On Stability of Regional Orthomodular Posets

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Abstract. The set of regions of a transition system, ordered by set inclusion, is an orthomodular poset, often referred to as quantum logic, here called *regional logic*. Regional logics, which result to be also regular and rich, are the main subject of investigation in this work.
Given a regular, rich logic L, one can build a condition/event transition system A, such that L embeds into the regional logic of A. Call *stable* a logic if the embedding is an isomorphism.
We give some necessary conditions for a logic to be stable, and show that under these, the embedding presents some stronger property. The full characterization of the class of stable logics is still an open problem. In particular, we show that any $\{0, 1\}$-pasting of n Boolean logics is stable, and that, whenever L contains n maximal Boolean sublogics sharing exactly one atom, L is stable.

1 Introduction

Regions of transition systems have been introduced by Ehrenfeucht and Rozenberg [7], [8] and applied to the *synthesis of net systems* in several ways, as described by Badouel, Darondeau and Bernardinello [2],[1].

We are interested in the order structure of regions both in case they are seen concretely as sets ordered by set inclusion and in case they are seen abstracly as *quantum logics*. While quantum logics [11] have been introduced as ordered structures representing *observations* and *measurement* operations of quantum physics, we use here their properties without going into the details of their relations with physics.

In more detail, we exploit a duality between *condition/event* transition systems and *rich* and *regular* quantum logics presented in [4]. In that contribution, with some difference in notation that will be explained in a remark at the end of section 2, we proved that, given a rich and regular quantum logic L (we use simply *logic* in what follows), one can build a condition/event transition system A, such that L embeds into the logic generated by the regions of A. In the present work, we explore the case in which this embedding is an isomorphism. So we call a logic *stable* if the embedding mentioned above is an isomorphism and we give

some necessary conditions for a logic to be stable, while the full characterization of the class remains an open problem.

The special cases analyzed in this paper exploit the fact that any quantum logic can be seen as a family of partially overlapping Boolean algebras. In particular, we show that any $\{0,1\}$-pasting ([11], p. 4) of n Boolean algebras, or Boolean sublogics, is stable, and that, whenever L contains n maximal Boolean sublogics sharing exactly one atom, L is stable.

The rest of the paper is structured as follows. In section 2, after preliminary definitions on transition systems, condition event transition systems (CETS) and regions, we present in section 2.2 basic notions on quantum logics, in particular on rich and regular logics, as well as the notion of two-value state. In section 3 we show that to any CETS it is possible to associate the quantum logic of its regions, and we show a synthesis procedure to associate a CETS to any rich and regular logic. Sections 4 and 5 constitute the original part of the paper: the first deals with some results towards the characterization of stable logics, the second considers some particular subclass of logics, and shows that they are stable. Section 6 concludes the paper with some remarks.

2 Preliminary Definitions and Notations

2.1 Transition systems

Transition systems are a class of automata representing the *global* behaviour of a system in opposition to the representation via *local states* or *local properties*. The relation between these two ways of representation is extensively reported in the literature, see [7], [8] and [9] as first papers on the subject and [1] for a complete survey. In this contribution, we will always consider *finite* transition systems.

Definition 1. *A* transition system *is a triple* $A = (Q, E, T)$ *where Q is a set of* states, *E is a set of* events *and $T \subseteq Q \times E \times Q$ is a set of* transitions. *We assume that the following conditions are respected:*

1. *the underlying graph of the* transition system *is connected;*
2. $\forall (q_1, e, q_2) \in T \quad q_1 \neq q_2$*;*
3. $\forall (q, e_1, q_1), (q, e_2, q_2) \in T \quad q_1 = q_2 \Rightarrow e_1 = e_2$*;*
4. $\forall e \in E \quad \exists (q_1, e, q_2) \in T$*.*

In some cases, we will drop axiom 1, which imposes connection.

A *region* of a transition system is a set of its states such that the occurrence of one of its events has the same *crossing relation*, namely entering, leaving the region itself, or otherwise neither of the two. This is formalised as follows.

Definition 2. *A* region *of a transition system $A = (Q, E, T)$ is a subset r of Q such that* $\forall e \in E, \ \forall (q_1, e, q_2), (q_3, e, q_4) \in T$*:*

1. $(q_1 \in r \ and \ q_2 \notin r) \ \Rightarrow \ (q_3 \in r \ and \ q_4 \notin r)$*; and*

2. $(q_1 \notin r \text{ and } q_2 \in r) \Rightarrow (q_3 \notin r \text{ and } q_4 \in r)$.

Given a transition system A, its set of regions will be denoted by $\mathcal{R}(A)$; given a state $q \in Q$, the set of regions containing q will be denoted by $\mathcal{R}_q(A)$ and, when the transition system that originates the regions is clear from the context, simply by $\mathcal{R}_q$. Note that the set of regions $\mathcal{R}(A)$ of a transition system $A = (Q, E, T)$ cannot be empty since at least the whole set of states Q is a region.

Definition 3. *Let $A = (Q, E, T)$ be a transition system. The* pre-set *and* post-set *operations, denoted respectively by the operators* $^\bullet(.)$ *and* $(.)^\bullet$, *applied to regions $r \in \mathcal{R}(A)$ and events $e \in E$ are defined by:*

1. $^\bullet r = \{e \in E \mid \exists (q_1, e, q_2) \in T \text{ such that } q_1 \notin r \text{ and } q_2 \in r\}$;
2. $r^\bullet = \{e \in E \mid \exists (q_1, e, q_2) \in T \text{ such that } q_1 \in r \text{ and } q_2 \notin r\}$;
3. $^\bullet e = \{r \in \mathcal{R}(A) \mid e \in r^\bullet\}$;
4. $e^\bullet = \{r \in \mathcal{R}(A) \mid e \in {}^\bullet r\}$.

Condition/event transition systems have been introduced as the class of transition systems isomorphic to the *sequential case graphs* of *condition/event net systems* [8], [10].

Definition 4. *A* Condition/Event Transition System *(CETS) is a transition system such that the following conditions are satisfied:*

1. $\forall q_1, q_2 \in Q \quad \mathcal{R}_{q_1} = \mathcal{R}_{q_2} \Rightarrow q_1 = q_2$;
2. $\forall q_1 \in Q, \forall e \in E \quad {}^\bullet e \subseteq \mathcal{R}_{q_1} \Rightarrow \exists q_2 \in Q: (q_1, e, q_2) \in T$;
3. $\forall q_1 \in Q, \forall e \in E \quad e^\bullet \subseteq \mathcal{R}_{q_1} \Rightarrow \exists q_2 \in Q: (q_1, e, q_2) \in T$.

Basic facts concerning regions of CETS [3] are listed in the following:

Proposition 1. *Let $A = (Q, E, T)$ be a CETS and $\mathcal{R}(A)$ its set of regions, then:*

1. $\emptyset \in \mathcal{R}(A)$;
2. $Q \in \mathcal{R}(A)$;
3. $r \in \mathcal{R}(A) \Rightarrow Q \setminus r \in \mathcal{R}(A)$;
4. $r_1, r_2 \in \mathcal{R}(A) \Rightarrow (r_1 \cap r_2 \in \mathcal{R}(A) \Leftrightarrow r_1 \cup r_2 \in \mathcal{R}(A))$.

We consider $\mathcal{R}(A)$ as enriched with the usual concrete structure. Elements, seen as subsets of A, are ordered by inclusion. Set union and intersection are here partial operations; $\mathcal{R}(A)$ is closed by set complement.

2.2 Quantum logics and states on a logic

We will follow the notation and definitions given in [11], but for the fact that we will consider only *finite* structures. In particular, the name used for the basic order strucure defined in this section will be *quantum logic*, or simply *logic*. In the literature, quantum logics are known as well as *orthomodular posets*. This class is larger than that of *orthomodular lattices* since the operators of *greatest lower bound*, denoted $\wedge$, and *lowest upper bound*, denoted $\vee$ — induced by the order relation — are not always defined. The following definition is taken from [11] (definition 1.1.1).

Definition 5. *A* quantum logic *(or* logic*)* $L = (L, \leq, 0, 1, (\,.\,)')$ *is a partially ordered finite set* $(L, \leq)$ *endowed with a least and a greatest element, denoted by* 0 *and* 1, *respectively, and a unary operation* $(\,.\,)'$ *(called* orthocomplement*), such that the following conditions are satisfied:*
$\forall\, x, y \in L$

1. $x \leq y \;\Rightarrow\; y' \leq x'$;
2. $(x')' = x$;
3. $x \leq y' \;\Rightarrow\; x \vee y \in L$;
4. $x \leq y \;\Rightarrow\; y = x \vee (x' \wedge y)$.

This latter condition is sometimes referred to as orthomodular law*.*

Let $x, y \in L$ be such that $x \leq y'$, then we say that they are *orthogonal*, denoted $x \perp y$.

A *sublogic* of L is a subset $\hat{L}$ of L that is itself a logic with respect to the restrictions of the operation $(.)'$ and the relation $\leq$ to $\hat{L}$. In particular, $x \in \hat{L} \Rightarrow x' \in \hat{L}$ and, $\forall x, y \in \hat{L} \;\; x \leq y' \Rightarrow x \vee y \in \hat{L}$. A sublogic is *Boolean* if it is a Boolean algebra.

An element a of a logic is an *atom* if, for any element b of L such that $b \leq a$ either $b = a$ or $b = 0$. Atoms are the least elements in the logic except for the bottom element. A logic is said to be *atomic* if any element, except the bottom, is greater or equal to some atom. Finite logics are atomic. Atomic logics present the practical advantage to be determined by the relations among their *atoms*, and thus allow for a concise representation.

We say that two elements x and y in L are *compatible*, denoted $x \,\$\, y$ if, and only if, there exist three mutually orthogonal elements $\hat{x}$, $\hat{y}$ and z in L such that $x = \hat{x} \vee z$ and $y = \hat{y} \vee z$. Intuitively, we can see maximal sets of mutually compatible elements in L as *maximal Boolean sublogics* of L. The following definition is taken from [11](definition 1.3.26).

Definition 6. *A logic L is called* regular *if, for any set $\{x, y, z\} \subseteq L$ of pairwise compatible elements, we have that $x \,\$\, (y \vee z)$.*

The relation between compatible subsets of a logic L and Boolean sublogics of L is put in evidence by the following proposition, to be found in [11] (proposition 1.3.29).

Proposition 2. *A logic L is* regular *if and only if every subset of pairwise compatible elements of L admits an enlargement to a* Boolean sublogic *of L.*

Example 1. The poset shown in figure 1 is a regular quantum logic. The set of its atoms is v, w, x, y, z. Examples of orthogonal pairs of elements are: $x \perp y$, $x \perp w$, $v \perp w$. It contains two maximal Boolean sublogics: $\{0, v, w, x, v', w', x', 1\}$ and $\{0, x, y, z, x', y', z', 1\}$, which intersect in the (non-maximal) Boolean sublogic $\{0, x, x', 1\}$.

Morphisms of logics are defined in such a way that they preserve order (and consequently orthogonality) and compatibility. The formal definition is borrowed from [11] (definition 1.2.7)
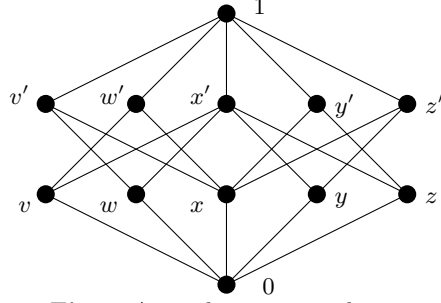
Fig. 1. A regular quantum logic.

Definition 7. *Let L_1 and L_2 be logics. A mapping $f : L_1 \to L_2$ is a* morphism *of logics if the following conditions are satisfied:*

1. *$f(0) = 0$;*
2. *$\forall x \in L_1 \; f(x') = f(x)'$;*
3. *$\forall x, y \in L_1 \; x \perp y \;\Rightarrow\; f(x \vee y) = f(x) \vee f(y)$.*

A morphism $f : L_1 \to L_2$ is an *isomorphism* if f is bijective, and f^{-1} is a morphism. Moreover, f is an *embedding* if $f(L_1)$ is a sublogic of L_2 and $f : L_1 \to f(L_1)$ is an isomorphism.

A crucial notion in the following is the concept of *two-valued state* which, in the case of regional logics, will allow to identify subsets of regions corresponding to states of the related condition/event transition systems. The following definition can be found in [11] (definition 2.1.1).

Definition 8. *A* two-valued state *on a quantum logic L is a mapping $s : L \to \{0,1\}$ such that:*

1. *$s(1) = 1$;*
2. *$\forall x, y \in L \;\; x \perp y \;\Rightarrow\; s(x \vee y) = s(x) + s(y)$*

An immediate consequence of the definition is that a state s preserves order.

In [11] a distinction is made between *states*, that is mappings defined on L whose co-domain is the interval $[0, 1]$, and two-valued states as in Def. 8 above. Since we are using exclusively two-valued states, in what follows we will not make this distinction and we will call *states* the two-valued states of Def. 8.

If L is atomic, then every state will assign 1 to exactly one atom per maximal Boolean sublogic.

Given a logic L, we denote by $\mathcal{S}(L)$ the set of all states on L and by $\mathcal{S}_x$ the set $\{s \in \mathcal{S}(L) \,|\, s(x) = 1\}$.

Definition 9. *Let L be a logic, and $X \subseteq L$, then the* up-closure *of X, denoted $\uparrow X$, is the set of all elements in L greater or equal to some element in X. Formally:*

$$\uparrow X := \{a \in L \mid \exists x \in X : x \le a\}$$

This definition is particularly useful, for it allows to represent states in a concise way. This representation relies on a result from [4] (proposition 29), slightly reformulated in the following proposition.

Proposition 3. *Any state s of a finite logic L is the characteristic function of a set $\uparrow X$, where X is a maximal set of pairwise incompatible atoms, such that it intersects each maximal set of mutually orthogonal atoms.*

Example 2. Consider the logic shown in figure 1. $v \not{S} y$, $v \not{S} z$, $w \not{S} y$ and $w \not{S} z$, and indeed its states are: $\uparrow\{v,y\}$, $\uparrow\{v,z\}$, $\uparrow\{w,y\}$, $\uparrow\{w,z\}$ as well as $\uparrow\{x\}$. $\mathcal{S}_v = \{\uparrow\{v,y\}, \uparrow\{v,z\}\}$, $\mathcal{S}_x = \{\uparrow\{x\}\}$.

Logics which have "enough" states, in such a way that the order relation can be re-constructed by the evaluation of the states, are called *rich*.

Definition 10. *Let L be a logic and $x, y \in L$. L is* rich *if:*

$$\mathcal{S}_x \subseteq \mathcal{S}_y \;\Rightarrow\; x \leq y.$$

The converse property: $x \leq y \;\Rightarrow\; \mathcal{S}_x \subseteq \mathcal{S}_y$ is a consequence of condition 2. in Def. 8 above. A characterisation of rich logics, which will be of use in the contributions of the present work, is provided by the following theorem ([11] section 2.4.12).

Theorem 1. *Let L be a logic. Then L is rich if and only if $\forall a, b \in L$ $a \not{S} b \Rightarrow \mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset$.*

Example 3. Consider again the logic shown in figure 1. It is rich, in fact, for example, $v \not{S} y$ and $\mathcal{S}_v \cap \mathcal{S}_y = \{\uparrow\{v,y\}\}$ since $\mathcal{S}_v = \{\uparrow\{v,y\}, \uparrow\{v,z\}\}$ and $\mathcal{S}_y = \{\uparrow\{v,y\}, \uparrow\{w,y\}\}$.

A particular type of logic is known as *concrete logic*. A logic is called concrete if it can be represented as a collection of subsets of a given set Ω. In this case, the order relation is the set inclusion between subsets of Ω while the orthocomplement $(\,.\,)'$ is the set complement in Ω.

Definition 11. *The tuple $(\Delta, \subseteq, \emptyset, \Omega, (.)')$, where Δ is a collection of subsets of a given set Ω and $A \in \Delta \Rightarrow (A)' = \Omega \setminus A$, is a* concrete logic *if and only if the following conditions are satisfied:*

1. *$\emptyset \in \Delta$;*
2. *$A \in \Delta \;\Rightarrow\; \Omega \setminus A \in \Delta$;*
3. *$\forall A_1, A_2 \in \Delta$ $A_1 \cap A_2 = \emptyset \;\Rightarrow\; A_1 \cup A_2 \in \Delta$.*

Note that in this *concrete* representation of a logic, orthogonality is equivalent to being pairwise disjoint, and two elements $A_1, A_2 \in \Delta$ will be compatible if and only if $A_1 \cap A_2 \in \Delta$. A more detailed presentation of *concrete logics* and the discussion on the satisfaction by *concrete logics* of properties 1-4 in definition 5 can be found in [11], p. 2.

As shown in section 3 below, the set of regions $\mathcal{R}(A)$ of a CETS A is a *concrete logic*. This is the reason why we are interested in concrete logics and in the following theorem (due to Stanley Gudder and reported in [11] as theorem 2.2.1) that relates richness to concreteness:

Theorem 2. *A logic L is isomorphic (as a logic) to a concrete logic if and only if it is rich.*

The proof of this theorem uses a property of *duality* between L and the set of its states $\mathcal{S}(L)$: each element in $x \in L$ can be represented by the set $\mathcal{S}_x$ of the $s \in \mathcal{S}(L)$ such that $s(x) = 1$. Conversely, we note that states are in fact characteristic functions, and can therefore be interpreted as subsets of L. We will use the same *duality* in the next sections.

Remark. In the present paper, we have partly changed notation and terminology with respect to [4]. In particular, here we use 'rich and regular logic' to denote what was called 'prime and coherent orthomodular poset'; moreover, 'two-valued state' or 'state' of a logic was called 'prime filter' to stress the connection with the concept of ultrafilter in Boolean algebras.

3   Regional Logics and Synthesis of Saturated Transition Systems

In the following, we recall how to associate to any CETS a concrete regular logic, and how to construct a CETS starting from a rich and regular logic; afterwards, we discuss the relations between these two transformations.

Let us consider the properties of regions recalled in proposition 1. By using these properties, it is possible to construct a logic starting from the regions of a transition system. More precisely, if $A = (Q, E, T)$ is a finite CETS and $\mathcal{R}(A)$ is its set of regions then $\mathcal{R}(A) = (\mathcal{R}(A), \subseteq, \emptyset, Q, (.)')$ (where $r \in \mathcal{R}(A) \Rightarrow (r)' = Q \setminus r$) is a rich and regular quantum logic as proved in [4]. Moreover, $\mathcal{R}(A)$ is a concrete logic as in Def. 11. We will say that $\mathcal{R}(A)$ is the *regional logic* of A, and that a logic L is *regional* if it is isomorphic to $\mathcal{R}(A)$ for some A.

In [4], a *synthesis procedure* allowing to construct a CETS starting from a rich and regular logic L was presented and set in categorical terms by showing the existence of two contravariant functors between the categories of CETS and rich and regular quantum logics.

In the following, besides illustrating the synthesis procedure, we show by means of examples that in general in both cases there is no isomorphism between L and $\mathcal{R}(A(L))$ as well as between A and $A(\mathcal{R}(A))$.

The core of the synthesis procedure is in interpreting the states of $\mathcal{S}(L)$ as the states of a transition system. A state of the transition system can be identified with the set of all regions containing it, and, by proposition 3, it coincides with the state of a logic, as in definition 8. In fact, the transition system associated to a logic L is constructed by taking $\mathcal{S}(L)$ as the set of states, and symmetric differences between states as events. Formally we have the following.

$$E(L) = \{\langle s_1 \setminus s_2, s_2 \setminus s_1 \rangle \mid s_1, s_2 \in \mathcal{S}(L), s_1 \neq s_2\}. \tag{1}$$

The set of transitions is now naturally defined as the set of all pairs of distinct states, each labelled by the corresponding ordered symmetric difference. In the

following, $[s_1, s_2]$ will denote $\langle s_1 \setminus s_2, s_2 \setminus s_1 \rangle$.

$$T(L) = \{(s_1, [s_1, s_2], s_2) \mid s_1, s_2 \in \mathcal{S}(L), s_1 \neq s_2\} \tag{2}$$

Of course, the same label can have several occurrences. We can now define the transition system

$$A(L) = (\mathcal{S}(L), E(L), T(L)). \tag{3}$$

The transition system $A(L)$ includes a transition for each ordered pair of states; hence we call it *saturated* of transitions. $A(L)$ is a CETS, as shown in [4].

At this point, two natural questions arise about the two opposite transformations and the possibility that they are inverse of each other:

1. Given a rich and regular logic L, build the CETS $A(L)$ and consider its regional logic $\mathcal{R}(A(L))$. Is $\mathcal{R}(A(L))$ isomorphic to L?.

2. Given a CETS A_0, construct the CETS associated to its regional logic $A(\mathcal{R}(A_0))$. Is $A(\mathcal{R}(A_0))$ isomorphic to A_0?

In the general case, L embeds into $\mathcal{R}(A(L))$. In fact, for each $x \in L$, $\mathcal{S}_x$ is a set of regions of $A(L)$ and the embedding is given by $\phi(x) = \mathcal{S}_x \subseteq \mathcal{S}(L)$.

The fact that $\phi(.)$ is an embedding of logics will be formally proved in the next section in proposition 4. However, $\phi(.)$ is not always an isomorphism, and this can be clarified by the following example.

Example 4. Consider the set $\Omega = \{1, 2, \ldots, 6\}$ and define Δ as the collection of the $X \in \mathcal{P}(\Omega)$ such that $|X|$ is an even number. Then $L = (\Delta, \subseteq, \emptyset, \Omega, (.)')$ is a regular concrete logic in which, for $x, y \in \Delta$, $x \$ y \Leftrightarrow x \cap y \in \Delta$. All the states of L are represented by the sets $\delta_i = \{x \in \Delta \mid i \in x\}$ for $i = \{1, 2, \ldots, 6\}$. The CETS associated to L, $A(L)$ has then six states and a set of transitions, computed as in eq. 2 above, whose labels are all distinct. This means that the regions of the CETS $A(L)$, defined as in eq. 3 above, are isomorphic to the power set $\mathcal{P}(\Omega)$ that strictly contains Δ.

Viceversa, by assuming as given a CETS $A = (Q, E, T)$, the general case shows that $\mathcal{S}(\mathcal{R}(A))$, and then also $A(\mathcal{R}(A))$, can contain states which do not correspond to any state in Q. This means that, in general, A is not isomorphic to $A(\mathcal{R}(A))$, as for example in the following.

Example 5. Consider the CETS $A = (Q, E, T)$ shown in figure 2. Its regions are the trivial ones, $\emptyset$ and Q, plus $x = \{1, 2, 5\}$, $y = \{1, 2, 6\}$, $z = \{1, 3, 5\}$, $w = \{1, 3, 6\}$ and the respective complements $\{3, 4, 6\} = (\{1, 2, 5\})'$, $\{3, 4, 5\} = (\{1, 2, 6\})'$, $\{2, 4, 6\} = (\{1, 3, 5\})'$ and $\{2, 4, 5\} = (\{1, 3, 6\})'$. The corresponding concrete logic is represented in figure 3 and its states are formed by choosing exactly one element from each complementary pair of disjoint non-trivial regions. Hence, there are sixteen states. By applying the *synthesis procedure* above to the logic $\mathcal{R}(A) = (\mathcal{R}(A), \subseteq, \emptyset, Q, (.)')$, we find that six out of the sixteen states in $\mathcal{S}(\mathcal{R}(A))$ correspond to the original states of A.
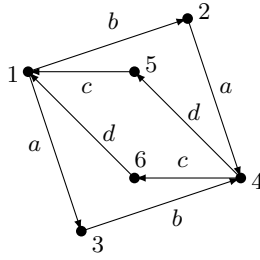
Fig. 2. A CETS with fewer states than its regional logic.

Concerning the relation between regions and events, we note that pre- and post-regions of an event $e \in E$ can be retrieved in $\mathcal{R}(A)$ as set differences between states of the logic. Remember that $\mathcal{R}_{q_1}$ is the set of regions containing q_1 and $\mathcal{R}_{q_2}$ is the set of regions containing q_2. If (q_1, e, q_2) is a transition in A, then $\mathcal{R}_{q_1} \setminus \mathcal{R}_{q_2}$ gives the set of regions from which e exits, namely the set of pre-regions of e; the difference in the other direction gives the set of post-regions. By definition of region, these differences are independent of the individual occurrence of e in A: let $(q_1, e, q_2), (q_3, e, q_4) \in T$. Then $\mathcal{R}_{q_1} \setminus \mathcal{R}_{q_2} = \mathcal{R}_{q_3} \setminus \mathcal{R}_{q_4} = {}^{\bullet}e$ and $\mathcal{R}_{q_2} \setminus \mathcal{R}_{q_1} = \mathcal{R}_{q_4} \setminus \mathcal{R}_{q_3} = e^{\bullet}$. Hence, these set differences allow us to identify different occurrences as corresponding to the same event, in other words, that two transitions carry the same label.

Example 6. For the CETS A in figure 2, the case can be exemplified by considering the two events labelled a, leading from state 1 to state 3 and from state 2 to state 4, respectively. In this case, $\mathcal{R}_1$ is the set composed by $\{1, 2, 5\}, \{1, 2, 6\}, \{1, 3, 5\}, \{1, 3, 6\}$ and $\mathcal{R}_3$ is composed by $\{3, 4, 6\}, \{3, 4, 5\}, \{1, 3, 5\}, \{1, 3, 6\}$. By computing the set differences as above, we find that: $\mathcal{R}_1 \setminus \mathcal{R}_3 = \mathcal{R}_2 \setminus \mathcal{R}_4$, the set of regions from which a exits. The symmetric case leads to $\mathcal{R}_3 \setminus \mathcal{R}_1 = \mathcal{R}_4 \setminus \mathcal{R}_2$.
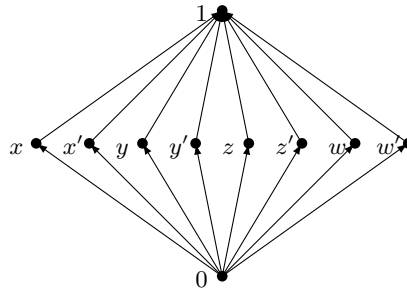

Fig. 3. A quantum logic composed by four complementary pairs; the regional logic of the CETS in figure 2.

With reference to the logic $\mathcal{R}(A)$ of the CETS A in figure 2 as represented in figure 3, we can compute, in terms of symmetric differences, the transitions

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between any couple of the sixteen states of the logic $\mathcal{R}(A)$ as in equation 2 above. We find, among many others, the transitions corresponding to the identical labels of A. For example, by considering the states in S_x and in S_y, we have that $S_x \setminus S_y = S_{y'} \setminus S_{x'}$.

4 Towards the characterization of Stable Regional Logics

In the previous section, we have considered two cases: the first one deals with the construction of a logic $\mathcal{R}(A)$ starting from a CETS A and the second one deals with the construction, or *synthesis*, of a CETS $A(L)$ starting from a regular and rich logic L. In both cases, by applying again the process and computing $A(\mathcal{R}(A))$ and $\mathcal{R}(A(L))$ we obtain two embeddings. The original CETS A embeds in terms of both states and transitions into the CETS resulting from the *synthesis procedure* applied to $\mathcal{R}(A)$, and the original logic L embeds into the concrete logic formed by taking Ω as the states of $A(L)$ and Δ as the set of regions of $A(L)$, as asserted by theorem 2. We say that the logic L is *stable* if this embedding is an isomorphism of logics.

Definition 12. *A quantum logic L is* stable *if there is an isomorphism of logics from L onto $\mathcal{R}(A(L))$.*

Our aim is to characterize the class of stable regional logics. In what follows, we present some new results towards such a goal. First we discuss two necessary conditions for a logic to be regional, and then we prove that one implies the other. Afterwards, in subsection 4.2, we prove that there is an embedding of logics from L to $\mathcal{R}(A(L))$, and that, under the same condition, this embedding is strong in the sense that it preserve also incompatibility.

4.1 Regional Logics are ETI and TIP

In [6] and [5] it is shown that regional logics satisfy the properties called TIP and ETI, respectively. In the following, after recalling the definitions, we show that any ETI logic is also a TIP logic.

The property TIP results from the translation in the abstract setting of quantum logic of a property of regions of transition systems, which we call *triple intersection property* and which is expressed by the following lemma, the proof of which can be found in [6].

Lemma 1. *Let A be a CETS, and let $a, b, c \in \mathcal{R}(A)$ be such that: $a \cap b = b \cap c = c \cap a$, then $z = a \cap b \cap c \in \mathcal{R}(A)$.*

The triple intersection property (TIP) for a logic L is then obtained by considering for any element in L the set of two-valued states selecting that element and from the fact that this set identifies a region of the CETS $A(L)$ synthesised from L.

Definition 13. *A logic L is* TIP *if*

$$\forall a, b, c \in L: \ \mathcal{S}_a \cap \mathcal{S}_b = \mathcal{S}_b \cap \mathcal{S}_c = \mathcal{S}_c \cap \mathcal{S}_a \quad \Rightarrow \quad \exists z \in L: \ \mathcal{S}_z = \mathcal{S}_a \cap \mathcal{S}_b \cap \mathcal{S}_c$$

As well as TIP, also ETI is inspired by a property of regions. If (q_1, e, q_2), (q_3, e, q_4), with $q_1 \neq q_3$ and $q_2 \neq q_4$, are two distinct transitions in a CETS $A = (Q, E, T)$, then $\mathcal{R}(A)$ must contain two incompatible regions, one containing q_1 and q_2 but not q_3, the other containing q_1 and q_3 but not q_2. Hence, events with multiple occurrences and pairs of incompatible regions are related. In this sense, we say that the events of an abstract logic L *testify incompatibility*; L is said to be ETI if, for any pair of incompatible regions, the set $E(L)$ contains an element witnessing this incompatibility.

Definition 14. *A logic L is* ETI *if* $\quad \forall a, b \in L : \quad a \not\$ b \;\Rightarrow$

$$\exists s_1 \in \mathcal{S}_a \cap \mathcal{S}_b, s_a \in \mathcal{S}_a \cap \mathcal{S}_{b'}, s_b \in \mathcal{S}_b \cap \mathcal{S}_{a'}, s_0 \in \mathcal{S}_{a'} \cap \mathcal{S}_{b'} : \; s_a \setminus s_1 = s_0 \setminus s_b$$

Example 7. An example of a concrete logic which is neither TIP nor ETI is the logic of the subsets of even cardinality of $\{1, 2, 3, 4, 5, 6\}$, already seen in example 4. This logic is regular and rich, but not regional as discussed in [4]. In order to see that L is not TIP, let us consider the elements in L : $\{1, 2\}, \{1, 3\}, \{1, 4\}$, then $\mathcal{S}_{\{1,2\}} \cap \mathcal{S}_{\{1,3\}} = \mathcal{S}_{\{1,3\}} \cap \mathcal{S}_{\{1,4\}} = \mathcal{S}_{\{1,4\}} \cap \mathcal{S}_{\{1,2\}} = \{\delta_1\}$, where δ_1 is the two-valued state of L selecting all the elements containing 1. It is then immediate to see that there is no z in L such that $\mathcal{S}_z = \{\delta_1\}$. L is not ETI since the symmetric differences among the six states of the associated transition systems are all different, and then it is not possible for a pair of incompatible elements of L to find pairs of equal symmetric differences.

Any regional logic is TIP, as shown in [6], and ETI, as shown in [5] Although we do not know yet if the two properties, ETI and TIP, coincide, we can prove that ETI implies than TIP.

Theorem 3. *Let L be a* ETI *logic. Then L is* TIP.

Proof. By contradiction, let L be ETI and not TIP. Not TIP means: $\exists a, b, c \in L$: $\mathcal{S}_a \cap \mathcal{S}_b = \mathcal{S}_b \cap \mathcal{S}_c = \mathcal{S}_c \cap \mathcal{S}_a \neq \emptyset$ and $\forall z \in L : \; \mathcal{S}_z \neq \mathcal{S}_a \cap \mathcal{S}_b \cap \mathcal{S}_c$. We have two cases.

First case: $a \$ b$. Then since $\mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset$, $a \not\perp b$ and then there exist three mutually orthogonal elements $\hat{a}$, $\hat{b}$ and x in L such that $a = \hat{a} \vee x$ and $b = \hat{b} \vee x$. This implies $\mathcal{S}_a$ to be the disjoint union of $\mathcal{S}_{\hat{a}}$ and $\mathcal{S}_x$, and $\mathcal{S}_b$ to be the disjoint union of $\mathcal{S}_{\hat{b}}$ and $\mathcal{S}_x$, then we get the contradiction: $\exists x : \mathcal{S}_x = \mathcal{S}_a \cap \mathcal{S}_b = \mathcal{S}_c \cap \mathcal{S}_a$.

Second case: $a \not\$ b$. Then, since L is ETI, there are four states: $s_1 \in \mathcal{S}_a \cap \mathcal{S}_b$, $s_a \in \mathcal{S}_a \cap \mathcal{S}_{b'}, s_b \in \mathcal{S}_b \cap \mathcal{S}_{a'}, s_0 \in \mathcal{S}_{a'} \cap \mathcal{S}_{b'}$ such that: $s_a \setminus s_1 = s_0 \setminus s_b$. Then $s_1 \in \mathcal{S}_a \cap \mathcal{S}_b$ implies $s_1 \in \mathcal{S}_c$. Since $\mathcal{S}_c$ is a region in $\mathcal{R}(A(L))$, then either $s_a \in \mathcal{S}_c$ and $s_b \notin \mathcal{S}_c$, or $s_b \in \mathcal{S}_c$ and $s_a \notin \mathcal{S}_c$. In any case this contradicts the hypothesis $\mathcal{S}_a \cap \mathcal{S}_b = \mathcal{S}_b \cap \mathcal{S}_c = \mathcal{S}_c \cap \mathcal{S}_a$. $\qquad\square$

4.2 Strong embedding of L into $\mathcal{R}(A(L))$

At this point, we stress the fact that given a regular, and rich logic L, and its synthesised CETS $A(L)$, the injection defined by $\phi(x) = \mathcal{S}_x$ is a morphism

of logics. Furthermore, we show that $\phi : L \to \mathcal{R}(A(L))$ is an embedding of logics. To check that ϕ is a morphism, we verify the three properties in Def. 7. First note that $\mathcal{S}_0 = \emptyset$ is the bottom element in $\mathcal{R}(A(L))$. Second, since any $x \in L$ is orthogonal to its complement x', from point 2. in Def 8, it stands that $\mathcal{S}_x \cap \mathcal{S}_{x'} = \emptyset$, and furthermore, $\forall x \in L : \mathcal{S}_x \cup \mathcal{S}_{x'} = \mathcal{S}(L)$, so $\mathcal{S}_{x'} = \mathcal{S}(L) \setminus \mathcal{S}_x$. The third point is a direct consequence of proposition 1.

On the other hand, put $\mathcal{S}_x = \mathcal{S}_y$, then L being rich, it holds from Def. 10 that $x \leq y$ and $y \leq x$, so ϕ is injective. We see in the following proposition, that ϕ is even an embedding.

Proposition 4. *Let L be a rich and regular logic, and $\phi : L \to \mathcal{R}(A(L))$ be defined by $\phi(x) = \mathcal{S}_x$ for all $x \in L$. Then $\psi \equiv \phi^{-1}|_{\phi(L)}$ is a logic morphism.*

Proof. Clearly, $\forall x \in L : \psi(\mathcal{S}_x) = x$. Now, since L is rich, the only element whose associated set of states is the bottom element, hence $\psi(\emptyset) = 0$. Also, $\psi(\mathcal{S}(L) \setminus \mathcal{S}_x) = \psi(\mathcal{S}_{x'}) = x'$, so it preserves orthocomplements. Finally, consider two disjoint $\mathcal{S}_x, \mathcal{S}_y$. Then $\mathcal{S}_x \subseteq \mathcal{S}(L) \setminus \mathcal{S}_y = \mathcal{S}_{y'}$, and since L is rich, $x \perp y$. From point 2. in Def. 8 it stands that $\{s \in \mathcal{S}(L) \mid s(x \vee y) = 1\} = \{s \in \mathcal{S}(L) \mid s(x) = 1\} \cup \{s \in \mathcal{S}(L) \mid s(y) = 1\} = \mathcal{S}_x \cup \mathcal{S}_y$. Hence $\psi(\mathcal{S}_x \cup \mathcal{S}_y) = x \vee y$. $\square$

Now, ϕ being an embedding means that L and $\mathcal{R}(A(L))$ would be isomorphic if ϕ was surjective. We have also seen in the previous section that L being ETI is a necessary condition for that. We shall now see that if L is ETI, the embedding verifies a stronger property, required (but not sufficient) for ϕ to be an isomorphism.

We remind the reader that logic morphisms preserve order, orthogonality and compatibility. Since ϕ is an embedding it shall also reflect these relations. However, in general this is only true when considering them restricted to the image $\phi(L)$. Indeed, if ϕ is an embedding then L is isomorphic to $\phi(L)$, but the lack of surjectivity might, in the general case, allow the images of two incompatible elements to be compatible. We shall explain this notion through an example.

Example 8. Consider the logic $L = \{0, u, u', v, v', 1\}$, and the Boolean logic B whose atoms are $\{a_1, a_2, a_3, a_4\}$. Then the mapping given by $\phi(u) = a_1 \vee a_2$, $\phi(u') = a_3 \vee a_4$, $\phi(v) = a_1 \vee a_3$, $\phi(v') = a_2 \vee a_4$ is indeed an embedding. Since $(a_1 \vee a_2)' = a_3 \vee a_4$ and $(a_1 \vee a_3)' = a_2 \vee a_4$, the sublogic $L_1 = \{0, a_1 \vee a_2, a_3 \vee a_4, a_1 \vee a_3, a_2 \vee a_4, 1\}$ of B is isomorphic to L. However, when considered in the whole of B, we see that a_1 is both $a_1 \leq a_1 \vee a_2$ and $a_1 \leq a_1 \vee a_3$, with $\{a_1, a_2, a_3\}$ mutually orthogonal in B. Thus $\phi(u) \$ \phi(v)$, whereas $u \not\$ v$. This does not prevent ϕ from being an embedding because, in fact $a_1, a_2, a_3 \notin L_1$.

This example should justify the following definition (see [11]).

Definition 15. *Let $\phi : L_1 \to L_2$ be an embedding between logics. Then ϕ is said to be a* strong embedding *if*

$$\forall a, b \in L_1 : a \$ b \Leftrightarrow \phi(a) \$ \phi(b)$$

For instance, a logic with incompatible elements cannot embed strongly into a Boolean logic.

We shall now prove that a logic L being ETI is a sufficient condition for $\phi : L \to \mathcal{R}(A(L))$ to be a strong embedding.

Theorem 4. *Let L be a rich and regular logic. If L is* ETI*, then the embedding $\phi : L \to \mathcal{R}(A(L))$ defined as $\phi(x) = \mathcal{S}_x$ is strong.*

Proof. We have already shown that ϕ preserves compatibility, it will therefore be sufficient to prove that it also preserves incompatibility. So let $a, b \in L$ verify $a \not\mathbin{\$} b$. Since L is ETI, $\exists s_1 \in \mathcal{S}_a \cap \mathcal{S}_b, s_a \in \mathcal{S}_a \cap \mathcal{S}_{b'}, s_b \in \mathcal{S}_b \cap \mathcal{S}_{a'}, s_0 \in \mathcal{S}_{a'} \cap \mathcal{S}_{b'} :$ $s_a \setminus s_1 = s_0 \setminus s_b$. Then $e = [s_1, s_a] \in E(L)$ will be a label in the saturated transition system $A(L)$, and the transitions $(s_1, e, s_a), (s_b, e, s_0) \in T(L)$ will prevent $\mathcal{S}_a \cap \mathcal{S}_b$ from being a region. Indeed, (s_1, e, s_a) crosses the border of $\mathcal{S}_a \cap \mathcal{S}_b$, whereas (s_b, e, s_0) does not. Since $\mathcal{R}(A(L))$ is a concrete logic, we have that $\mathcal{S}_a = \phi(a) \not\mathbin{\$} \phi(b) = \mathcal{S}_b$. $\square$

This result implies, in particular, that if new regions are produced by the synthesis procedure, these cannot be contained in the image by ϕ of an atom. Orthocomplementation implies therefore that they cannot contain the images of coatoms (maximal elements except for the top). Thus, the possible lack of surjectivity of ϕ is narrowed down.

5 Classes of stable Regional Logics

In this section we look at a few subclasses of concrete logics, and show that they are stable.

To start with the simplest example, let L be a finite Boolean logic with k atoms. Then L is a regular rich logic, isomorphic to the power set of $\{1, \cdots, k\}$, in which singletons correspond to atoms. L has exactly k states, each corresponding to $\Upsilon\{x\}$, where x is an atom of L.

This implies that all the ordered symmetric differences between states differ in at least one atom, so that each transition in $A(L)$ carries a unique label, and all subsets of states are regions. Hence, L and $\mathcal{R}(A(L))$ are isomorphic, and L is stable.

In the next cases, we will use the notion of *restriction* of a transition system to a subset of events. We will say that $A_1 = (S, E_1, T_1)$ is a restriction of $A = (S, E, T)$ if they have the same set of states, $E_1 \subseteq E$, and T_1 is obtained by removing from T all the transitions labelled by events in $E \setminus E_1$. From the definition of regions, it follows immediately that $\mathcal{R}(A) \subseteq \mathcal{R}(A_1)$.

The next case we consider is that of a logic obtained as the so-called $\{0, 1\}$-pasting of two (or more) logics. This construction is strictly related to a construction on transition systems, so we briefly deviate from the main route, in order to clarify the correspondence.

Given two transition systems, A_1 and A_2, with disjoint sets of events, we can build a new one by putting them side-by-side and letting them work in parallel.

Definition 16. *Let $A_i = (Q_i, E_i, T_i)$ be a CETS for $i = 1, 2$, with $E_1 \cap E_2 = \emptyset$. Define*

$$A_1 \,\|\, A_2 = (Q_1 \times Q_2, E_1 \cup E_2, T)$$

where

$$\begin{aligned} T = &\{((q_1, q_2), e, (q_1', q_2)) \mid (q_1, e, q_1') \in T_1\} \quad \cup \\ &\{((q_1, q_2), e, (q_1, q_2')) \mid (q_2, e, q_2') \in T_2\} \end{aligned}$$

By construction, for each region r of A_1, the set $r \times Q_2$ is a region of $A_1 \,\|\, A_2$, and, for each region r of A_2, the same holds for the set $Q_1 \times r$. For any region r of $A_1 \,\|\, A_2$, the projection of its states on the first component must be a region of A_1 (and symmetrically for the projection on the second component), because in any transition only one of the two components of a state will change; hence, the full set of non-trivial regions of $A_1 \,\|\, A_2$ is given by $\{r \times Q_2 \mid r \in \mathcal{R}(A_1)\} \cup \{Q_1 \times r \mid r \in \mathcal{R}(A_2)\}$. We now introduce the corresponding operation on logics, found as {0-1}-pasting in the literature. We use the same symbol as for the operation on transition systems. It will always be clear by the context which is being applied.

Definition 17. *Let L_1 and L_2 be two logics with $0_i, 1_i$ for $i = 1, 2$ the respective least and greatest elements. Define on $L_1 \cup L_2$ the equivalence relation*

$$\sim := \{(a, a) \mid a \in L_1 \cup L_2\} \cup \{(0_1, 0_2), (1_1, 1_2)\}.$$

Then $L_1 \,\|\, L_2 = (L_1 \cup L_2)/\!\sim$ is the {0-1}-pasting of L_1 and L_2. It is their disjoint union but for identification of 0_1 with 0_2, and 1_1 with 1_2.

The *{0-1}-pasting* of L_1 and L_2, $L_1 \,\|\, L_2$, is again a logic.

Without loss of generality, in what follows we will simplify the notation concerning the *{0-1}-pasting* of L_1 and L_2 by indicating by the same element 0 and, respectively, 1 the bottom and top elements in *both* L_1 and L_2. As a useful consequence, the set of states $\mathcal{S}(L_1 \,\|\, L_2)$ of $L_1 \,\|\, L_2$, seen as membership functions, will simply be composed by the union $s_1 \cup s_2$ for any pair of states $s_1 \in \mathcal{S}(L_1)$ and $s_2 \in \mathcal{S}(L_2)$.

The next lemma shows that the parallel product of transition systems and the $\{0, 1\}$-pasting of logics are strictly related. The lemma follows from the characterization of regions of $A_1 \,\|\, A_2$ discussed above.

Lemma 2. *Let $A_i = (Q_i, E_i, T_i)$ be a CETS for $i = 1, 2$, with $E_1 \cap E_2 = \emptyset$. Then $\mathcal{R}(A_1 \,\|\, A_2)$ and $\mathcal{R}(A_1) \,\|\, \mathcal{R}(A_2)$ are isomorphic logics.*

Proposition 5. *Let L_1 and L_2 be stable regional logics. Then $L = L_1 \,\|\, L_2$ is a stable regional logic.*

Proof. Consider $A(L) = (\mathcal{S}(L), E(L), T(L))$, the synthesis as defined in section 3. Every state in L is obtained by taking the union of a state of L_1 and a state L_2, and every such union is a state in L:

$$\mathcal{S}(L) = \{s_1 \cup s_2 \mid s_1 \in \mathcal{S}(L_1), s_2 \in \mathcal{S}(L_2)\}.$$

All these unions are distinct, so we can represent $\mathcal{S}(L)$ as $\mathcal{S}(L_1) \times \mathcal{S}(L_2)$.

We will now define a transition system on $\mathcal{S}(L)$, by taking a subset of the events and transitions of the saturated transition system. Define

$$
\begin{aligned}
E_M =&\{[(s_1, s_2), (s_1', s_2)] \mid s_1, s_1' \in \mathcal{S}(L_1), s_2 \in \mathcal{S}(L_2)\} \quad \cup \\
&\{[(s_1, s_2), (s_1, s_2')] \mid s_1 \in \mathcal{S}(L_1), s_2, s_2' \in \mathcal{S}(L_2)\}
\end{aligned}
$$

We must now take all the transitions corresponding to labels in E.

$$
\begin{aligned}
T_M =&\{((s_1, s_2), [(s_1, s_2), (s_1', s_2)], (s_1', s_2))\} \quad \cup \\
&\{((s_1, s_2), [(s_1, s_2), (s_1, s_2')], (s_1, s_2'))\}.
\end{aligned}
$$

The transition system $A_M(L) = (\mathcal{S}(L), E_M, T_M)$ is a restriction of $A(L)$, such that $E_M \subseteq E$, hence, its set of regions is a superset of $R(A(L))$. On the other hand, each region of $A_M(L)$ is either of the form $\mathcal{S}(L_1) \times r$, for $r \in L_2$, or of the form $r \times \mathcal{S}(L_2)$, for $r \in L_1$. Hence, after theorem 2, there is an isomorphism between L and $\mathcal{R}(A_M(L))$, and since $\mathcal{R}(A(L)) \subseteq \mathcal{R}(A_M(L)) \simeq \mathcal{R}(A(L_1)) \cup \mathcal{R}(A(L_2) \subseteq \mathcal{R}(A(L))$, there is a bijection between L and $\mathcal{R}(A(L))$, and L is stable. □

The construction and the argument above can be easily generalised to the case of the $\{0, 1\}$-pasting of K logics.

Let us now suppose that L is a rich, regular quantum logic with exactly two maximal Boolean sublogics, B_1 and B_2, such that $B_1 \cap B_2 = B$, with $B = \{0, x, x', 1\}$, and x is an atom of L (hence x' is a co-atom). Note that B is a Boolean logic (it corresponds to what is called the *centre* of L in [11]). An example of such a logic has been given in figure 1 and discussed in examples 1, 2, and 3.

Proposition 6. *Let L be a logic such that $L = B_1 \cup B_2$, B_1, B_2 are Boolean sublogics, and $B = B_1 \cap B_2 = \{0, x, x', 1\}$. Then L is stable.*

Proof. There are two kinds of states of L. States of the first kind are built by choosing one atom a of L in $B_1 \setminus B_2$, one atom b of L in $B_2 \setminus B_1$, and the up-closure of $\{a, b\}$ in L. The only state of the second kind is the up-closure of $\{x\}$. We will now define a generalised transition system, so that it is a restriction of $A(L)$, by selecting a subset of events, and the corresponding transitions.

Let us denote each state in $A(L)$ of the first kind by the ordered pair (a, b), with a and b atoms of L. Let x, y, w, and z be variables ranging on the atoms of L. Then we select the events associated to transitions from, say, (x, y) to (w, y), and those associated to transitions from (x, y) to (x, z).

Let $EM(L)$ be this set of events, and define $TM(L)$ as the set of all transitions in $A(L)$ with labels in $EM(L)$. Finally, define $A_M(L) = (\mathcal{S}(L), EM(L), TM(L))$. Clearly, $A_M(L)$ is a generalised transition system, which is a restriction of $A(L)$.

$A_M(L)$ is not connected because the state $\Uparrow\{x\}$, with x the atom of L common to B_1 and B_2, is isolated.

Each transition in $A(L)$ starting from, or leading to, $\Uparrow\{x\}$ carries a unique label, which has no other occurrence. Hence, the singleton formed by this state is a region in $A(L)$, as it is in $A_M(L)$.

The rest of $A_M(L)$ can be seen as the parallel product of $A(B_1)$, without the states corresponding to atoms shared with B_2, and the analogous substructure of $A(B_2)$. We can then apply the same argument as in the case of the $\{0,1\}$-pasting given in proposition 5 above.

The construction can be generalised to the case of a family of K Boolean logics, such that their pairwise intersections coincide, and consist in a Boolean logic $B = \{0, x, x', 1\}$, and x is an atom of L. A state of $A(L)$ is either the up-closure of $\{x\}$ or the up-closure of a set built by choosing exactly one atom from each B_i.

6 Conclusions

With the results presented in this paper we have done a further step towards the characterization of the rich and regular quantum logics, which result to be isomorphic to the orthomodular posets of the regions of the CETSs synthesised starting from the logics themself.

We are particularly interested in such a characterization because it is the basis for founding a logic of distributed systems. While the regions of a single sequential component constitute a Boolean logic, the presence of concurrency leads to a family of partially overlapping Boolean sublogics, each one corresponding to a sequential component.

References

1. Eric Badouel, Luca Bernardinello, and Philippe Darondeau. *Petri Net Synthesis*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2015.
2. Eric Badouel and Philippe Darondeau. Theory of regions. In Wolfgang Reisig and Grzegorz Rozenberg, editors, *Petri Nets*, volume 1491 of *Lecture Notes in Computer Science*, pages 529–586. Springer, 1996.
3. Luca Bernardinello. Synthesis of net systems. In Marco Ajmone Marsan, editor, *Application and Theory of Petri Nets 1993, 14th International Conference, Chicago, Illinois, USA, June 21-25, 1993, Proceedings*, volume 691 of *Lecture Notes in Computer Science*, pages 89–105. Springer, 1993.
4. Luca Bernardinello, Carlo Ferigato, and Lucia Pomello. An algebraic model of observable properties in distributed systems. *Theor. Comput. Sci.*, 290(1):637–668, 2003.
5. Luca Bernardinello, Carlo Ferigato, Lucia Pomello, and Adrián Puerto Aubel. Synthesis of transition systems from concrete quantum logics. Submitted. Draft available at `http://www.mc3.disco.unimib.it/pub/bfpp_sts.pdf`, 2017.

6. Luca Bernardinello, Lucia Pomello, and Stefania Rombolà. On orthomodular posets generated by transition systems. *Electr. Notes Theor. Comput. Sci.*, 270(1):147–154, 2011.

7. Andrzej Ehrenfeucht and Grzegorz Rozenberg. Partial (set) 2-structures. part I: basic notions and the representation problem. *Acta Inf.*, 27(4):315–342, 1990.

8. Andrzej Ehrenfeucht and Grzegorz Rozenberg. Partial (set) 2-structures. part II: state spaces of concurrent systems. *Acta Inf.*, 27(4):343–368, 1990.

9. Mogens Nielsen, Grzegorz Rozenberg, and P. S. Thiagarajan. Elementary transition systems. *Theor. Comput. Sci.*, 96(1):3–33, 1992.

10. C. A. Petri. Concepts of net theory. In *MFCS*, pages 137–146. Mathematical Institute of the Slovak Academy of Sciences, 1973.

11. Pavel Pták and Sylvia Pulmannová. *Orthomodular Structures as Quantum Logics*. Kluwer Academic Publishers, 1991.