbf{L^\infty}(\mathbb{R}^+\times[0,m];\mathbb{R}^+)$ such that*

$$\begin{aligned} & p_l(t,0+) \text{ is well defined and equals } p_g\left(\tau_g(t,0-)\right) \text{ for a.e. } t \in \mathbb{R}^+\,,\\ & p_l(t,m-) \text{ is well defined and equals } p_g\left(\tau_g(t,m+)\right) \text{ for a.e. } t \in \mathbb{R}^+\,,\\ & p_l(t,x) = \left(1 - \tfrac{x}{m}\right)\,p_g\left(\tau_g(t,0-)\right) + \tfrac{x}{m}\,p_g\left(\tau_g(t,m+)\right) \text{ for a.e. } t \in \mathbb{R}^+\,. \end{aligned}$$

The proof of Theorem 2.4 is deferred to Section 3.

Observe that from the Eulerian coordinates' point of view, the locations of the boundaries of the liquid phase can be recovered through a time integration:

$$\begin{aligned} a^\eta(t) &= a_o + \int_0^t v_g^\eta\,(s,0-)\,ds & \qquad a(t) &= a_o + \int_0^t v_l\,(s)\,ds \\ b^\eta(t) &= b_o + \int_0^t v_g^\eta\,(s,m+)\,ds & \qquad b(t) &= b_o + \int_0^t v_l\,(s)\,ds\,. \end{aligned}$$

The boundaries of the two phases turns out to be Lipschitz continuous functions and moreover $a^\eta \to a$, $b^\eta \to b$ uniformly on compact sets as $\eta \to +\infty$. Moreover, point 2. in Definition 2.2 justifies the usual relation between the acceleration of the droplet and the pressure difference on its sides.

3 Technical Details

We collect below a few facts about he p-system in Lagrangian coordinates

$$\left\{ \begin{array}{l} \tau_t - v_x = 0 \\ v_t + \left[p(\tau)\right]_x = 0\,. \end{array} \right. \qquad t \in [0,+\infty[\ ,\ x \in \mathbb{R}$$

The eigenvalues are

$$\lambda_1(\tau,v) = -\sqrt{-p'(\tau)} \qquad\qquad \lambda_2(\tau,v) = \sqrt{-p'(\tau)} \tag{3.1}$$

so that the Lax shock and rarefaction curves in the (p,v)–plane are

$$\begin{aligned} S_1(u,\sigma) &= \left[\begin{array}{l} p-\sigma \\ v - \sqrt{\left(\tau\,(p-\sigma) - \tau(p)\right)\sigma} \end{array}\right] & \text{if } \sigma &< 0 \\ R_1(u,\sigma) &= \left[\begin{array}{l} p-\sigma \\ v - \displaystyle\int_p^{p-\sigma} \sqrt{-\tau'(\pi)}\,\mathrm{d}\pi \end{array}\right] & \text{if } \sigma &> 0 \\ S_2(u,\sigma) &= \left[\begin{array}{l} p+\sigma \\ v - \sqrt{-\left(\tau\,(p+\sigma) - \tau(p)\right)\sigma} \end{array}\right] & \text{if } \sigma &< 0 \\ R_2(u,\sigma) &= \left[\begin{array}{l} p+\sigma \\ v + \displaystyle\int_p^{p+\sigma} \sqrt{-\tau'(\pi)}\,\mathrm{d}\pi \end{array}\right] & \text{if } \sigma &> 0 \end{aligned} \tag{3.2}$$

Fix a state $\bar{u}_l$ and using the pressure (1.5), the Lax curves take the form

$$\mathcal{L}_1(u,\sigma) = \left[\begin{array}{c} p-\sigma \\ v+\frac{\sigma}{\eta} \end{array}\right] \qquad \text{and} \qquad \mathcal{L}_2(u,\sigma) = \left[\begin{array}{c} p+\sigma \\ v+\frac{\sigma}{\eta} \end{array}\right]. \tag{3.3}$$

Proof of Theorem 2.3. In view of [4, Theorem 2.5], it is sufficient to prove only the estimates at 1. and (2.2). To this aim, we construct approximate solutions through an algorithm different from that in [4]. Once a subsequence of these approximate solutions is proved to converge, by the properties of the Standard Riemann Semigroup [3], it is known that the present approximations converge to the solution constructed in [4].

Definition of the Algorithm. Fix $\varepsilon > 0$. We approximate the initial datum u^o by a sequence u^o_ε of piecewise constant initial data with a finite number of discontinuities such that $\left\|u^o_\varepsilon - u^o\right\|_{\mathbf{L}^1} \leq \varepsilon$. At each point of jump in the approximate initial condition, we solve the corresponding Riemann problem using the front tracking algorithm as stated in [3, Chapter 7]. We approximate each rarefaction wave by a rarefaction fan by means of (non-entropic) shock waves. Similarly to what happens in the general case [3, Chapter 5], there exists a constant $\delta^o > 0$ such that each of the above Riemann problems has a unique approximate solution as long as $\mathrm{TV}(u^o) < \delta^o$.

This construction can be extended up to the first time t_1 at which two waves interact. At time t_1, the so constructed functions are piecewise constant with a finite number of discontinuities. We can thus iterate the previous construction at any subsequent interaction, provided suitable upper bounds on the total variation of the approximate solutions are available. These bounds essentially rely on the interaction estimates below.

As it is usual in this contest, see [3], we may assume that no more than 2 waves interact at any interaction point. Moreover, rarefaction waves, once arisen, are not further split even if their size exceeds the threshold ε after subsequent interactions, with other waves or with the phase boundaries. Besides, we call $\hat\lambda^\eta$ an upper bound for the moduli of the propagation speeds of all waves.

Specific to the present construction, is our choice to parametrize the Lax curves as in (3.2) and, hence, waves' sizes are measured through the pressure variation σ between the two states on the sides of the wave.

Interaction Estimates. We recall the classical Glimm interaction estimates, see [3, Chapter 7, formulæ (7.31)- (7.32)], which holds for any smooth parametrization:

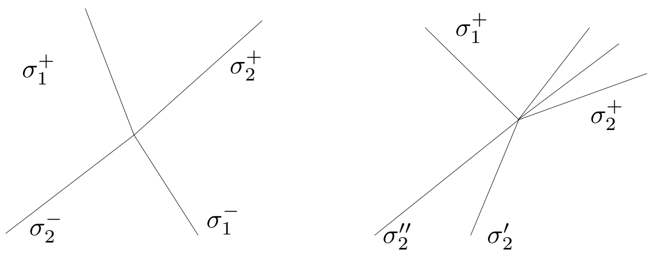

Figure 1: Left, an interaction between waves of different families. Right, an interaction between waves of the same family.

$$\begin{array}{rcl} \left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| & \leq & C \cdot \left|\sigma_1^- \sigma_2^-\right| \\ \left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| & \leq & C \cdot \left|\sigma' \sigma''\right| \end{array} \tag{3.4}$$

where we used the notation described in Figure 1. The estimates on the waves' sizes in the case of interactions involving the interface are as follows. In the case of Figure 2,

$$\left|\sigma_1^+\right| \leq C \left|\sigma_2^-\right| \qquad \text{and} \qquad \left|\sigma_2^+\right| \leq C \left|\sigma_2^-\right| \tag{3.5}$$

while in the case of Figure 3,

$$\left|\sigma_1^+\right| + \left|\sigma_2^+\right| = \left|\sigma_2^-\right|. \tag{3.6}$$

Note that the constant C above depends only on quantities related to the gas phase, is uniformly bounded when u_g varies in a compact set and, in particular, is independent from η.

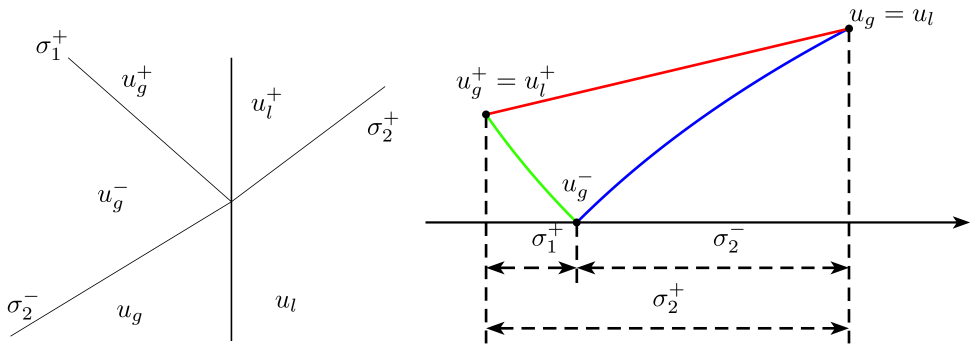

Figure 2: Notation used in the interaction estimates involving the phase interface.

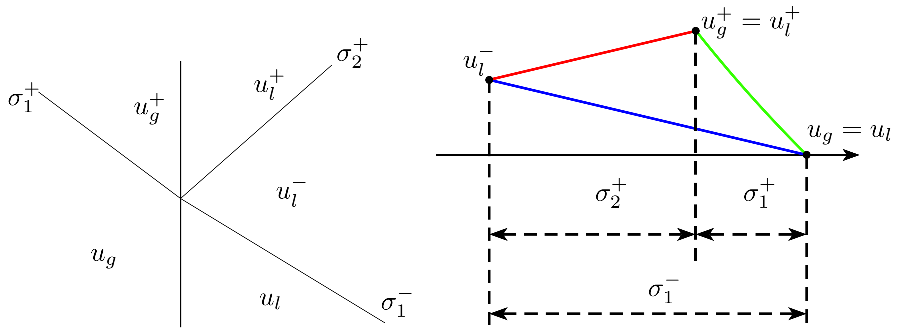

Figure 3: Interaction estimate at the gas–liquid interface in the (p, v) plane. Note that the two states u_g^+ and u_l^+ have different specific volumes but, due to the interface conditions, share the same pressure and the same speed.

Bounds on the Total Variation. We follow the classical techniques based on Glimm functionals, see [3]. To this aim, introduce the quantities

$$
\begin{array}{lll}
V_g^- = \sum_{i=1}^{2} \sum_{x_\alpha < 0} K_i^{g-} \left|\sigma_{i,\alpha}\right| & V_l = K_l \sum_{i=1}^{2} \sum_{x_\alpha \in [0,m]} \left|\sigma_{i,\alpha}\right| & V_g^+ = \sum_{i=1}^{2} \sum_{x_\alpha > m} K_i^{g+} \left|\sigma_{i,\alpha}\right| \\
Q_g^- = \sum_{\mathcal{A}_g^-} \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right| & & Q_g^+ = \sum_{\mathcal{A}_g^+} \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right| \\
\Upsilon_g^- = V_g^- + H\, Q_g^- & & \Upsilon_g^+ = V_g^+ + H\, Q_g^+
\end{array}
$$

where $\mathcal{A}_g^-$, respectively $\mathcal{A}_g^+$, is the set of pairs of approaching waves both with support in $x < 0$, respectively $x > m$. The weights $K_i^{g\pm}$, K_l and H are defined below. They are positive numbers independent from η.

Define $\Upsilon = \Upsilon_g^- + V_l + \Upsilon_g^+$ and consider the various case:

Case 1: An interaction in the interior of the liquid phase. then, thanks to the choice (1.5), we have that $\Delta\Upsilon = \Delta V_l = 0$.

Case 2: An interaction at the interface. Consider first the case of Figure 2. Then,

$$
\begin{array}{rcl}
\Delta\Upsilon & = & \Delta\Upsilon_g^- + \Delta V_l \\
& = & \Delta V_g^- + H\,\Delta Q_g^- + \Delta V_l \\
& \leq & K_1^{g-}\left|\sigma_1^+\right| - K_2^{g-}\left|\sigma_2^-\right| + H\left|\sigma_1^+\right| \sum_{x_\alpha<0} \left|\sigma_\alpha\right| + K_l\left|\sigma_2^+\right|
\end{array}
$$

$$\le \left(\left(K_1^{g-} + K_l + H \sum_{x_\alpha<0} |\sigma_\alpha|\right) C - K_2^{g-}\right) \left|\sigma_2^-\right|,$$

while in the symmetric situation we get

$$\Delta\Upsilon \le \left(\left(K_2^{g+} + K_l + H \sum_{x_\alpha>m} |\sigma_\alpha|\right) C - K_1^{g+}\right) \left|\sigma_1^-\right|.$$

In the case of Figure 3:

$$\begin{aligned}
\Delta\Upsilon &= \Delta\Upsilon_g^- + \Delta V_l \\
&= \Delta V_g^- + H\,\Delta Q_g^- + \Delta V_l \\
&\le K_1^{g-}\left|\sigma_1^+\right| + H\left|\sigma_1^+\right| \sum_{x_\alpha<0} |\sigma_\alpha| + K_l\left|\sigma_2^+\right| - K_l\left|\sigma_1^-\right| \\
&= K_1^{g-}\left|\sigma_1^+\right| + H\left|\sigma_1^+\right| \sum_{x_\alpha<0} |\sigma_\alpha| - K_l\left|\sigma_1^+\right| \\
&\le \left(K_1^{g-} + H \sum_{x_\alpha<0} |\sigma_\alpha| - K_l\right)\left|\sigma_1^+\right|,
\end{aligned}$$

while in the symmetric situation we get

$$\Delta\Upsilon \le \left(K_2^{g+} + H \sum_{x_\alpha>m} |\sigma_\alpha| - K_l\right)\left|\sigma_2^+\right|,$$

Case 3: An interaction in the interior of the left gas phase. Then, the classical estimates (3.4) ensure that the standard Glimm functional decreases, i.e.,

$$\begin{aligned}
\Delta\Upsilon &= \Delta\Upsilon_g^- \\
&= \Delta V_g^- + H\Delta Q_g^- \\
&\le (K_1^{g-} + K_2^{g-})\,C\left|\sigma_1^-\,\sigma_2^-\right| + H(C\delta - 1)\left|\sigma_1^-\,\sigma_2^-\right| \\
&\le \Big((K_1^{g-} + K_2^{g-})\,C - H/2\Big)\left|\sigma_1^-\,\sigma_2^-\right| \qquad \text{and} \\
\Delta\Upsilon &= \Delta\Upsilon_g^- \\
&= \Delta V_g^- + H\Delta Q_g^- \\
&\le (K_1^{g-} + K_2^{g-})\,C\left|\sigma'\,\sigma''\right| + H(C\delta - 1)\left|\sigma'\,\sigma''\right| \\
&\le \Big((K_1^{g-} + K_2^{g-})\,C - H/2\Big)\left|\sigma'\,\sigma''\right|
\end{aligned}$$

in the two cases of Figure 1, provided δ is sufficiently small and H is sufficiently large.

We now choose $K_1^{g-} = K_2^{g+} = 1$, $K_l = 2$, $K_1^{g+} = K_2^{g-} = 4C$, $H = 4(1+2C)C$ and finally $\delta = \min\left\{1/(2C), 1/(2H)\right\}$, obtaining that $\Delta\Upsilon \le 0$ at any interaction. Indeed, in the different cases considered above, we have:

$$
\begin{array}{lrcl}
\text{Case 1:} & \Delta\Upsilon & = & 0\\
\text{Case 2, Figure 2, left boundary:} & \Delta\Upsilon & \leq & -\frac{C}{2}\left|\sigma_2^-\right|\\
\text{Case 2, right boundary:} & \Delta\Upsilon & \leq & -\frac{C}{2}\left|\sigma_1^-\right|\\
\text{Case 2, Figure 3, left boundary:} & \Delta\Upsilon & \leq & -\frac{1}{2}\left|\sigma_1^+\right|\\
\text{Case 2, right boundary:} & \Delta\Upsilon & \leq & -\frac{1}{2}\left|\sigma_2^+\right|\\
\text{Case 3, different families:} & \Delta\Upsilon & \leq & -C\left|\sigma_1^-\,\sigma_2^-\right|\\
\text{Case 3, same family:} & \Delta\Upsilon & \leq & -C\left|\sigma'\,\sigma''\right|.
\end{array}
\tag{3.7}
$$

This implies that the map $t \to \Upsilon(t)$ decreases along the wave front tracking approximate solution, independently from ε. This, thanks to the pressure law (1.5) and to the *ad hoc* choice of the parametrization (3.3), leads to the following bounds uniform in ε:

$$
\begin{array}{rcccl}
\mathrm{TV}\left(p_l^\varepsilon(t)\right) & \leq & \Upsilon(t) & \leq & \Upsilon(0)\,;\\
\eta^2\,\mathrm{TV}\left(\tau_l^\varepsilon(t)\right) & \leq & \Upsilon(t) & \leq & \Upsilon(0)\,;\\
\eta\,\mathrm{TV}\left(v_l^\varepsilon(t)\right) & \leq & \Upsilon(t) & \leq & \Upsilon(0)\,.
\end{array}
\tag{3.8}
$$

Bounds on the Number of Interactions. To be sure that the algorithm can be continued for all times one needs to prove that interaction times do not accumulate in finite time.

Fix a positive ε and refer to the ε–approximate wave front tracking solution u^ε defined above. Assume there exists a first time $t_\infty > 0$ such that the point (t_∞, x_∞) is the limit of a sequence (t_n, x_n) of interaction points, with $t_n < t_{n+1}$ for all n.

Claim: $\boldsymbol{x_\infty \notin \,]0, m[}$. By contradiction, assume that $x_\infty \in \,]0, m[$. Then, for suitable positive Δt and Δx, the trapezoid

$$
\mathcal{T} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \colon t \in [t_\infty - \Delta t, t_\infty] \text{ and } |x - x_\infty| \leq \Delta x + \hat\lambda^\eta (t_\infty - t)\right\}
$$

is contained in $\mathbb{R}^+ \times [0, m]$. No waves can enter the sides of $\mathcal{T}$ and a finite number of waves is supported along the lower side of $\mathcal{T}$. Hence, only a finite number of interactions may take place inside $\mathcal{T}$ and no new waves are created therein, due to (1.5) in $\mathbb{R}^+ \times [0, m]$. This contradicts (t_∞, x_∞) being the limit of an infinite sequence of distinct interaction points, proving the claim.

We can thus assume that $x_\infty \geq m$.

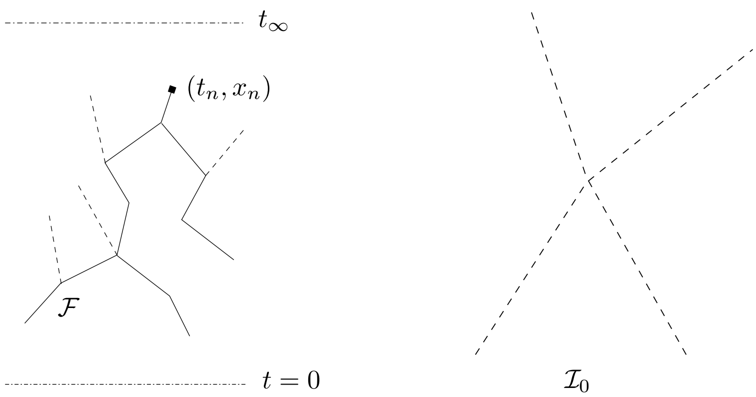

Figure 4: Left, an example of the set $\mathcal{F}$. Right, an interaction point in $\mathcal{I}_0$. Solid lines represent waves in $\mathcal{F}$, dashed ones are waves not in $\mathcal{F}$.

Call $\mathcal{F}$ the set of segments in $[0, t_\infty] \times [m, +\infty[$ that support discontinuities in u^ε and can be connected forward in time along discontinuities in u^ε to one of the points (t_n, x_n), see Figure 4.

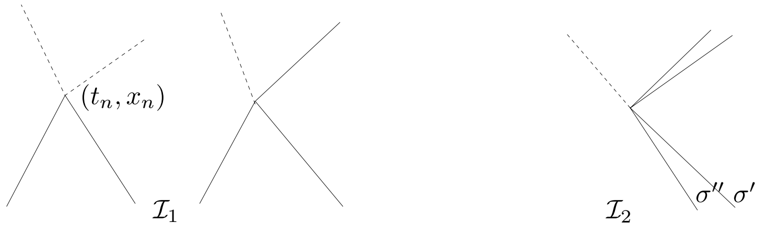

Figure 5: Left and center, examples of interaction point in the set $\mathcal{I}_1$. Right, an interaction point in $\mathcal{I}_2$. Solid lines represent waves in $\mathcal{F}$, dashed ones are waves not in $\mathcal{F}$.

Call $\mathcal{I}$ the set of all interaction points in u^ε in the strip $[0, t_\infty] \times [m, +\infty[$. With reference to figures 5 and 6, remark that $\mathcal{I}$ can be partitioned as follows:

$$
\begin{aligned}
\mathcal{I}_0 &= \Big\{(t,x) \in \mathcal{I} \colon x > m \text{ and no wave entering } (t,x) \text{ is in } \mathcal{F}\Big\} \\
\mathcal{I}_1 &= \left\{(t,x) \in \mathcal{I} \colon \begin{array}{l} x > m, \text{ at most 1 wave exiting } (t,x) \text{ is in } \mathcal{F} \\ \text{and the two interacting waves are in } \mathcal{F} \end{array}\right\} \\
\mathcal{I}_2 &= \Big\{(t,x) \in \mathcal{I} \colon x > m \text{ and at least 2 outgoing waves of the same family are in } \mathcal{F}\Big\} \\
\mathcal{I}_3 &= \left\{(t,x) \in \mathcal{I} \colon \begin{array}{l} x = m, \text{ the wave entering } (t,x) \text{ from } \mathbb{R}^+ \times [m, +\infty[\text{ is in } \mathcal{F} \\ \text{and the outgoing wave is not in } \mathcal{F} \end{array}\right\} \\
\mathcal{I}_\infty &= \left\{(t,x) \in \mathcal{I} \colon \begin{array}{l} x > m, \text{ the 2 waves exiting } (t,x) \text{ are one of the first family,} \\ \text{one of the second and both belong to } \mathcal{F} \end{array}\right\} \\
&\quad \cup \Big\{(t,x) \in \mathcal{I} \colon x = m \text{ and the outgoing wave in } \mathbb{R}^+ \times [m, +\infty[\text{ is in } \mathcal{F}\Big\}
\end{aligned}
$$

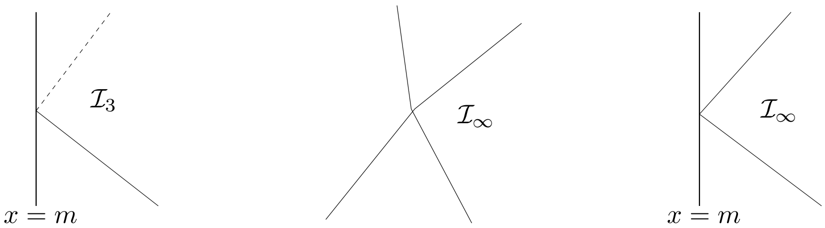

Figure 6: Left, an example of interaction point in $\mathcal{I}_3$. Center, respectively right, an interaction point in $\mathcal{I}_\infty$ at $x > m$, respectively along the boundary. Solid lines represent waves in $\mathcal{F}$, dashed ones are waves not in $\mathcal{F}$, the thick line is the (right) phase boundary.

Claim: The set $\mathcal{I}_2$ is finite. Refer to Figure 5, right. When two waves of the same family arise, they are rarefactions with total size bigger than ε. Hence, the interacting waves are shocks of the same family with sizes σ', σ'' satisfying $\left|\sigma' \sigma''\right| > \varepsilon / C$. Hence, $\Delta\Upsilon < -\varepsilon$ and this can happen only a finite number of times.

Claim: The sets $\mathcal{I}_1$ and $\mathcal{I}_3$ are finite. Call $\mathcal{Z}(\tau)$ the number of segments in $\mathcal{F}$ that intersect the line $t = \tau$. $\mathcal{Z}$ may increase only at the interactions in $\mathcal{I}_2$, hence a finite number of times. Then, at each interaction in $\mathcal{I}_1$ and in $\mathcal{I}_3$ it decreases at least by 1. Thus, $\mathcal{I}_1$ and $\mathcal{I}_3$ are finite.

Thus, in the partition above of $\mathcal{I}$, only the sets $\mathcal{I}_0$ and $\mathcal{I}_\infty$ can be infinite. Let $\tau_\infty \in]0, t_\infty[$ be such that all the interactions in $\mathcal{I}$ in the time interval $[\tau_\infty, t_\infty[$ are in $\mathcal{I}_0$ or in $\mathcal{I}_\infty$.

Claim: $\boldsymbol{x_\infty > m}$. Let $t_n \in]\tau_\infty, t_\infty[$. Then, $x_n \geq m$ and it is possible to follow waves in $\mathcal{F}$ only of the second family converging to x_∞. Waves of the second family have positive speed, hence $x_\infty > x_n \geq m$.

Claim: (t_∞, x_∞) does not exist. Let (t_n, x_n) be sufficiently near to t_∞, x_∞, so that $x_n - m > \hat\lambda^\eta(t_\infty - t_n)$ and $t_n > \tau_\infty$. Then, the phase boundary $x = m$ can not be reached following waves of the first family in $\mathcal{F}$ that exit (t_n, x_n). On the other side, by the definition of $\mathcal{F}$, (t_∞, x_∞) should be reached starting from (t_n, x_n) following waves only of the first and only of the second family in $\mathcal{F}$ that connect points in $\mathcal{I}_\infty$. However, following waves of the first family one reaches a strictly decreasing sequence of points x_{n_j}, so that $x_\infty = \inf_j x_{n_j} < x_n$, whereas along waves of the second family one reaches a strictly increasing sequence of points x_{n_i}, so that $x_\infty = \sup_i x_{n_i} > x_n$. This contradicts the existence of (t_∞, x_∞).

Lipschitz Continuity in Time. Let $u^\varepsilon(t) = (p^\varepsilon, v^\varepsilon)(t)$ denote the ε-solution constructed above. Introduce the following upper bound for the characteristic speeds in the gas phase $\Lambda = \left\|\sqrt{p'}\right\|_{\mathbf{C^0}}$ and note that the characteristic speed in the liquid phase is η. Then, by the above definition of the approximate solution and by the parametrization (3.3), see also [3, Chapter 7, formula (7.9)], if $t_1 < t_2$,

$$\begin{aligned}\left\|v^\varepsilon(t_1) - v^\varepsilon(t_2)\right\|_{\mathbf{L^1}} &\leq \left(\Upsilon_g^-\left(u^\varepsilon(t_1)\right) + \Upsilon_g^+\left(u^\varepsilon(t_1)\right)\right)\Lambda\,|t_1 - t_2| + V_l\left(u^\varepsilon(t)\right)\,|t_1 - t_2| \\ &\leq \Upsilon(u_o^\varepsilon)(1+\Lambda)|t_1 - t_2|\,. \end{aligned} \tag{3.9}$$

Passing now to the pressure, the same computations lead to:

$$\begin{aligned}\left\|p^\varepsilon(t_1) - p^\varepsilon(t_2)\right\|_{\mathbf{L^1}(\mathbb{R}\setminus[0,m])} &\leq \left(\Upsilon_g^-\left(u^\varepsilon(t_1)\right) + \Upsilon_g^+\left(u^\varepsilon(t_1)\right)\right)\Lambda\,|t_1 - t_2| \\ \left\|p^\varepsilon(t_1) - p^\varepsilon(t_2)\right\|_{\mathbf{L^1}([0,m])} &\leq \eta\, V_l\left(u^\varepsilon(t)\right)\,|t_1 - t_2|\end{aligned}$$

and the parametrization (3.2)–(3.3) give the following bounds on the specific volume

$$\begin{aligned}\left\|\tau_g^\varepsilon(t_1) - \tau_g^\varepsilon(t_2)\right\|_{\mathbf{L^1}(\mathbb{R}\setminus[0,m])} &\leq C\left(\Upsilon_g^-\left(u^\varepsilon(t_1)\right) + \Upsilon_g^+\left(u^\varepsilon(t_1)\right)\right)\Lambda\,|t_1 - t_2| \\ &\leq C\,\Upsilon(u_o^\varepsilon)\Lambda|t_1 - t_2| \qquad (3.10)\\ \left\|\tau_l^\varepsilon(t_1) - \tau_l^\varepsilon(t_2)\right\|_{\mathbf{L^1}([0,m])} &\leq \frac{1}{\eta}V_l\left(u^\varepsilon(t)\right)\,|t_1 - t_2| \;\leq\; \frac{1}{\eta}\,\Upsilon(u_o^\varepsilon)|t_1 - t_2|\,. \qquad (3.11)\end{aligned}$$

The Limit. Let ε_k be a sequence converging to 0 as $k \to +\infty$. Then, the sequence u^{ε_k} of ε_k-wave front tracking solutions satisfies Helly Compactness Theorem. By the uniqueness of the Riemann Semigroup constructed in [4, Theorem 2.5], any of its converging subsequences converge to the unique semigroup of entropy solutions constructed therein. Then, the $\mathbf{L^1}$-lower semicontinuity of the total variation allows to pass the bounds (3.8) to the limit $k \to +\infty$, proving both inequalities in (2.2).

Concerning point 1, we have to show that there is a constant δ_g independent from η and a positive δ_l^η, such that all functions in the domain

$$\left\{u \in \bar u + (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R}) \colon\ \mathrm{TV}(u_g) < \delta_g \text{ and } \mathrm{TV}(u_l) < \delta_l^\eta\right\}$$

satisfy $\Upsilon < \delta$. By standard properties of the Riemann problem, it is easy to show that if the total variation of the initial data is sufficiently small, independently from η, then $\Upsilon_g^- + \Upsilon_g^+ < \delta/2$. Concerning the liquid phase, the choice (1.5) allows to compute the exact solution to any Riemann problem and obtain the estimate

$$V_l = K_l\ \mathrm{TV}\left(p^\eta\left(\tau_l(0+)\right)\right) = \frac{K_l}{2}\left[\eta^2\,\mathrm{TV}\left(\tau_l(0)\right) + \eta\,\mathrm{TV}\left(v_l(0)\right)\right]$$

where we have also used (1.5). Therefore, choosing $\delta_l^\eta < \delta/(K_l\,\eta^2)$ implies $\Upsilon < \delta$.

Point 9. By **(p)** and (3.1), the line $x = \bar{x}$ is noncharacteristic. Fix η, ε and a point $x > m$, the case $x < 0$ being entirely similar. The estimate we now prove is an analogous to [5, Formula (14.5.19)] in the present setting.

Introduce the functional

$$\Xi_x(t) = \mathrm{TV}\left(p_g\left(\tau_g^{\varepsilon,\eta}(\cdot,x)_{\big|[0,t]}\right)\right) + \sum_{x_\alpha\in]m,x[}\left|\sigma_{2,\alpha}\right| + \sum_{x_\alpha\in]m,+\infty[}\left|\sigma_{1,\alpha}\right| + 4\,\Upsilon(t)$$

and observe that it is non increasing in time. Indeed, considering all the possible interactions as above, we have:

Case 1: Clearly, $\Delta\Xi_x = 0$.

Case 2: When the interaction is against the left boundary, in both cases of Figure 2 and Figure 3, clearly $\Delta\Xi_x = \Delta\Upsilon < 0$. When the interaction is as in against the right boundary and the incoming wave comes from the gas phase, by (3.5) and (3.7), we have

$$\Delta\Xi = \left|\sigma_2^+\right| + 4\Delta\Upsilon \leq -C\left|\sigma_1^-\right| < 0\,.$$

When the interaction is against the right boundary and the incoming wave comes from the liquid phase, by (3.6) and (3.7), we have:

$$\Delta\Xi = \left|\sigma_2^+\right| + 4\,\Delta\Upsilon \leq -\left|\sigma_2^+\right| < 0\,.$$

Case 3: If the interaction is in the interior of the left gas phase, then clearly $\Delta\Xi_x = \Delta\Upsilon < 0$. In the case of an interaction between waves of different families at an interaction point $\bar{x} \in\,]m, x[$, by (3.4) and (3.7), we have:

$$\Delta\Xi = \left|\sigma_2^+\right| - \left|\sigma_2^-\right| - 4\,C\left|\sigma_1^-\,\sigma_2^-\right| \leq -3\,C\left|\sigma_1^-\,\sigma_2^-\right| < 0\,.$$

If $\bar{x} > x$, then, by (3.4) and (3.7),

$$\Delta\Xi = \left|\sigma_1^+\right| - \left|\sigma_1^-\right| - 4\,C\left|\sigma_1^-\,\sigma_2^-\right| \leq -3\,C\left|\sigma_1^-\,\sigma_2^-\right| < 0$$

and entirely analogous estimates hold when the interacting waves belong to the same family.

Case 4: We have now to consider also the case of a wave passing through x. Then, by the definition of Ξ, $\Delta\Xi = 0$.

The monotonicity of Ξ ensures the following estimate on the total variation of the ε-wave front tracking approximate solution constructed above:

$$\mathrm{TV}\left(u_g^{\varepsilon,\eta}(\cdot,x)\right) \leq \mathcal{K}_0\,\Xi(0) \leq \mathcal{K}_1\,\Upsilon\left(u^{\varepsilon,\eta}(0+)\right) \leq \mathcal{K}_2\,\Delta$$

where $\mathcal{K}_1, \mathcal{K}_2$ are constants independent from the initial data, from η and from ε. Fix now x_1 and x_2 in the same gas phase. Then, similarly to [5, Formula (14.4.7)],

$$\begin{aligned}
\int_0^t \left\|u_g^{\varepsilon,\eta}(s,x_2) - u_g^{\varepsilon,\eta}(s,x_1)\right\| \mathrm{d}s \quad &\leq \quad \frac{1}{\inf\sqrt{-p_g'}}\left(\sup_{x\in\mathbb{R}\setminus[0,m]} \mathrm{TV}\left(u_g^{\varepsilon,\eta}(\cdot,x)\right)\right)|x_2 - x_1| \\
&\leq \quad \mathcal{K}_3\,\Delta\,|x_2 - x_1|
\end{aligned}$$

for a suitable $\mathcal{K}_3$ independent from the initial data, from η and from ε. By the same arguments used in the paragraph above, we may pass to the limit $\varepsilon \to 0$ obtaining for a.e. x_1, x_2 the Lipschitz estimate (2.4), completing the proof of Theorem 2.3. □

Proof of Theorem 2.4. Point 1. is a direct consequence of 1 in Theorem 2.3, with $\delta = \delta_g$.

Let η_k be a sequence with $\lim_{k\to+\infty} \eta_k = +\infty$. Helly Compactness Theorem [3, Chapter 2, Theorem 2.4] can be applied thanks to the Lipschitz estimates 4. in Theorem 2.3 and ensures that a subsequence of $S_t^{\eta_k} u$ converges a.e. to a function $u_* \in \mathbf{C^{0,1}}\left(\mathbb{R}^+; \bar{u} + \mathbf{L^1}(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R})\right)$, where $u_* = (\tau_*, v_*)$, proving point 2.

By (2.2), for all $t \in \mathbb{R}^+$, the function u_* is constant for $x \in [0, m]$. Hence, the bounds at 4. in Theorem 2.3 ensure that $\tau^*(t, x) = \bar{\tau}$ for all $(t, x) \in \mathbb{R}^+ \times [0, m]$ and that $v \in \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}^+; \mathbb{R})$, completing the proof of point 3.

To prove point 4., the case $\bar{x} \in [0, m]$ is immediate by the bounds (2.2) which ensure that the limit is independent from x in the liquid phase. Assume that $\bar{x} \in \mathbb{R} \setminus [0, m]$ and note that passing to the limit $\eta \to +\infty$ in 9 of Theorem 2.3, we have

$$\int_0^t \left\| u_*(s, x_2) - u_*(s, x_1) \right\| \mathrm{d}s \le L \left| x_2 - x_1 \right| \tag{3.12}$$

for a.e. x_1, x_2 in the same gas phase. Consider the case of the right trace, the other case being entirely similar. Let x_n be a sequence converging to $\bar{x}$ from the right of points where 9. in Theorem 2.3 applies. Using (3.12),

$$\begin{aligned}
& \int_0^t \left\| u_g^\eta(s, \bar{x}+) - u_*(s, \bar{x}+) \right\| \mathrm{d}s \\
\le\; & \int_0^t \left\| u_g^\eta(s, \bar{x}+) - u_g^\eta(s, x_n) \right\| \mathrm{d}s + \int_0^t \left\| u_g^\eta(s, x_n) - u_*(s, x_n) \right\| \mathrm{d}s \\
& + \int_0^t \left\| u_*(s, x_n) - u_*(s, \bar{x}+) \right\| \mathrm{d}s \\
\le\; & 2\, L |\bar{x} - x_n| + \int_0^t \left\| u_g^\eta(s, x_n) - u_*(s, x_n) \right\| \mathrm{d}s
\end{aligned}$$

To complete the proof of point 4., pass now to the $\limsup$ as $\eta \to +\infty$ and then to the limit as $n \to +\infty$.

To prove point 5., note that the regularity condition (2.1) clearly holds. Point 1. in Definition 2.2 is proved passing to the limit in the definition of weak entropy solution, which is possible by the convergence proved above. Recall now that the integral formulation of (1.3) implies

$$\int_0^m \left(v_l^\eta(t, x) - \bar{v}_l \right) \mathrm{d}x = \int_0^t \left(p_g\left(\tau_g^\eta(s, 0-)\right) - p_g\left(\tau_g^\eta(s, m+)\right) \right) \mathrm{d}s$$

and, thanks to the convergence of the traces proved above, in the limit $\eta \to +\infty$ we have

$$v_l(t) - \bar{v}_l = \int_0^t \frac{1}{m} \Big(p_g\left(\tau_g(s, 0-)\right) - p_g\left(\tau_g(s, m+)\right) \Big) \mathrm{d}s\ .$$

which is the integral formulation of point 2. in Definition 2.2. Finally, point 3. in the same definition immediately follows from the convergence of the traces proved above.

To prove point 6., define

$$\pi^\eta(t, x) = \begin{cases} p_g\left(\tau_g^\eta(t, x)\right) & x \in [0, m] \\ p^\eta\left(\tau_l^\eta(t, x)\right) & x \in \mathbb{R} \setminus [0, m] \end{cases}$$

and let η_n be an arbitrary real sequence converging to $+\infty$. Note that the sequence π^{η_n} is uniformly bounded in $\mathbf{L}^\infty$ by (2.2), hence it admits a subsequence which is weak* convergent to a limit $\pi \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^+)$. Passing to the limit $n \to +\infty$ in the definition of weak solution, we have

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left(v_*(t, x)\, \partial_t \varphi(t, x) + \pi(t, x)\, \partial_x \varphi(t, x) \right) \mathrm{d}x\, \mathrm{d}t = 0 \tag{3.13}$$

for all $\varphi \in \mathbf{C_c^1}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$. Call now $p_l = \pi_{|[0,m]}$ and remark that in $[0, m]$ the following equality is satisfied in the sense of distributions:

$$\partial_t v_l + \partial_x p_l = 0$$

showing that, by 3., $\partial_x p_l(t, \cdot)$ is constant in x and, hence, the map $x \to p_l(t, x)$ is linear. The existence of the traces immediately follows. Moreover, by (3.13), necessarily $\pi(t, 0-) = p_l(t, 0+)$ and $\pi(t, m+) = p_l(t, m-)$, which shows also the linear interpolation formula and, hence, that p_l is independent from the choice of the sequence η_n □

Acknowledgment: The authors acknowledge the support by the German Research Foundation (DFG) in the framework of the Collaborative Research Center Transregio 75 *Droplet Dynamics under Extreme Ambient Conditions*, by the Elite program for postDocs of the Baden-Württemberg Foundation, by the PRIN-2009 project *Fluid dynamics hyperbolic Equations and conservation Laws* and by the research program *Conservation Laws and Applications* of the University of Brescia.

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