FOURTH ORDER EQUATIONS OF CRITICAL SOBOLEV GROWTH. ENERGY FUNCTION AND SOLUTIONS OF BOUNDED ENERGY IN THE CONFORMALLY FLAT CASE

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Abstract. Given (M, g) a smooth compact Riemannian manifold of dimension $n \geq 5$, we consider equations like

$$P_g u = u^{2^\sharp - 1} ,$$

where $P_g u = \Delta_g^2 u + \alpha \Delta_g u + a_\alpha u$ is a Paneitz-Branson type operator with constant coefficients α and a_α, u is required to be positive, and $2^\sharp = \frac{2n}{n-4}$ is critical from the Sobolev viewpoint. We define the energy function E_m as the infimum of $E(u) = \|u\|_{2^\sharp}^{2^\sharp}$ over the u's which are solutions of the above equation. We prove that $E_m(\alpha) \to +\infty$ as $\alpha \to +\infty$. In particular, for any $\Lambda > 0$, there exists $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$, the above equation does not have a solution of energy less than or equal to Λ.

In 1983, Paneitz [23] introduced a conformally fourth order operator defined on 4-dimensional Riemannian manifolds. Branson [1] generalized the definition to n-dimensional Riemannian manifolds, $n \geq 5$. Such operators have a geometrical meaning. While the conformal Laplacian is associated to the scalar curvature, the Paneitz-Branson operator is associated to a notion of Q-curvature. Possible references are Chang [2] and Chang-Yang [3]. When the manifold (M, g) is Einstein, the Paneitz-Branson operator PB_g has constant coefficients. It expresses as

$$PB_g(u) = \Delta_g^2 u + \bar{\alpha} \Delta_g u + \bar{a} u , \tag{0.1}$$

where $\Delta_g = -div_g \nabla$ and, if S_g is the scalar curvature of g,

$$\bar{\alpha} = \frac{n^2 - 2n - 4}{2n(n-1)} S_g \quad \text{and} \quad \bar{a} = \frac{(n-4)(n^2-4)}{16n(n-1)^2} S_g^2$$

are real numbers. In particular,

$$\frac{\bar{\alpha}^2}{4} - \bar{a} = \frac{S_g^2}{n^2(n-1)^2} .$$

The Paneitz-Branson operator when the manifold is Einstein is a special case of what we usually refer to as a Paneitz-Branson type operator with constant coefficients, namely an operator which expresses as

$$P_g u = \Delta_g^2 u + \alpha \Delta_g u + a u , \tag{0.2}$$

Date: March 2002. Revised May 2002.
1991 *Mathematics Subject Classification.* 58E35, 35J35.
Key words and phrases. Paneitz-Branson type operators, Energy function.

where α, a are real numbers. We let in this article (M,g) be a smooth compact conformally flat Riemannian n-manifold, $n \geq 5$, and consider equations as

$$P_g u = u^{2^\sharp - 1} ,$$

where P_g is a Paneitz-Branson type operator with constant coefficients, u is required to be positive, and $2^\sharp = \frac{2n}{n-4}$ is critical from the Sobolev viewpoint. In order to fix ideas, we concentrate our attention on the equation

$$\left(\Delta_g + \frac{\alpha}{2}\right)^2 u = u^{2^\sharp - 1} , \qquad (E_\alpha)$$

where $\alpha > 0$. We let H_2^2 be the Sobolev space consisting of functions u in L^2 which are such that $|\nabla u|$ and $|\nabla^2 u|$ are also in L^2, and let

$$\mathcal{S}_\alpha = \left\{ u \in H_2^2 \text{ s.t. } u \text{ is a solution of } (E_\alpha) \right\} .$$

It is easily seen that the constant function $\overline{u}_\alpha = (\alpha^2/4)^{(n-4)/8}$ is in $\mathcal{S}_\alpha$ for any α. In particular, $\mathcal{S}_\alpha \neq \emptyset$. Extending to fourth order equations the notion of energy function introduced by Hebey [15] for second order equations, we define the energy function E_m of (E_α) by

$$E_m(\alpha) = \inf_{u \in \mathcal{S}_\alpha} E(u) ,$$

where $E(u) = \int_M |u|^{2^\sharp} dv_g$ is the energy of u. It is easily seen that $E_m(\alpha) > 0$ for any $\alpha > 0$. Our main result is as follows. An extension of this result to Paneitz-Branson operators with constant coefficients as in (0.1)-(0.2) is in section 2.

Theorem 0.1. *Let (M,g) be a smooth compact conformally flat Riemannian manifold of dimension $n \geq 5$. Then*

$$\lim_{\alpha \to +\infty} E_m(\alpha) = +\infty .$$

In particular, for any $\Lambda > 0$, there exists $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$, equation (E_α) does not have a solution of energy less than or equal to Λ.

As we will see below, there are several manifolds with the property that (E_α) has nonconstant solutions for arbitrary large α's, and with the property that $E_m(\alpha)$ is not realized by the constant solution $\overline{u}_\alpha$. Such a remark is important since, if not, then Theorem 0.1 is trivial. Theorem 0.1 in the easier case of second order operators was proved by Druet-Hebey-Vaugon [9].

Let K_0 be the sharp constant in the Euclidean Sobolev inequality

$$\|\varphi\|_{2^\sharp} \leq K_0 \|\Delta\varphi\|_2 ,$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is smooth with compact support. The value of K_0 was computed by Edmunds-Fortunato-Janelli [10], Lieb [18], and Lions [20]. We get that

$$K_0^{-2} = \pi^2 n(n-4)(n^2-4)\Gamma\left(\frac{n}{2}\right)^{4/n} \Gamma(n)^{-4/n} ,$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, $x > 0$, is the Euler function. The answer to the sharp constant problem for the H_2^2-Sobolev space, recently obtained by Hebey [16], reads as the existence of some α such that for any $u \in H_2^2(M)$,

$$\left(\int_M |u|^{2^\sharp} dv_g\right)^{2/2^\sharp} \leq K_0^2 \int_M (P_g u)\, u dv_g ,$$

where $P_g u$ is the left hand side in equation (E_α). This is in turn equivalent, the proof of such a claim is not very difficult, to the existence of some α such that $E_m(\alpha) \geq K_0^{-n/2}$. Such a statement requires the understanding of the asymptotic behavior of a sequence of solutions of (E_α) which blows up with one bubble. The more general Theorem 0.1 requires the understanding of the more difficult situation where the sequence blows up with an arbitrary large number of bubbles.

Fourth order equations like equation (E_α) have been intensively investigated in recent years. Among others, possible references are Chang [2], Chang-Yang [3], Djadli-Hebey-Ledoux [4], Djadli-Malchiodi-Ould Ahmedou [5], [6], Esposito-Robert [11], Felli [12], Gursky [13], Hebey [16], Hebey-Robert [17], Lin [19], Robert [24], Van der Vorst [25], [26], and Xu-Yang [27], [28].

Section 1 of this paper is devoted to the proof that there are several manifolds with the property that (E_α) has nonconstant solutions for arbitrary large α's, and such that $E_m(\alpha)$ is not realized by the constant solution $\overline{u}_\alpha$. In section 2 we discuss a possible extension of Theorem 0.1. Sections 3 to 8 are devoted to the proof of this extension, and thus, to the proof of Theorem 0.1.

1. Nonconstant solutions

We claim that there are several manifolds with the property that (E_α) has smooth positive nonconstant solutions for arbitrary large α's, and such that $E_m(\alpha)$ is not realized by the constant solution $\overline{u}_\alpha$. We prove the result for the unit sphere S^n in odd dimension, and for products $S^1 \times M$ where M is arbitrary.

1.1. **The case of S^n.** We let (S^n, h) be the unit n-sphere. We claim that for n odd, equation (E_{α_k}) on S^n possesses a smooth positive nonconstant solution for a sequence (α_k) such that $\alpha_k \to +\infty$ as $k \to +\infty$, with the additional property that $E_m(\alpha_k)$ is not realized by the constant solution $\overline{u}_{\alpha_k}$. Writing that $n = 2m+1$, we let $\{z_j\}$, $j = 1, \dots, m+1$, be the natural complex coordinates on $\mathbb{C}^{m+1}$. Given k integer, we let G_k be the subgroup of $O(n+1)$ generated by

$$z_j \to e^{\frac{2i\pi}{k}} z_j \ ,$$

where $j = 1, \dots, m+1$. We let also $\overline{u}$ be a smooth nonconstant function on S^n having the property that $\overline{u} \circ \sigma = \overline{u}$ for any k and any $\sigma \in G_k$. For instance, $\overline{u}(z_1, \dots, z_{m+1}) = |z_1|^2$. It is easily seen that G_k acts freely on S^n. We let P_k be the quotient manifold S^n/G_k, and h_k be the quotient metric on P_k. We let also $\overline{u}_k = \overline{u}/G_k$ be the quotient function induced by $\overline{u}$ on P_k. We know from Hebey [16] that there exists B such that for any smooth function u on P_k,

$$\|u\|_{2^\sharp}^2 \leq K_0^2 \int_{P_k} (\Delta_{h_k} u)^2 \, dv_{h_k} + B K_0 \|\nabla u\|_2^2 + \frac{B^2}{4} \|u\|_2^2 \ ,$$

where K_0 is the sharp constant in the Euclidean inequality $\|\varphi\|_{2^\sharp} \leq K_0 \|\Delta \varphi\|_2$, φ smooth with compact support. The value of K_0 was computed by Edmunds-Fortunato-Janelli [10], Lieb [18], and Lions [20]. We let $B_0(h_k)$ be the smallest constant B in this inequality. Then,

$$\|u\|_{2^\sharp}^2 \leq K_0^2 \int_{P_k} (\Delta_{h_k} u)^2 \, dv_{h_k} + B_0(h_k) K_0 \|\nabla u\|_2^2 + \frac{B_0(h_k)^2}{4} \|u\|_2^2 \ .$$

Taking $u = 1$, it is easily seen that $B_0(h_k) \geq 2V_{h_k}^{-2/n}$, where V_{h_k} is the volume of P_k with respect to h_k. First, we claim that for k sufficiently large, $B_0(h_k) > 2V_{h_k}^{-2/n}$. If not the case, then for any k,

$$\|\overline{u}_k\|_{2^\sharp}^2 \leq K_0^2 \int_{P_k} (\Delta_{h_k}\overline{u}_k)^2 \, dv_{h_k} + 2K_0 V_{h_k}^{-2/n} \|\nabla \overline{u}_k\|_2^2 + V_{h_k}^{-4/n} \|\overline{u}_k\|_2^2 .$$

Noting that

$$\int_{P_k} |T\overline{u}_k|^p \, dv_{h_k} = \frac{1}{k} \int_{S^n} |T\overline{u}|^p \, dv_h ,$$

where p is any real number, and T is either the identity operator, the gradient operator, or the Laplace-Beltrami operator, we get that, for any k,

$$\|\overline{u}\|_{2^\sharp}^2 \leq \frac{K_0^2}{k^{4/n}} \int_{S^n} (\Delta_h \overline{u})^2 \, dv_h + \frac{2K_0 \omega_n^{-2/n}}{k^{2/n}} \|\nabla \overline{u}\|_2^2 + \omega_n^{-4/n} \|\overline{u}\|_2^2 ,$$

where ω_n is the volume of the unit sphere. Letting $k \to +\infty$, this implies that

$$\left(\int_{S^n} |\overline{u}|^{2^\sharp} dv_h \right)^{2/2^\sharp} \leq \frac{1}{\omega_n^{4/n}} \int_{S^n} \overline{u}^2 dv_h$$

and this is impossible since $\overline{u}$ is nonconstant. The above claim is proved, and $B_0(h_k) > 2V_{h_k}^{-2/n}$ for k sufficiently large. We let now α_k be any real number such that $2V_{h_k}^{-2/n} < \alpha_k < B_0(h_k)$, and let

$$\lambda_k = \inf_{u \in H_2^2 \setminus \{0\}} \frac{\int_{P_k} \left(P_{h_k}^k u\right) u dv_{h_k}}{\|u\|_{2^\sharp}^2} ,$$

where

$$P_{h_k}^k u = \left(\Delta_{h_k} + \frac{\hat{\alpha}_k}{2} \right)^2 u$$

and $\hat{\alpha}_k = \alpha_k K_0^{-1}$. Since $\alpha_k < B_0(h_k)$, we get with the definition of $B_0(h_k)$ that $\lambda_k < K_0^{-2}$. Then it follows from basic arguments, as developed for instance in Djadli-Hebey-Ledoux [4], that there exists a minimizer u_k for λ_k. This minimizer can be chosen positive and smooth. Clearly, u_k is nonconstant. If not the case, then

$$\frac{\alpha_k^2 V_{h_k}^{4/n}}{4} = \lambda_k K_0^2 .$$

Since $2V_{h_k}^{-2/n} < \alpha_k$, the left hand side in this equation is greater than 1. Noting that the right hand side is less than 1, we get a contradiction. Up to a multiplicative positive constant, u_k is a solution of

$$\left(\Delta_{h_k} + \frac{\hat{\alpha}_k}{2} \right)^2 u_k = u_k^{2^\sharp - 1} .$$

If $\tilde{u}_k$ is the smooth positive function on S^n defined by the relation $\tilde{u}_k / G_k = u_k$, then $\tilde{u}_k$ is a nonconstant solution of $(E_{\hat{\alpha}_k})$ on S^n. Since $V_{h_k}^{-1} \to +\infty$ as $k \to +\infty$, we have that $\hat{\alpha}_k \to +\infty$ as $k \to +\infty$. Summarizing, we proved that for n odd, equation $(E_{\hat{\alpha}_k})$ on S^n possesses a smooth positive nonconstant solution $\hat{u}_k$ for a sequence $(\hat{\alpha}_k)$ such that $\hat{\alpha}_k \to +\infty$ as $k \to +\infty$. Noting that $E(\hat{u}_k) < E(\overline{u}_{\alpha_k})$, this proves the first claim we made in this subsection.

1.2. **The case of $S^1 \times M$.** We let (M,g) be any smooth compact Riemannian manifold of dimension $n-1$, and let $S^1(t)$ be the circle in $\mathbb{R}^2$ of center 0 and radius $t>0$. We let $M_t = S^1(t)\times M$, and $g_t = h_t + g$ be the product metric on M_t. We claim that equation (E_{α_k}) on M_1 possesses a smooth positive nonconstant solution for a sequence (α_k) such that $\alpha_k \to +\infty$ as $k\to+\infty$, with the additional property that $E_m(\alpha_k)$ is not realized by the constant solution $\overline{u}_{\alpha_k}$. Given k integer, we let G_k be the subgroup of $O(2)$ generated by

$$z \to e^{\frac{2i\pi}{k}} z\ .$$

We regard G_k as acting on M_t by $(x,y)\to(\sigma(x),y)$, and $M_t/G_k = M_{t/k}$. We let $\overline{u}$ be a smooth nonconstant function on M, and let $\overline{u}_t$ be the function it induces on M_t by $\overline{u}_t(x,y) = \overline{u}(y)$. Then $\overline{u}_t\circ\sigma = \overline{u}_t$ for all $\sigma\in G_k$. We know from Hebey [16] that there exists B such that for any smooth function u on M_t,

$$\|u\|_{2^\sharp}^2 \le K_0^2\int_{M_t}\left(\Delta_{g_t}u\right)^2 dv_{g_t} + BK_0\|\nabla u\|_2^2 + \frac{B^2}{4}\|u\|_2^2\ ,$$

where K_0 is the sharp constant in the Euclidean inequality $\|\varphi\|_{2^\sharp}\le K_0\|\Delta\varphi\|_2$, φ smooth with compact support. We let $B_0(g_t)$ be the smallest constant B in this inequality. Then,

$$\|u\|_{2^\sharp}^2 \le K_0^2\int_{P_k}\left(\Delta_{g_t}u\right)^2 dv_{g_t} + B_0(g_t)K_0\|\nabla u\|_2^2 + \frac{B_0(g_t)^2}{4}\|u\|_2^2\ .$$

Taking $u=1$, it is easily seen that $B_0(g_t)\ge 2V_{g_t}^{-2/n}$, where V_{g_t} is the volume of M_t with respect to g_t. First, we claim that for k sufficiently large, $B_0(g_{1/k}) > 2V_{g_{1/k}}^{-2/n}$. If not the case, then for any k,

$$\|\overline{u}_{1/k}\|_{2^\sharp}^2 \le K_0^2\int_{M_{1/k}}\left(\Delta_{g_{1/k}}\overline{u}_{1/k}\right)^2 dv_{g_{1/k}} + 2K_0V_{g_{1/k}}^{-2/n}\|\nabla\overline{u}_{1/k}\|_2^2 + V_{g_{1/k}}^{-4/n}\|\overline{u}_{1/k}\|_2^2\ .$$

Noting that

$$\int_{M_{1/k}}\left|T\overline{u}_{1/k}\right|^p dv_{g_{1/k}} = \frac{1}{k}\int_{M_1}\left|T\overline{u}_1\right|^p dv_{g_1}\ ,$$

where p is any real number, and T is either the identity operator, the gradient operator, or the Laplace-Beltrami operator, we get that, for any k,

$$\|\overline{u}_1\|_{2^\sharp}^2 \le \frac{K_0^2}{k^{4/n}}\int_{M_1}\left(\Delta_{g_1}\overline{u}_1\right)^2 dv_{g_1} + \frac{2K_0V_{g_1}^{-2/n}}{k^{2/n}}\|\nabla\overline{u}_1\|_2^2 + V_{g_1}^{-4/n}\|\overline{u}_1\|_2^2\ .$$

Hence,

$$\|\overline{u}\|_{2^\sharp}^2 \le \frac{K_0^2(2\pi)^{4/n}}{k^{4/n}}\int_M\left(\Delta_g\overline{u}\right)^2 dv_g + \frac{2(2\pi)^{2/n}K_0V_g^{-2/n}}{k^{2/n}}\|\nabla\overline{u}\|_2^2 + V_g^{-4/n}\|\overline{u}\|_2^2\ ,$$

where V_g is the volume of M with respect to g. Letting $k\to+\infty$, this implies that

$$\left(\int_M|\overline{u}|^{2^\sharp}dv_g\right)^{2/2^\sharp} \le \frac{1}{V_g^{4/n}}\int_M\overline{u}^2 dv_g$$

and this is impossible since $\overline{u}$ is nonconstant. The above claim is proved, and $B_0(g_{1/k}) > 2V_{g_{1/k}}^{-2/n}$ for k sufficiently large. We let now α_k be any real number such that $2V_{g_{1/k}}^{-2/n} < \alpha_k < B_0(g_{1/k})$, and let

$$\lambda_k = \inf_{u \in H_2^2 \setminus \{0\}} \frac{\int_{M_{1/k}} \left(P_{g_{1/k}}^k u\right) u dv_{g_{1/k}}}{\|u\|_{2^\sharp}^2} ,$$

where

$$P_{g_{1/k}}^k u = \left(\Delta_{g_{1/k}} + \frac{\hat{\alpha}_k}{2}\right)^2 u$$

and $\hat{\alpha}_k = \alpha_k K_0^{-1}$. Since $\alpha_k < B_0(h_k)$, we get with the definition of $B_0(h_k)$ that $\lambda_k < K_0^{-2}$. As above, it follows from basic arguments that there exists a minimizer u_k for λ_k. This minimizer can be chosen positive and smooth. Clearly, u_k is nonconstant. If not the case, then

$$\frac{\alpha_k^2 V_{g_{1/k}}^{4/n}}{4} = \lambda_k K_0^2 .$$

Since $2V_{g_{1/k}}^{-2/n} < \alpha_k$, the left hand side in this equation is greater than 1. Noting that the right hand side is less than 1, we get a contradiction. Up to a multiplicative positive constant, u_k is a solution of

$$\left(\Delta_{g_{1/k}} + \frac{\hat{\alpha}_k}{2}\right)^2 u_k = u_k^{2^\sharp - 1} .$$

If $\tilde{u}_k$ is the smooth positive function on M_1 defined by the relation $\tilde{u}_k / G_k = u_k$, then $\tilde{u}_k$ is a nonconstant solution of $(E_{\hat{\alpha}_k})$ on M_1. Since $V_{g_{1/k}}^{-1} \to +\infty$ as $k \to +\infty$, we have that $\hat{\alpha}_k \to +\infty$ as $k \to +\infty$. Summarizing, we proved that equation $(E_{\hat{\alpha}_k})$ on M_1 possesses a smooth positive nonconstant solution $\hat{u}_k$ for a sequence $(\hat{\alpha}_k)$ such that $\hat{\alpha}_k \to +\infty$ as $k \to +\infty$. Noting that $E(\hat{u}_k) < E(\overline{u}_{\alpha_k})$, this proves the first claim we made in this subsection.

2. Extending Theorem 0.1 to a more general equation

Theorem 0.1 can be extended to more general equations than (E_α). Given (M, g) smooth, compact, conformally flat and of dimension $n \geq 5$, we consider the equation

$$\Delta_g^2 u + \alpha \Delta_g u + a_\alpha u = u^{2^\sharp - 1} , \qquad (E'_\alpha)$$

where Δ_g and $2^\sharp$ are as above, and where $\alpha, a_\alpha > 0$. Equation (E'_α) reduces to equation (E_α) when $a_\alpha = \alpha^2/4$. We let $\mathcal{S}'_\alpha$ be the set of functions u in H_2^2 which are such that u is a solution of (E'_α), and define the energy function E'_m of (E'_α) by

$$E'_m(\alpha) = \inf_{u \in \mathcal{S}'_\alpha} E(u) ,$$

where $E(u)$ is as above. We assume that:

(A1) $a_\alpha \leq \frac{\alpha^2}{4}$ for all α, and

(A2) $\frac{a_\alpha}{\alpha} \to +\infty$ as $\alpha \to +\infty$.

These assumptions are clearly satisfied when dealing with (E_α), since in this case $a_\alpha = \alpha^2/4$. We claim that when (A1) and (A2) are satisfied,

$$\lim_{\alpha \to +\infty} E'_m(\alpha) = +\infty . \qquad (2.1)$$

In particular, it follows from (2.1) that for any $\Lambda > 0$, there exists $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$, equation (E'_α) does not have a solution of energy less than or equal to Λ. As an easy remark, such a result is false without any assumption on the behaviour of a_α. For instance, it is easily checked that $E'_m(\alpha) \leq a_\alpha^{n/4} V_g$ where V_g is the volume of M with respect to g, so that $E'_m(\alpha)$ is bounded if a_α is bounded. As another remark, if we assume in addition that a_α is increasing in α, then, with only slight modifications of the arguments developed in section 1, we get that there are several manifolds with the property that (E_α) has smooth positive nonconstant solutions for arbitrary large α's. As in section 1, such a result holds for the unit sphere in odd dimension, and for products $S^1 \times M$. A key point in getting (2.1) is the decomposition

$$\Delta_g^2 u + \alpha \Delta_g u + a_\alpha u = \left(\Delta_g + c_\alpha\right)\left(\Delta_g + d_\alpha\right) \ , \tag{2.2}$$

where c_α and d_α are positive constants given by

$$c_\alpha = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - a_\alpha} \quad \text{and} \quad d_\alpha = \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - a_\alpha} \ . \tag{2.3}$$

The rest of this paper is devoted to the proof of (2.1). Since (2.1) is more general than Theorem 0.1, this will prove Theorem 0.1.

3. Geometrical blow-up points

Given (M,g) smooth, compact, of dimension $n \geq 5$, we let (u_α) be a sequence of smooth positive solutions of equation (E_α). As a remark, it easily follows from the developments in Van der Vorst [25] or Djadli-Hebey-Ledoux [4] that a solution in H_2^2 of equation (E_α) is smooth. We assume that for some $\Lambda > 0$, $E(u_\alpha) \leq \Lambda$ for all α, and that (A1) and (A2) of section 2 hold. We let

$$\tilde{u}_\alpha = \frac{1}{\|u_\alpha\|_{2^\sharp}} u_\alpha$$

so that $\|\tilde{u}_\alpha\|_{2^\sharp} = 1$. Then,

$$\Delta_g^2 \tilde{u}_\alpha + \alpha \Delta_g \tilde{u}_\alpha + a_\alpha \tilde{u}_\alpha = \lambda_\alpha \tilde{u}_\alpha^{2^\sharp - 1} \ , \tag{$\tilde{E}_\alpha$}$$

where $\lambda_\alpha = \|u_\alpha\|_{2^\sharp}^{8/(n-4)}$. In particular, $\lambda_\alpha \leq \Lambda^{4/n}$. Multiplying $(\tilde{E}_\alpha)$ by $\tilde{u}_\alpha$ and integrating, we see that

$$\lim_{\alpha \to +\infty} \|\tilde{u}_\alpha\|_{H_1^2} = 0 \ ,$$

where $\|.\|_{H_1^2}$ is the standard norm of the Sobolev space $H_1^2(M)$ (see for instance Hebey [14]). In particular, blow-up occurs as $\alpha \to +\infty$. Following standard terminology, we say that x_0 is a concentration point for the $\tilde{u}_\alpha$'s if for any $\delta > 0$,

$$\liminf_{\alpha \to +\infty} \int_{B_{x_0}(\delta)} \tilde{u}_\alpha^{2^\sharp} dv_g > 0 \ ,$$

where $B_{x_0}(\delta)$ is the geodesic ball in M of center x_0 and radius δ. The $\tilde{u}_\alpha$'s have at least one concentration point. We claim that the two following propositions hold: up to a subsequence,

(P1) the $\tilde{u}_\alpha$'s have a finite number of concentration points, and

(P2) $\tilde{u}_\alpha \to 0$ in $C^0_{loc}(M \backslash \mathcal{S})$ as $\alpha \to +\infty$,

where $\mathcal{S}$ is the set of the concentration points of the $\tilde{u}_\alpha$'s. The rest of this section is devoted to the proof of (P1) and (P2).

Propositions (P1) and (P2) are easy to prove when discussing second order equations. There are a little bit more tricky when discussing fourth order equations. We borrow ideas from Druet [7]. We start with the following theoretical construction by induction. First, we let $x_\alpha^1 \in M$ be such that

$$\tilde{u}_\alpha(x_\alpha^1) = \max_{x \in M} \tilde{u}_\alpha(x) \, .$$

Clearly, $\tilde{u}_\alpha(x_\alpha^1) \to +\infty$ as $\alpha \to +\infty$. Assuming that $x_\alpha^1, \dots, x_\alpha^i$ are known, we let m_α^i be the function

$$m_\alpha^i(x) = \left(\inf_{j=1,\dots,i} d_g(x_\alpha^j, x) \right)^{\frac{n-4}{2}} \tilde{u}_\alpha(x) \, ,$$

where d_g is the distance with respect to g. If

$$\limsup_{\alpha \to +\infty} \left(\max_{x \in M} m_\alpha^i(x) \right) < +\infty$$

we end up the process. If not, we add one point and let x_α^{i+1} be such that

$$m_\alpha^i(x_\alpha^{i+1}) = \max_{x \in M} m_\alpha^i(x) \, .$$

We also extract a subsequence so that $m_\alpha^i(x_\alpha^{i+1}) \to +\infty$ as $\alpha \to +\infty$. We let S_α be the set of the x_α^i's we get with such a process. We let also $m_\alpha^0 = \tilde{u}_\alpha$.

Our first claim is that there exists N integer and $C > 0$ such that, up to a subsequence,

$$S_\alpha = \left\{ x_\alpha^1, \dots, x_\alpha^N \right\} \tag{3.1}$$

and

$$\left(\inf_{i=1,\dots,N} d_g(x_\alpha^i, x) \right)^{\frac{n-4}{2}} \tilde{u}_\alpha(x) \le C \tag{3.2}$$

for any α and any x in M. In order to prove this claim, we assume that we have k such x_α^i's and, for $i = 1, \dots, k$, we let μ_α^i be such that

$$\tilde{u}_\alpha(x_\alpha^i) = (\mu_\alpha^i)^{-\frac{n-4}{2}} \, .$$

It is clear that $\mu_\alpha^i \to +\infty$ as $\alpha \to +\infty$. Given $\delta > 0$ less than the injectivity radius of (M, g), we let v_α^i be the function defined on $B_0(\delta/\mu_\alpha^i)$, the Euclidean ball of center 0 and radius δ/μ_α^i, by

$$v_\alpha^i(x) = (\mu_\alpha^i)^{\frac{n-4}{2}} \tilde{u}_\alpha \left(\exp_{x_\alpha^i}(\mu_\alpha^i x) \right) \, ,$$

where $\exp_{x_\alpha^i}$ is the exponential map at x_α^i. By construction,

$$\max_{x \in M} m_\alpha^{i-1}(x) = \min_{j<i} \left(\frac{d_g(x_\alpha^i, x_\alpha^j)}{\mu_\alpha^i} \right)^{\frac{n-4}{2}}$$

and this quantity goes to $+\infty$ as $\alpha \to +\infty$. It easily follows that for all $i = 1, \dots, k$, and all $j < i$,

$$\lim_{\alpha \to +\infty} \frac{d_g(x_\alpha^i, x_\alpha^j)}{\mu_\alpha^i} = +\infty \tag{3.3}$$

and that either

$$\lim_{\alpha \to +\infty} \frac{d_g(x_\alpha^i, x_\alpha^j)}{\mu_\alpha^j} = +\infty \tag{3.4}$$

or

$$\frac{d_g(x_\alpha^i, x_\alpha^j)}{\mu_\alpha^j} = O(1) \quad \text{and} \quad \frac{\mu_\alpha^i}{\mu_\alpha^j} = o(1) \; . \tag{3.5}$$

In order to see that either (3.4) or (3.5) hold, just note that

$$\frac{\mu_\alpha^i}{\mu_\alpha^j} = \frac{d_g(x_\alpha^i, x_\alpha^j)}{\mu_\alpha^j} \times \frac{\mu_\alpha^i}{d_g(x_\alpha^i, x_\alpha^j)} \; .$$

Given $x \in B_0(\delta/\mu_\alpha^i)$ we write that

$$v_\alpha^i(x) = \frac{u_\alpha\left(\exp_{x_\alpha^i}(\mu_\alpha^i x)\right)}{u_\alpha(x_\alpha^i)} = \frac{m_\alpha^{i-1}\left(\exp_{x_\alpha^i}(\mu_\alpha^i x)\right)}{D_\alpha^i\left(\exp_{x_\alpha^i}(\mu_\alpha^i x)\right)u_\alpha(x_\alpha^i)} \; ,$$

where

$$D_\alpha^i(x) = \min_{j<i} d_g(x_\alpha^j, x)^{\frac{n-4}{2}} \; .$$

Noting that

$$\begin{aligned} d_g\left(x_\alpha^j, \exp_{x_\alpha^i}(\mu_\alpha^i x)\right) &\geq d_g(x_\alpha^j, x_\alpha^i) - \mu_\alpha^i|x| \\ &\geq d_g(x_\alpha^j, x_\alpha^i)\left(1 - \frac{\mu_\alpha^i}{d_g(x_\alpha^j, x_\alpha^i)}|x|\right) \end{aligned}$$

we get with (3.3) that for any compact subset K of $\mathbb{R}^n$, and any $x \in K$,

$$d_g\left(x_\alpha^j, \exp_{x_\alpha^i}(\mu_\alpha^i x)\right) \geq \frac{1}{2} d_g(x_\alpha^j, x_\alpha^i)$$

as soon as $\alpha \gg 1$. Since in addition $m_\alpha^{i-1}(y) \leq m_\alpha^{i-1}(x_\alpha^i)$ for all y in M, we get that for any compact subset K of $\mathbb{R}^n$, and any $x \in K$,

$$v_\alpha^i(x) \leq 2^{\frac{n-4}{2}} \frac{m_\alpha^{i-1}(x_\alpha^i)}{D_\alpha^i(x_\alpha^i)u_\alpha(x_\alpha^i)} = 2^{\frac{n-4}{2}}$$

provided that $\alpha \gg 1$. It follows that the v_α^i's are bounded on any compact subset of $\mathbb{R}^n$. Now we let g_α be the Riemannian metric given by

$$g_\alpha(x) = \left(\exp_{x_\alpha^i}^\star g\right)(\mu_\alpha^i x) \; .$$

Let ξ be the Euclidean metric. Clearly, for any compact subset K of $\mathbb{R}^n$, $g_\alpha \to \xi$ in $C^2(K)$ as $\alpha \to +\infty$. Moreover, it is easily checked that

$$\Delta_{g_\alpha}^2 v_\alpha^i + \alpha\theta_\alpha^i \Delta_{g_\alpha} v_\alpha^i + a_\alpha \tilde{\theta}_\alpha^i v_\alpha^i = \lambda_\alpha (v_\alpha^i)^{2^\sharp - 1} \; , \tag{3.6}$$

where $\theta_\alpha^i = (\mu_\alpha^i)^2$ and $\tilde{\theta}_\alpha^i = (\mu_\alpha^i)^4$. Equation (3.6) can be written as

$$\left[\left(\Delta_{g_\alpha} + c_\alpha\theta_\alpha^i\right) \circ \left(\Delta_{g_\alpha} + d_\alpha\theta_\alpha^i\right)\right] v_\alpha^i = \lambda_\alpha (v_\alpha^i)^{2^\sharp - 1} \; , \tag{3.7}$$

where c_α and d_α are given by (2.3). We let

$$w_\alpha^i = \Delta_{g_\alpha} v_\alpha^i + d_\alpha \theta_\alpha^i v_\alpha^i \; .$$

Noting that

$$w_\alpha^i(x) = (\mu_\alpha^i)^{\frac{n}{2}}\left(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha\right)\left(\exp_{x_\alpha^i}(\mu_\alpha^i x)\right) \tag{3.8}$$

and that

$$\left(\Delta_g + c_\alpha\right)(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha) > 0$$

we easily get that $w_\alpha^i > 0$. Coming back to (3.7) it follows that

$$\Delta_{g_\alpha} w_\alpha^i \le \lambda_\alpha (v_\alpha^i)^{2^\sharp - 1} .$$

Given $\varepsilon > 0$ we write that

$$\begin{aligned} \Delta_{g_\alpha} (w_\alpha^i)^{1+\varepsilon} &= (1+\varepsilon)(w_\alpha^i)^\varepsilon \Delta_{g_\alpha} w_\alpha^i - \varepsilon(1+\varepsilon)|\nabla w_\alpha^i|^2 (w_\alpha^i)^{\varepsilon-1} \\ &\le (1+\varepsilon)(w_\alpha^i)^\varepsilon \Delta_{g_\alpha} w_\alpha^i \le (1+\varepsilon)\lambda_\alpha (v_\alpha^i)^{2^\sharp-1}(w_\alpha^i)^\varepsilon . \end{aligned}$$

Let $R > 0$ be given. Since the v_α^i's are bounded on any compact subset of $\mathbb{R}^n$, and since the λ_α's are bounded, we get that there exists $C > 0$, independent of α, such that

$$\Delta_{g_\alpha} (w_\alpha^i)^{1+\varepsilon} \le C (w_\alpha^i)^\varepsilon$$

in $B_0(3R)$. Applying the De Giorgi-Nash-Moser iterative scheme, with $\varepsilon > 0$ small, we can write that for any p, there exists $C(p) > 0$, independent of α, such that

$$\max_{x \in B_0(R)} (w_\alpha^i)^{1+\varepsilon}(x) \le C(p) \left(\|(w_\alpha^i)^{1+\varepsilon}\|_{L^p(B_0(2R))} + \|(w_\alpha^i)^\varepsilon\|_{L^{2/\varepsilon}(B_0(2R))} \right) .$$

Taking $p = 2/(1+\varepsilon)$, it follows that

$$\max_{x \in B_0(R)} (w_\alpha^i)^{1+\varepsilon}(x) \le C \|w_\alpha^i\|_{L^2(B_0(2R))}^\varepsilon \left(1 + \|w_\alpha^i\|_{L^2(B_0(2R))} \right) . \tag{3.9}$$

Independently, we easily get with (3.8) that

$$\begin{aligned} \int_{B_0(2R)} (w_\alpha^i)^2 dv_{g_\alpha} &= \int_{B_{x_\alpha^i}(2R\mu_\alpha^i)} \left(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha \right)^2 dv_g \\ &\le \int_M \left(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha \right)^2 dv_g . \end{aligned}$$

Multiplying $(\tilde{E}_\alpha)$ by $\tilde{u}_\alpha$ and integrating over M,

$$\int_M (\Delta_g \tilde{u}_\alpha)^2 dv_g + \alpha \int_M |\nabla \tilde{u}_\alpha|^2 dv_g + a_\alpha \int_M \tilde{u}_\alpha^2 dv_g = \lambda_\alpha \tag{3.10}$$

so that

$$\int_M (\Delta_g \tilde{u}_\alpha)^2 dv_g = O(1) \quad \text{and} \quad a_\alpha \int_M \tilde{u}_\alpha^2 dv_g = O(1) .$$

Noting that $d_\alpha \le \sqrt{a_\alpha}$, it follows from the above equations that

$$\int_{B_0(2R)} (w_\alpha^i)^2 dv_{g_\alpha} = O(1)$$

and then, thanks to (3.9), that the w_α^i's are bounded in $B_0(R)$. Since $R > 0$ is arbitrary, we have proved that the w_α^i's are bounded on any compact subset of $\mathbb{R}^n$. Coming back to the v_α^i's, mimicking what has been done above, we let $\varepsilon > 0$, and write once again that

$$\begin{aligned} \Delta_{g_\alpha} (v_\alpha^i)^{1+\varepsilon} &= (1+\varepsilon)(v_\alpha^i)^\varepsilon \Delta_{g_\alpha} v_\alpha^i - \varepsilon(1+\varepsilon)|\nabla v_\alpha^i|^2 (v_\alpha^i)^{\varepsilon-1} \\ &\le (1+\varepsilon)(v_\alpha^i)^\varepsilon \Delta_{g_\alpha} v_\alpha^i \le (1+\varepsilon) w_\alpha^i (v_\alpha^i)^\varepsilon . \end{aligned}$$

Since the w_α^i's are bounded on any compact subset of $\mathbb{R}^n$, it follows from this equation and the De Giorgi-Nash-Moser iterative scheme that for any $R > 0$, and $\varepsilon > 0$ sufficiently small, there exists $C > 0$, independent of α, such that

$$\max_{x \in B_0(R)} (v_\alpha^i)^{1+\varepsilon}(x) \le C \|v_\alpha^i\|_{L^2(B_0(2R))}^\varepsilon \left(1 + \|v_\alpha^i\|_{L^2(B_0(2R))} \right) .$$

Since $v_\alpha^i(0) = 1$, we have proved that for any $R > 0$, there exists $C_R > 0$, independent of α, such that for any α,

$$\int_{B_0(R)} (v_\alpha^i)^2 dv_{g_\alpha} \geq C_R \ . \tag{3.11}$$

Independently, it is easily seen that

$$\tilde{\theta}_\alpha^i \int_{B_0(R)} (v_\alpha^i)^2 dv_{g_\alpha} = \int_{B_{x_\alpha^i}(R\mu_\alpha^i)} \tilde{u}_\alpha^2 dv_g \ .$$

Hence, thanks to (3.10) and (3.11), the $a_\alpha \tilde{\theta}_\alpha^i$'s are bounded. Since $d_\alpha \leq \sqrt{a_\alpha}$, it comes that the $d_\alpha \theta_\alpha^i$'s are also bounded. Noting that

$$\Delta_{g_\alpha} v_\alpha^i + d_\alpha \theta_\alpha^i v_\alpha^i = w_\alpha^i$$

and thanks to standard elliptic theory, we then get that the v_α^i's are bounded in $C^{1,s}_{loc}(\mathbb{R}^n)$, $0 < s < 1$. In particular, there exists $v \in C^1(\mathbb{R}^n)$ such that, up to a subsequence, $v_\alpha^i \to v$ in $C^1_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$. From this and (3.10), noting that

$$\theta_\alpha^i \int_{B_0(R)} |\nabla v_\alpha^i|^2 dv_{g_\alpha} = \int_{B_{x_\alpha^i}(R\mu_\alpha^i)} |\nabla \tilde{u}_\alpha|^2 dv_g$$

we easily get that

$$\alpha \theta_\alpha^i \int_{B_0(R)} |\nabla v|^2 dx = O(1) \ .$$

Since

$$\int_{B_0(R)} (v_\alpha^i)^{2^\sharp} dv_{g_\alpha} = \int_{B_{x_\alpha^i}(R\mu_\alpha^i)} \tilde{u}_\alpha^{2^\sharp} dv_g \leq 1$$

we must have that $\int_{B_0(R)} |\nabla v|^2 dx > 0$. It follows that the $\alpha\theta_\alpha^i$'s are bounded. Then, up to a subsequence, we can assume that

$$\lim_{\alpha \to +\infty} \alpha \theta_\alpha^i = \lambda^i \quad \text{and} \quad \lim_{\alpha \to +\infty} a_\alpha \tilde{\theta}_\alpha^i = \mu^i \ .$$

Clearly, the $c_\alpha \theta_\alpha^i$'s and $d_\alpha \theta_\alpha^i$'s are also bounded. Coming back to (3.7), and thanks to standard elliptic theory, we get that the v_α^i's are bounded in $C^{4,s}_{loc}(\mathbb{R}^n)$, $0 < s < 1$. In particular, still up to a subsequence, we can assume that $v_\alpha^i \to v$ in $C^4_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$. Here, $v \in C^4(\mathbb{R}^n)$, and $v(0) = 1$. We can also assume that v is in $\mathcal{D}_2^2(\mathbb{R}^n)$, where $\mathcal{D}_2^2(\mathbb{R}^n)$ is the homogeneous Euclidean Sobolev space of order two for integration and order two for differenciation, and that $\lambda_\alpha \to \lambda_\infty$ as $\alpha \to +\infty$. Passing to the limit $\alpha \to +\infty$ in (3.6), it follows that

$$\Delta^2 v + \lambda^i \Delta v + \mu^i v = \lambda_\infty v^{2^\sharp - 1} \ .$$

Thanks to the result of section 4 we then get that $\lambda^i = \mu^i = 0$, so that

$$\Delta^2 v = \lambda_\infty v^{2^\sharp - 1} \ .$$

As a remark, $\lambda_\infty > 0$, since if not, $\tilde{u}_\alpha \to 0$ in $H_2^2(M)$ as $\alpha \to +\infty$, contradicting the normalisation condition $\|\tilde{u}_\alpha\|_{2^\sharp} = 1$. Thanks to the work of Lin [19], see also Hebey-Robert [17], we then get that

$$\lambda_\infty^{1/(2^\sharp - 2)} v(x) = c_n \left(\frac{\lambda_0}{1 + \lambda_0^2 |x - x_0|^2} \right)^{\frac{n-4}{2}} \ ,$$

where $\lambda_0 > 0$, $x_0 \in \mathbb{R}^n$, and $c_n = \left(n(n-4)(n^2-4)\right)^{(n-4)/8}$. In particular,

$$\int_{\mathbb{R}^n} v^{2^\sharp} dx = \frac{1}{(\lambda_\infty K_0^2)^{\frac{n}{4}}} ,$$

where, as in section 1, K_0 is the sharp constant in the Euclidean Sobolev inequality $\|\varphi\|_{2^\sharp} \le K_0 \|\Delta\varphi\|_2$. Then we can write that

$$\int_{B_{x_\alpha^i}(R\mu_\alpha^i)} \tilde{u}_\alpha^{2^\sharp} dv_g = \int_{B_0(R)} (v_\alpha^i)^{2^\sharp} dv_{g_\alpha} = \frac{1}{(\lambda_\infty K_0^2)^{\frac{n}{4}}} + o(1) + \varepsilon_R , \tag{3.12}$$

where $o(1) \to 0$ as $\alpha \to +\infty$, and $\varepsilon_R \to 0$ as $R \to +\infty$. Still in the process of proving (3.1) and (3.2), we now prove that the local energies carried by the x_α^i's can be added. Given $R > 0$, and m integer, we let

$$\Omega_\alpha^m = \bigcup_{i=1}^m B_{x_\alpha^i}(R\mu_\alpha^i) .$$

Obviously $\int_{\Omega_\alpha^m} \tilde{u}_\alpha^{2^\sharp} dv_g \le 1$ since $\Omega_\alpha^m \subset M$. We let

$$\tilde{\Omega}_\alpha^m = \Omega_\alpha^{m-1} \backslash \cup_{i=1}^{m-1} \left(B_{x_\alpha^i}(R\mu_\alpha^i) \backslash B_{x_\alpha^m}(R\mu_\alpha^m)\right) .$$

Then

$$\tilde{\Omega}_\alpha^m \subset \cup_{i=1}^{m-1} \left(B_{x_\alpha^i}(R\mu_\alpha^i) \cap B_{x_\alpha^m}(R\mu_\alpha^m)\right) \tag{3.13}$$

and

$$\int_{\Omega_\alpha^m} \tilde{u}_\alpha^{2^\sharp} dv_g = \int_{B_{x_\alpha^m}(R\mu_\alpha^m)} \tilde{u}_\alpha^{2^\sharp} dv_g + \int_{\Omega_\alpha^{m-1}} \tilde{u}_\alpha^{2^\sharp} dv_g - \int_{\tilde{\Omega}_\alpha^m} \tilde{u}_\alpha^{2^\sharp} dv_g . \tag{3.14}$$

We investigate the last term in the right hand side of (3.14). Thanks to (3.13),

$$\int_{\tilde{\Omega}_\alpha^m} \tilde{u}_\alpha^{2^\sharp} dv_g \le \sum_{i=1}^{m-1} \int_{B_{x_\alpha^i}(R\mu_\alpha^i) \cap B_{x_\alpha^m}(R\mu_\alpha^m)} \tilde{u}_\alpha^{2^\sharp} dv_g .$$

Let $i < m$. We know from (3.3) that

$$\lim_{\alpha \to +\infty} \frac{d_g(x_\alpha^i, x_\alpha^m)}{\mu_\alpha^m} = +\infty .$$

If in addition

$$\lim_{\alpha \to +\infty} \frac{d_g(x_\alpha^i, x_\alpha^m)}{\mu_\alpha^i} = +\infty \tag{3.15}$$

then $B_{x_\alpha^i}(R\mu_\alpha^i) \bigcap B_{x_\alpha^m}(R\mu_\alpha^m) = \emptyset$. If (3.15) is false, then, thanks to (3.5),

$$\frac{d_g(x_\alpha^i, x_\alpha^m)}{\mu_\alpha^i} = O(1) \quad \text{and} \quad \mu_\alpha^m = o(\mu_\alpha^i) . \tag{3.16}$$

We let $\mathcal{R}_\alpha = B_0(R) \bigcap \tilde{\mathcal{R}}_\alpha$ where

$$\tilde{\mathcal{R}}_\alpha = \frac{1}{\mu_\alpha^i} \exp_{x_\alpha^i}^{-1} \left(B_{x_\alpha^m}(R\mu_\alpha^m)\right) .$$

Then, since the v_α^i's are bounded on compact subsets of $\mathbb{R}^n$,

$$\int_{B_{x_\alpha^i}(R\mu_\alpha^i) \cap B_{x_\alpha^m}(R\mu_\alpha^m)} \tilde{u}_\alpha^{2^\sharp} dv_g = \int_{\mathcal{R}_\alpha} (v_\alpha^i)^{2^\sharp} dv_{g_\alpha} \le C \left|\mathcal{R}_\alpha\right| ,$$

where $|\mathcal{R}_\alpha|$ is the Euclidean volume of $\mathcal{R}_\alpha$, and $C > 0$ is independent of α. It is easily seen that $|\mathcal{R}_\alpha| \leq C\left(\mu_\alpha^m(\mu_\alpha^i)^{-1}\right)^n$, where $C > 0$ is independent of α, so that, thanks to (3.16), $|\mathcal{R}_\alpha| = o(1)$. Summarizing, we always have that

$$\int_{\tilde{\Omega}_\alpha^m} \tilde{u}_\alpha^{2^\sharp} dv_g = o(1)$$

and, coming back to (3.14), we have proved that

$$\int_{\Omega_\alpha^m} \tilde{u}_\alpha^{2^\sharp} dv_g = \int_{B_{x_\alpha^m}(R\mu_\alpha^m)} \tilde{u}_\alpha^{2^\sharp} dv_g + \int_{\Omega_\alpha^{m-1}} \tilde{u}_\alpha^{2^\sharp} dv_g + o(1) \ .$$

By induction on m, this implies that

$$\int_{\Omega_\alpha^k} \tilde{u}_\alpha^{2^\sharp} dv_g = \sum_{i=1}^k \int_{B_{x_\alpha^i}(R\mu_\alpha^i)} \tilde{u}_\alpha^{2^\sharp} dv_g + o(1)$$

as soon as we have k sequences (x_α^i), $i = 1, \ldots, k$. Thanks to (3.12), this implies in turn that

$$\int_{\Omega_\alpha^k} \tilde{u}_\alpha^{2^\sharp} dv_g = \frac{k}{(\lambda_\infty K_0^2)^{\frac{n}{4}}} + o(1) + \varepsilon_R \ ,$$

where $o(1) \to 0$ as $\alpha \to +\infty$, and $\varepsilon_R \to 0$ as $R \to +\infty$. Letting $\alpha \to +\infty$, and then $R \to +\infty$, we get that

$$\frac{k}{(\lambda_\infty K_0^2)^{\frac{n}{4}}} \leq 1$$

so that $k \leq (\lambda_\infty K_0^2)^{n/4}$. This proves (3.1) and (3.2).

Up to a subsequence, we can assume that for $i = 1, \ldots, N$, $x_\alpha^i \to x^i$ as $\alpha \to +\infty$. We let

$$\hat{\mathcal{S}} = \left\{x^1, \ldots, x^p\right\}$$

be the limit set, here $p \leq N$, and claim that

$$\tilde{u}_\alpha \to 0 \ \text{ in } C_{loc}^0\left(M\backslash\hat{\mathcal{S}}\right) \tag{3.17}$$

as $\alpha \to +\infty$. We let $x \in M\backslash\hat{\mathcal{S}}$, and $R > 0$ such that $B_x(4R) \subset M\backslash\hat{\mathcal{S}}$. It follows from (3.2) that $\tilde{u}_\alpha \leq C$ in $B_x(3R)$, where $C > 0$ is independent of α. We let $\tilde{v}_\alpha$ be such that

$$\tilde{v}_\alpha = \Delta_g\tilde{u}_\alpha + d_\alpha\tilde{u}_\alpha \ ,$$

where d_α is as in (2.3). It is easily seen that the $\tilde{v}_\alpha$'s are positive and bounded in $L^2(M)$. Since $\Delta_g\tilde{v}_\alpha \leq \lambda_\alpha\tilde{u}_\alpha^{2^\sharp-1}$, we get with the De Giorgi-Nash-Moser iterative scheme that the $\tilde{v}_\alpha$ are bounded in $B_x(2R)$. Given $\varepsilon > 0$, it follows that

$$\begin{aligned} \Delta_g\tilde{u}_\alpha^{1+\varepsilon} &= (1+\varepsilon)\tilde{u}_\alpha^\varepsilon\Delta_g\tilde{u}_\alpha - \varepsilon(1+\varepsilon)|\nabla\tilde{u}_\alpha|^2\tilde{u}_\alpha^{\varepsilon-1} \\ &\leq (1+\varepsilon)\tilde{u}_\alpha^\varepsilon\Delta_g\tilde{u}_\alpha \leq C(\varepsilon)\tilde{u}_\alpha^\varepsilon \ . \end{aligned}$$

Applying once again the De Giorgi-Nash-Moser iterative scheme, we get that

$$\sup_{y\in B_x(R)} \tilde{u}_\alpha^{1+\varepsilon}(y) \leq C\left[\|\tilde{u}_\alpha\|_2^{(1+\varepsilon)} + \|\tilde{u}_\alpha\|_2^\varepsilon\right] \ .$$

Since $\tilde{u}_\alpha \to 0$ in $L^2(M)$ as $\alpha \to +\infty$, this proves (3.17).

Now we claim that (P1) and (P2) hold. It suffices to prove that $\mathcal{S} = \hat{\mathcal{S}}$. It easily follows from (3.17) that $\mathcal{S} \subset \hat{\mathcal{S}}$. Conversely,

$$\int_{B_{x_\alpha^i}(\mu_\alpha^i)} \tilde{u}_\alpha^{2^\sharp} dv_g = \int_{B_0(1)} (v_\alpha^i)^{2^\sharp} dv_{g_\alpha}$$

and we have seen that

$$\lim_{\alpha\to+\infty} \int_{B_0(1)} (v_\alpha^i)^{2^\sharp} dv_{g_\alpha} = \int_{B_0(1)} v^{2^\sharp} dx \ ,$$

where, for some $\lambda_1, \lambda_2 > 0$ and some $x_0 \in \mathbb{R}^n$,

$$\lambda_\infty^{1/(2^\sharp-2)} v(x) = \left(\frac{\lambda_1}{1+\lambda_2^2|x-x_0|^2}\right)^{\frac{n-4}{2}} \ .$$

In particular, $\int_{B_0(1)} v^{2^\sharp} dx > 0$. Noting that for $\delta > 0$, and $\alpha \gg 1$,

$$\int_{B_{x_\alpha^i}(\mu_\alpha^i)} \tilde{u}_\alpha^{2^\sharp} dv_g \le \int_{B_{x^i}(\delta)} \tilde{u}_\alpha^{2^\sharp} dv_g$$

we get that $\hat{\mathcal{S}} \subset \mathcal{S}$. Hence, $\hat{\mathcal{S}} = \mathcal{S}$, and (P1) and (P2) are proved.

4. A Pohozaev type nonexistence result

Let $\mathcal{D}_2^2(\mathbb{R}^n)$ be the homogeneous Euclidean Sobolev space defined as the completion of $C_c^\infty(\mathbb{R}^n)$, the set of smooth functions with compact support, with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^n} (\Delta u)^2 \, dx \ .$$

Given $\lambda, \mu \ge 0$, we let $\Phi_{\lambda,\mu}$ be the functional

$$\Phi_{\lambda,\mu}(u) = \lambda \int_{\mathbb{R}^n} |\nabla u|^2 dx + \mu \int_{\mathbb{R}^n} u^2 dx \ .$$

We assume that there exists $u \in \mathcal{D}_2^2(\mathbb{R}^n)$, of class C^4 and nonnegative, solution of the equation

$$\Delta^2 u + \lambda \Delta u + \mu u = u^{2^\sharp - 1} \tag{4.1}$$

and such that $\Phi_{\lambda,\mu}(u) < +\infty$. Then we claim that either $\lambda = \mu = 0$, or $u \equiv 0$. The rest of this section is devoted to the proof of this rather elementary claim.

We start with the preliminary simple remark that if u is a C^1-function in $\mathbb{R}^n$ with the property that u belongs to some $L^p(\mathbb{R}^n)$, $p \ge 1$, and that $|\nabla u| \in L^2(\mathbb{R}^n)$, then $u \in L^{2^\star}(\mathbb{R}^n)$ where $2^\star = 2n/(n-2)$. Indeed, it is well known that there exists $C > 0$ such that for any $r > 0$, and any $u \in C^1(B_0(r))$,

$$\left(\int_{B_0(r)} |u - \overline{u}_r|^{2^\star} dx\right)^{1/2^\star} \le C \int_{B_0(r)} |\nabla u|^2 dx \ ,$$

where

$$\overline{u}_r = \frac{1}{|B_0(r)|} \int_{B_0(r)} u dx$$

and $|B_0(r)|$ is the volume of the ball $B_0(r)$ of center 0 and radius r. A more general statement in the Riemannian context is in Maheux and Saloff-Coste [21]. Assuming that $u \in L^p(\mathbb{R}^n)$, $p \geq 1$, we can write that

$$\begin{aligned} |\overline{u}_r| &\leq \frac{1}{|B_0(r)|}\int_{B_0(r)} |u| dx \\ &\leq \frac{1}{|B_0(r)|}\left(\int_{B_0(r)} |u|^p dx\right)^{1/p} |B_0(r)|^{1-\frac{1}{p}} \\ &\leq \frac{C}{|B_0(r)|^{1/p}} , \end{aligned}$$

where $C > 0$ is independent of r. Hence, $\overline{u}_r \to 0$ as $r \to +\infty$. We fix $R > 0$. Since $|\nabla u| \in L^2(\mathbb{R}^n)$, we can write that for r large,

$$\left(\int_{B_0(R)} |u - \overline{u}_r|^{2^\star} dx\right)^{1/2^\star} \leq C \int_{\mathbb{R}^n} |\nabla u|^2 dx .$$

Letting $r \to +\infty$, and then $R \to +\infty$, this gives that $u \in L^{2^\star}(\mathbb{R}^n)$ where $2^\star$ is as above. If $u \in \mathcal{D}_2^2(\mathbb{R}^n)$, then $u \in L^{2^\sharp}(\mathbb{R}^n)$. It follows that we have proved that for u as above, solution of (4.1),

$$\Phi_{\lambda,\mu}(u) < +\infty \text{ and } \lambda \neq 0 \quad \Rightarrow \quad u \in L^{2^\star}(\mathbb{R}^n) . \tag{4.2}$$

Another very simple remark is that $|\nabla u| \in L^{2^\star}(\mathbb{R}^n)$. Indeed, thanks to Kato's identity, if φ is a smooth function, then $|\nabla|\nabla\varphi|| \leq |\nabla^2\varphi|$ a.e. Hence, if (u_i) is a sequence in $C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} \int_{\mathbb{R}^n} ||\nabla u_i| - |\nabla u_j||^{2^\star} dx &\leq \int_{\mathbb{R}^n} |\nabla(u_i - u_j)|^{2^\star} dx \\ &\leq C\int_{\mathbb{R}^n} \left|\nabla^2(u_i - u_j)\right|^2 dx \\ &= C\int_{\mathbb{R}^n} (\Delta(u_i - u_j))^2 dx , \end{aligned}$$

where $C > 0$ is the constant for the Sobolev inequality corresponding to the embedding $\mathcal{D}_1^2(\mathbb{R}^n) \subset L^{2^\star}(\mathbb{R}^n)$, and $\mathcal{D}_1^2(\mathbb{R}^n)$ is the homogeneous Sobolev space consisting of the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|\nabla u\|_2$. In particular, C is independent of i and j. This easily gives that $|\nabla u| \in L^{2^\star}(\mathbb{R}^n)$.

Now we let η, $0 \leq \eta \leq 1$, be a smooth function in $\mathbb{R}^n$ such that

$$\eta = 1 \text{ in } B_0(1) \text{ and } \eta = 0 \text{ in } \mathbb{R}^n \backslash B_0(2) .$$

Given $R > 0$, we let also

$$\eta_R(x) = \eta\left(\frac{x}{R}\right) .$$

We consider the Pohozaev type identity as presented in Motron [22], and we plugg $\eta_R u$ into this identity, where u is a solution of (4.1). Then we get that

$$\int_{\mathbb{R}^n} \Delta^2(\eta_R u)\, x^k \partial_k(\eta_R u)\, dx + \frac{n-4}{2}\int_{\mathbb{R}^n} (\Delta(\eta_R u))^2 dx = 0 , \tag{4.3}$$

where x^k is the kth coordinate of x in $\mathbb{R}^n$, and the Einstein summation convention is used so that there is a sum over k in the first term of this equation. We want to prove that if $\Phi_{\lambda,\mu}(u) < +\infty$ and $\lambda \neq 0$ or $\mu \neq 0$, then $u \equiv 0$. We assume in what follows that $\Phi_{\lambda,\mu}(u) < +\infty$ and $\lambda \neq 0$ or $\mu \neq 0$.

We start with the computation of the second term in the left hand side of (4.3). It is easily seen that

$$\begin{aligned}\int_{\mathbb{R}^n} \left(\Delta\left(\eta_R u\right)\right)^2 dx = &\int_{\mathbb{R}^n} \left(\Delta \eta_R\right)^2 u^2 dx + 4\int_{\mathbb{R}^n} \left(\nabla\eta_R \nabla u\right)^2 dx\\ &+\int_{\mathbb{R}^n} \eta_R^2 \left(\Delta u\right)^2 dx - 4\int_{\mathbb{R}^n} \left(\nabla \eta_R \nabla u\right)\left(\Delta\eta_R\right) u dx\\ &+2\int_{\mathbb{R}^n} \eta_R \left(\Delta\eta_R\right) u \left(\Delta u\right) dx - 4\int_{\mathbb{R}^n} \eta_R\left(\nabla\eta_R\nabla u\right)\Delta u dx\ ,\end{aligned}$$

where, for two functions φ and ψ, $(\nabla\varphi\nabla\psi)$ is the scalar product of $\nabla\varphi$ and $\nabla\psi$. Integrating by parts, it is easily seen that

$$\begin{aligned}\int_{\mathbb{R}^n} \eta_R^2 \left(\Delta u\right)^2 dx \quad = \quad &\int_{\mathbb{R}^n} \eta_R^2 u \Delta^2 u dx - \int_{\mathbb{R}^n} \left(\Delta\eta_R^2\right) u \left(\Delta u\right) dx\\ &+4\int_{\mathbb{R}^n} \eta_R \left(\nabla\eta_R \nabla u\right)\Delta u dx\ .\end{aligned}$$

By equation (4.1), integrating by parts,

$$\begin{aligned}\int_{\mathbb{R}^n} \eta_R^2 u\Delta^2 u dx \quad = \quad &\int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp} dx - \lambda\int_{\mathbb{R}^n} \eta_R^2 |\nabla u|^2 dx\\ &-\mu\int_{\mathbb{R}^n} \eta_R^2 u^2 dx - \lambda\int_{\mathbb{R}^n}\left(\nabla\eta_R^2\nabla u\right) u dx\ .\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^n} \left(\Delta\left(\eta_R u\right)\right)^2 dx = \int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp} dx - \lambda\int_{\mathbb{R}^n} \eta_R^2|\nabla u|^2 dx - \mu\int_{\mathbb{R}^n}\eta_R^2 u^2 dx\\ &- \lambda\int_{\mathbb{R}^n}\left(\nabla\eta_R^2\nabla u\right) u dx - \int_{\mathbb{R}^n}\left(\Delta\eta_R^2\right) u\left(\Delta u\right) dx + 4\int_{\mathbb{R}^n}\left(\nabla\eta_R\nabla u\right)^2 dx\\ &+ \int_{\mathbb{R}^n}\left(\Delta\eta_R\right)^2 u^2 dx - 4\int_{\mathbb{R}^n}\left(\nabla\eta_R\nabla u\right)\left(\Delta\eta_R\right) u dx\\ &+ 2\int_{\mathbb{R}^n}\eta_R\left(\Delta\eta_R\right) u \left(\Delta u\right) dx\ .\end{aligned} \tag{4.4}$$

It is easily checked that for $p = 1, 2$,

$$\int_{\mathbb{R}^n} \left(\Delta\eta_R^p\right)^2 u^2 dx = \varepsilon_R\ , \tag{4.5}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. Thanks to Hölder's inequality, we can indeed write that

$$\int_{\mathbb{R}^n} \left(\Delta\eta_R^p\right)^2 u^2 dx \le \left(\int_{\mathcal{A}_R} |\Delta\eta_R^p|^{n/2} dx\right)^{4/n}\left(\int_{\mathcal{A}_R} u^{2^\sharp} dx\right)^{(n-4)/n}\ ,$$

where $\mathcal{A}_R = B_0(2R)\backslash B_0(R)$. Noting that $|\Delta\eta_R^p| \le CR^{-2}$ for some $C > 0$ independent of R, and that $u \in L^{2^\sharp}(\mathbb{R}^n)$, we get (4.5). In particular, since

$$\int_{\mathbb{R}^n} \left|\Delta\eta_R^2\right| u \left|\Delta u\right| dx \le \sqrt{\int_{\mathbb{R}^n} \left(\Delta\eta_R^2\right)^2 u^2 dx}\sqrt{\int_{\mathbb{R}^n} \left(\Delta u\right)^2 dx}$$

$$\int_{\mathbb{R}^n} \eta_R \left|\Delta\eta_R\right| u \left|\Delta u\right| dx \le \sqrt{\int_{\mathbb{R}^n} \left(\Delta\eta_R\right)^2 u^2 dx}\sqrt{\int_{\mathbb{R}^n} \left(\Delta u\right)^2 dx}$$

and $\Delta u \in L^2(\mathbb{R}^n)$, we have also proved that

$$\int_{\mathbb{R}^n} \left(\Delta\eta_R^2\right) u \left(\Delta u\right) dx = \varepsilon_R \ \text{ and } \int_{\mathbb{R}^n} \eta_R \left(\Delta\eta_R\right) u \left(\Delta u\right) dx = \varepsilon_R \ , \tag{4.6}$$

where ε_R is as above. Similarly, thanks to Hölder's inequality, we can write that

$$\int_{\mathbb{R}^n} \left(\nabla\eta_R\nabla u\right)^2 dx \le \left(\int_{\mathcal{A}_R} \left|\nabla\eta_R\right|^n dx\right)^{2/n} \left(\int_{\mathcal{A}_R} \left|\nabla u\right|^{2^\star} dx\right)^{(n-2)/n} ,$$

where $2^\star = 2n/(n-2)$. Noting that $|\nabla\eta_R| \le CR^{-1}$ for some $C > 0$ independent of R, and that $|\nabla u| \in L^{2^\star}(\mathbb{R}^n)$, we get that

$$\int_{\mathbb{R}^n} \left(\nabla\eta_R\nabla u\right)^2 dx = \varepsilon_R \ , \tag{4.7}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. Then, writing that

$$\int_{\mathbb{R}^n} \left|\left(\nabla\eta_R\nabla u\right)\right| \left|\Delta\eta_R\right| u dx \le \sqrt{\int_{\mathbb{R}^n} \left(\nabla\eta_R\nabla u\right)^2 dx}\sqrt{\int_{\mathbb{R}^n} \left(\Delta\eta_R\right)^2 u^2 dx}$$

we get that

$$\int_{\mathbb{R}^n} \left(\nabla\eta_R\nabla u\right)\left(\Delta\eta_R\right) u dx = \varepsilon_R \ , \tag{4.8}$$

where ε_R is as above. At last, we claim that

$$\lambda\int_{\mathbb{R}^n} \left(\nabla\eta_R^2\nabla u\right) u dx = \varepsilon_{\lambda,R} \ , \tag{4.9}$$

where $\varepsilon_{\lambda,R} = 0$ if $\lambda = 0$, and $\varepsilon_{\lambda,R} \to 0$ as $R \to +\infty$ if $\lambda \ne 0$. Indeed, if $\lambda \ne 0$, then $|\nabla u| \in L^2(\mathbb{R}^n)$. According to what we said at the beginning of this section, see (4.2), it follows that $u \in L^{2^\star}(\mathbb{R}^n)$. Then, thanks to Hölder's inequalities, we can write that

$$\int_{\mathbb{R}^n} \left|\left(\nabla\eta_R^2\nabla u\right)\right| u dx \le \sqrt{\int_{\mathbb{R}^n} |\nabla\eta_R^2|^2 u^2 dx}\sqrt{\int_{\mathbb{R}^n} |\nabla u|^2 dx}$$

and that

$$\int_{\mathbb{R}^n} |\nabla\eta_R^2|^2 u^2 dx \le \left(\int_{\mathcal{A}_R} |\nabla\eta_R^2|^n\right)^{2/n} \left(\int_{\mathcal{A}_R} u^{2^\star} dx\right)^{(n-2)/n} .$$

Noting that $|\nabla\eta_R^2| \le CR^{-1}$ for some $C > 0$ independent of R, we get (4.9). Then, plugging (4.5)-(4.9) into (4.4), we get that

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\Delta\left(\eta_R u\right)\right)^2 dx &= \int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp} dx - \lambda\int_{\mathbb{R}^n} \eta_R^2 |\nabla u|^2 dx \\ &- \mu\int_{\mathbb{R}^n} \eta_R^2 u^2 dx + \varepsilon_{\lambda,R} + \varepsilon_R \ , \end{aligned} \tag{4.10}$$

where $\varepsilon_{\lambda,R}$ and ε_R are as above.

Now we compute the first term in the left hand side of (4.3). It is easily checked that

$$\begin{aligned}\Delta^2(\eta_R u) = \eta_R\Delta^2 u + u\Delta^2\eta_R + 2(\Delta\eta_R)(\Delta u)\\ -2\Delta(\nabla\eta_R\nabla u) - 2(\nabla\eta_R\nabla\Delta u) - 2(\nabla u\nabla\Delta\eta_R)\ .\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^n}\Delta^2\left(\eta_R u\right)x^k\partial_k\left(\eta_R u\right)dx\\ &= \int_{\mathbb{R}^n}\eta_R^2(\Delta^2 u)x^k\partial_k u dx + \int_{\mathbb{R}^n}u\eta_R(\Delta^2\eta_R)x^k\partial_k u dx\\ &+ 2\int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)\eta_R x^k\partial_k u dx - 2\int_{\mathbb{R}^n}\eta_R\left(\Delta(\nabla\eta_R\nabla u)\right)x^k\partial_k u dx\\ &- 2\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)\eta_R x^k\partial_k u dx - 2\int_{\mathbb{R}^n}(\nabla u\nabla\Delta\eta_R)\eta_R x^k\partial_k u dx \qquad (4.11)\\ &+ \int_{\mathbb{R}^n}\eta_R u(\Delta^2 u)x^k\partial_k\eta_R dx + \int_{\mathbb{R}^n}u^2(\Delta^2\eta_R)x^k\partial_k\eta_R dx\\ &+ 2\int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)ux^k\partial_k\eta_R dx - 2\int_{\mathbb{R}^n}\left(\Delta(\nabla\eta_R\nabla u)\right)ux^k\partial_k\eta_R dx\\ &- 2\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)ux^k\partial_k\eta_R dx - 2\int_{\mathbb{R}^n}(\nabla u\nabla\Delta\eta_R)ux^k\partial_k\eta_R dx\ .\end{aligned}$$

Noting that $|\Delta^2\eta_R| \le CR^{-4}$ for some $C > 0$ independent of R, and that $|x| \le 2R$ in $\mathcal{A}_R = B_0(2R)\backslash B_0(R)$, we can write that

$$\left|\int_{\mathbb{R}^n}u\eta_R(\Delta^2\eta_R)x^k\partial_k u dx\right| \le \frac{C}{R^3}\int_{\mathcal{A}_R}u|\nabla u|dx\ .$$

Thanks to Hölder's inequality,

$$\frac{1}{R^3}\int_{\mathcal{A}_R}u|\nabla u|dx \le \sqrt{\frac{1}{R^2}\int_{\mathcal{A}_R}|\nabla u|^2 dx}\sqrt{\frac{1}{R^4}\int_{\mathcal{A}_R}u^2 dx}$$

and

$$\begin{aligned}\frac{1}{R^2}\int_{\mathcal{A}_R}|\nabla u|^2 dx \le \frac{1}{R^2}\left|\mathcal{A}_R\right|^{\frac{2}{n}}\left(\int_{\mathcal{A}_R}|\nabla u|^{2^\star}dx\right)^{2/2^\star}\\ \frac{1}{R^4}\int_{\mathcal{A}_R}u^2 dx \le \frac{1}{R^4}\left|\mathcal{A}_R\right|^{\frac{4}{n}}\left(\int_{\mathcal{A}_R}u^{2^\sharp}dx\right)^{2/2^\sharp}\ .\end{aligned}$$

Since $|\mathcal{A}_R| \le CR^n$, $u \in L^{2^\sharp}(\mathbb{R}^n)$ and $|\nabla u| \in L^{2^\star}(\mathbb{R}^n)$, it follows that

$$\int_{\mathbb{R}^n}u\eta_R(\Delta^2\eta_R)x^k\partial_k u dx = \varepsilon_R\ , \qquad (4.12)$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. In a similar way, we can write that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)\eta_R x^k\partial_k u dx\right| &\le \frac{C}{R}\int_{\mathcal{A}_R}|\nabla u||\Delta u|dx\\ &\le C\sqrt{\int_{\mathcal{A}_R}(\Delta u)^2 dx}\sqrt{\frac{1}{R^2}\int_{\mathcal{A}_R}|\nabla u|^2 dx}\end{aligned}$$

so that, here again,

$$\int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)\eta_R x^k\partial_k u dx = \varepsilon_R\ . \tag{4.13}$$

Noting that

$$\left|\int_{\mathbb{R}^n}(\nabla u\nabla\Delta\eta_R)\eta_R x^k\partial_k u dx\right| \le \frac{C}{R^2}\int_{\mathcal{A}_R}|\nabla u|^2 dx$$

we get that

$$\int_{\mathbb{R}^n}(\nabla u\nabla\Delta\eta_R)\eta_R x^k\partial_k u dx = \varepsilon_R\ . \tag{4.14}$$

Noting that

$$\left|\int_{\mathbb{R}^n}u^2(\Delta^2\eta_R)x^k\partial_k\eta_R dx\right| \le \frac{C}{R^4}\int_{\mathcal{A}_R}u^2 dx$$

we get that

$$\int_{\mathbb{R}^n}u^2(\Delta^2\eta_R)x^k\partial_k\eta_R dx = \varepsilon_R\ . \tag{4.15}$$

Similarly, we can write that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)u x^k\partial_k\eta_R dx\right| &\le \frac{C}{R^2}\int_{\mathcal{A}_R}u|\Delta u|dx\\ &\le C\sqrt{\int_{\mathcal{A}_R}(\Delta u)^2dx}\sqrt{\frac{1}{R^4}\int_{\mathcal{A}_R}u^2dx}\end{aligned}$$

so that, as above, we get that

$$\int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)u x^k\partial_k\eta_R dx = \varepsilon_R\ . \tag{4.16}$$

Noting that

$$\left|\int_{\mathbb{R}^n}(\nabla u\nabla\Delta\eta_R)u x^k\partial_k\eta_R dx\right| \le \frac{C}{R^3}\int_{\mathcal{A}_R}u|\nabla u|dx$$

we also have that

$$\int_{\mathbb{R}^n}(\nabla u\nabla\Delta\eta_R)u x^k\partial_k\eta_R dx = \varepsilon_R\ . \tag{4.17}$$

Independently, integrating by parts,

$$\begin{aligned}&\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)\eta_R x^k\partial_k u dx\\ &= \int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)\eta_R x^k\partial_k u dx - \int_{\mathbb{R}^n}(\Delta u)\left(\nabla\eta_R\nabla(\eta_R x^k\partial_k u)\right)dx\\ &= \int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)\eta_R x^k\partial_k u dx - \int_{\mathbb{R}^n}|\nabla\eta_R|^2(\Delta u)x^k\partial_k u dx\\ &\quad - \int_{\mathbb{R}^n}\eta_R(\Delta u)(\nabla\eta_R\nabla u)dx - \int_{\mathbb{R}^n}\eta_R(\Delta u)\nabla^2u(x,\nabla\eta_R)dx\ .\end{aligned}$$

Noting that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}|\nabla\eta_R|^2(\Delta u)x^k\partial_k u dx\right| &\le \frac{C}{R}\int_{\mathcal{A}_R}|\nabla u||\Delta u|dx\\ \left|\int_{\mathbb{R}^n}\eta_R(\Delta u)(\nabla\eta_R\nabla u)dx\right| &\le \frac{C}{R}\int_{\mathcal{A}_R}|\nabla u||\Delta u|dx\end{aligned}$$

and thanks to (4.13), we get that

$$\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)\eta_R x^k\partial_k u dx=\varepsilon_R-\int_{\mathbb{R}^n}\eta_R(\Delta u)\nabla^2 u(x,\nabla\eta_R)dx\ .$$

Noting that $|\Delta u|\leq\sqrt{n}|\nabla^2 u|$, we have that

$$\left|\int_{\mathbb{R}^n}\eta_R(\Delta u)\nabla^2 u(x,\nabla\eta_R)dx\right|\leq C\int_{\mathcal{A}_R}|\nabla^2 u|^2dx\ .$$

Multiplying the Bochner-Lichnerowicz-Weitzenböck formula

$$\langle\Delta du,du\rangle=\frac{1}{2}\Delta|\nabla u|^2+|\nabla^2 u|^2$$

by η_R, and integrating over $\mathbb{R}^n$, it is easily seen that $|\nabla^2 u|\in L^2(\mathbb{R}^n)$. Hence,

$$\int_{\mathcal{A}_R}|\nabla^2 u|^2dx=\varepsilon_R$$

and we get that

$$\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)\eta_R x^k\partial_k u dx=\varepsilon_R\ . \tag{4.18}$$

In a similar way,

$$\begin{aligned}\int_{\mathbb{R}^n}\eta_R\left(\Delta(\nabla\eta_R\nabla u)\right)x^k\partial_k u dx&=\int_{\mathbb{R}^n}(\nabla\Delta\eta_R\nabla u)\eta_R x^k\partial_k u dx\\&+\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)\eta_R x^k\partial_k u dx-2\int_{\mathbb{R}^n}(\nabla^2\eta_R\nabla^2 u)\eta_R x^k\partial_k u dx\ .\end{aligned}$$

Noting that

$$\left|\int_{\mathbb{R}^n}(\nabla\Delta\eta_R\nabla u)\eta_R x^k\partial_k u dx\right|\leq\frac{C}{R^2}\int_{\mathcal{A}_R}|\nabla u|^2dx$$

and that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}(\nabla^2\eta_R\nabla^2 u)\eta_R x^k\partial_k u dx\right|&\leq\frac{C}{R}\int_{\mathcal{A}_R}|\nabla u||\nabla^2 u|dx\\&\leq C\sqrt{\int_{\mathcal{A}_R}|\nabla^2 u|^2dx}\sqrt{\frac{1}{R^2}\int_{\mathcal{A}_R}|\nabla u|^2dx}\end{aligned}$$

we get with (4.18) that

$$\int_{\mathbb{R}^n}\eta_R\left(\Delta(\nabla\eta_R\nabla u)\right)x^k\partial_k u dx=\varepsilon_R\ . \tag{4.19}$$

Similar computations give that

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\Delta(\nabla\eta_R\nabla u)\right)ux^k\partial_k\eta_R dx&=\int_{\mathbb{R}^n}(\nabla\Delta\eta_R\nabla u)ux^k\partial_k\eta_R dx\\&+\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)ux^k\partial_k\eta_R dx-2\int_{\mathbb{R}^n}(\nabla^2\eta_R\nabla^2 u)ux^k\partial_k\eta_R dx\ .\end{aligned}$$

We can write that

$$\left|\int_{\mathbb{R}^n}(\nabla\Delta\eta_R\nabla u)ux^k\partial_k\eta_R dx\right|\leq\frac{C}{R^3}\int_{\mathcal{A}_R}u|\nabla u|dx=\varepsilon_R$$

and that

$$\begin{aligned}\left|\int_{\mathbb{R}^n}(\nabla^2\eta_R\nabla^2 u)ux^k\partial_k\eta_R dx\right| &\le \frac{C}{R^2}\int_{\mathcal{A}_R}u|\nabla^2 u|dx\\ &\le C\sqrt{\int_{\mathcal{A}_R}|\nabla^2 u|^2dx}\sqrt{\frac{1}{R^4}\int_{\mathcal{A}_R}u^2dx}=\varepsilon_R\ .\end{aligned}$$

Integrating by parts,

$$\begin{aligned}&\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)ux^k\partial_k\eta_R dx\\ &=\int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)ux^k\partial_k\eta_R dx-\int_{\mathbb{R}^n}(\Delta u)\left(\nabla\eta_R\nabla(ux^k\partial_k\eta_R)\right)dx\\ &=\int_{\mathbb{R}^n}(\Delta\eta_R)(\Delta u)ux^k\partial_k\eta_R dx-\int_{\mathbb{R}^n}(\Delta u)(\nabla\eta_R\nabla u)x^k\partial_k\eta_R dx\\ &\quad-\int_{\mathbb{R}^n}u(\Delta u)|\nabla\eta_R|^2dx-\int_{\mathbb{R}^n}u(\Delta u)\nabla^2\eta_R(x,\nabla\eta_R)dx\ .\end{aligned}$$

We can write that

$$\left|\int_{\mathbb{R}^n}(\Delta u)(\nabla\eta_R\nabla u)x^k\partial_k\eta_R dx\right|\le\frac{C}{R}\int_{\mathcal{A}_R}|\nabla u||\Delta u|dx$$

and that

$$\left|\int_{\mathbb{R}^n}u(\Delta u)|\nabla\eta_R|^2dx\right|+\left|\int_{\mathbb{R}^n}u(\Delta u)\nabla^2\eta_R(x,\nabla\eta_R)dx\right|\le\frac{C}{R^2}\int_{\mathcal{A}_R}u|\Delta u|dx\ .$$

Since we also have (4.16), we get that

$$\int_{\mathbb{R}^n}\left(\Delta(\nabla\eta_R\nabla u)\right)ux^k\partial_k\eta_R dx=\varepsilon_R \tag{4.20}$$

and that

$$\int_{\mathbb{R}^n}(\nabla\eta_R\nabla\Delta u)ux^k\partial_k\eta_R dx=\varepsilon_R\ . \tag{4.21}$$

At last, we can write that

$$\begin{aligned}&\int_{\mathbb{R}^n}\eta_R u(\Delta^2 u)x^k\partial_k\eta_R dx=\int_{\mathbb{R}^n}(\Delta u)\Delta(u\eta_R x^k\partial_k\eta_R)dx\\ &=\int_{\mathbb{R}^n}\eta_R(x^k\partial_k\eta_R)(\Delta u)^2dx+\int_{\mathbb{R}^n}u(\Delta u)\Delta(\eta_R x^k\partial_k\eta_R)dx\\ &\quad-2\int_{\mathbb{R}^n}\left(\nabla(\eta_R x^k\partial_k\eta_R)\nabla u\right)(\Delta u)dx\ .\end{aligned}$$

It is easily seen that

$$\left|\Delta(\eta_R x^k\partial_k\eta_R)\right|\le\frac{C}{R^2}\quad\text{and}\quad\left|\nabla(\eta_R x^k\partial_k\eta_R)\right|\le\frac{C}{R}$$

for some $C>0$ independent of R. Hence,

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\eta_R u(\Delta^2 u)x^k\partial_k\eta_R dx\right|&\le C\int_{\mathcal{A}_R}(\Delta u)^2dx\\ &+\frac{C}{R^2}\int_{\mathcal{A}_R}u|\Delta u|dx+\frac{C}{R}\int_{\mathcal{A}_R}|\nabla u||\Delta u|dx\end{aligned}$$

and we get with the above developments that

$$\int_{\mathbb{R}^n} \eta_R u(\Delta^2 u) x^k \partial_k \eta_R dx = \varepsilon_R \ . \tag{4.22}$$

Plugging (4.12)-(4.22) into (4.11), we get that

$$\int_{\mathbb{R}^n} \Delta^2 \left(\eta_R u\right) x^k \partial_k \left(\eta_R u\right) dx = \int_{\mathbb{R}^n} \eta_R^2 (\Delta^2 u) x^k \partial_k u dx + \varepsilon_R \ , \tag{4.23}$$

where, as above, $\varepsilon_R \to 0$ as $R \to +\infty$. By (4.1),

$$\begin{aligned} \int_{\mathbb{R}^n} \eta_R^2 (\Delta^2 u) x^k \partial_k u dx = & \int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp - 1} x^k \partial_k u dx \\ & - \lambda \int_{\mathbb{R}^n} \eta_R^2 (\Delta u) x^k \partial_k u dx - \mu \int_{\mathbb{R}^n} \eta_R^2 u x^k \partial_k u dx \ . \end{aligned} \tag{4.24}$$

Integrating by parts, it is easily seen that

$$2^\sharp \int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp - 1} x^k \partial_k u dx = -n \int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp} dx - \int_{\mathbb{R}^n} u^{2^\sharp} x^k \partial_k \eta_R^2 dx \ .$$

Noting that

$$\left| \int_{\mathbb{R}^n} u^{2^\sharp} x^k \partial_k \eta_R^2 dx \right| \le C \int_{\mathcal{A}_R} u^{2^\sharp} dx$$

so that

$$\int_{\mathbb{R}^n} u^{2^\sharp} x^k \partial_k \eta_R^2 dx = \varepsilon_R$$

we get that

$$\int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp - 1} x^k \partial_k u dx = -\frac{n-4}{2} \int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp} dx + \varepsilon_R \ . \tag{4.25}$$

Similarly, it is easily checked that

$$\int_{\mathbb{R}^n} \eta_R^2 u x^k \partial_k u dx = -\frac{n}{2} \int_{\mathbb{R}^n} \eta_R^2 u^2 dx - \frac{1}{2} \int_{\mathbb{R}^n} u^2 x^k \partial_k \eta_R^2 dx \ .$$

If $\mu \neq 0$, $u \in L^2(\mathbb{R}^n)$. Noting that

$$\left| \int_{\mathbb{R}^n} u^2 x^k \partial_k \eta_R^2 dx \right| \le C \int_{\mathcal{A}_R} u^2 dx$$

it follows that

$$\mu \int_{\mathbb{R}^n} u^2 x^k \partial_k \eta_R^2 dx = \varepsilon_{\mu,R} \ ,$$

where $\varepsilon_{\mu,R} = 0$ if $\mu = 0$, and $\varepsilon_{\mu,R} \to 0$ as $R \to +\infty$ if $\mu \neq 0$. Hence,

$$\mu \int_{\mathbb{R}^n} \eta_R^2 u x^k \partial_k u dx = -\frac{n\mu}{2} \int_{\mathbb{R}^n} \eta_R^2 u^2 dx + \varepsilon_{\mu,R} \ . \tag{4.26}$$

Integrating by parts,

$$\begin{aligned} \int_{\mathbb{R}^n} \eta_R^2 (\Delta u) x^k \partial_k u dx = & \int_{\mathbb{R}^n} (\nabla \eta_R^2 \nabla u) x^k \partial_k u \\ & + \int_{\mathbb{R}^n} \eta_R^2 |\nabla u|^2 dx + \int_{\mathbb{R}^n} \eta_R^2 \nabla^2 u(x, \nabla u) dx \end{aligned}$$

and it is easily seen that

$$\int_{\mathbb{R}^n} \eta_R \nabla^2 u(x, \nabla u) dx = -\frac{n}{2} \int_{\mathbb{R}^n} \eta_R^2 |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 x^k \partial_k \eta_R^2 dx \ .$$

If $\lambda \neq 0$, $|\nabla u| \in L^2(\mathbb{R}^n)$. Noting that

$$\left|\int_{\mathbb{R}^n} (\nabla \eta_R^2 \nabla u) x^k \partial_k u\right| + \frac{1}{2}\left|\int_{\mathbb{R}^n} |\nabla u|^2 x^k \partial_k \eta_R^2 dx\right| \leq C \int_{\mathcal{A}_R} |\nabla u|^2 dx$$

we get that

$$\lambda \int_{\mathbb{R}^n} \eta_R^2 (\Delta u) x^k \partial_k u dx = -\frac{(n-2)\lambda}{2} \int_{\mathbb{R}^n} \eta_R^2 |\nabla u|^2 dx + \varepsilon_{\lambda,R} , \tag{4.27}$$

where $\varepsilon_{\lambda,R} = 0$ if $\lambda = 0$, and $\varepsilon_{\lambda,R} \to 0$ as $R \to +\infty$ if $\lambda \neq 0$. Plugging (4.24)-(4.27) into (4.23), it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \Delta^2 \left(\eta_R u\right) x^k \partial_k \left(\eta_R u\right) dx = &-\frac{n-4}{2} \int_{\mathbb{R}^n} \eta_R^2 u^{2^\sharp} dx + \frac{n\mu}{2} \int_{\mathbb{R}^n} \eta_R^2 u^2 dx \\ &+ \frac{(n-2)\lambda}{2} \int_{\mathbb{R}^n} \eta_R^2 |\nabla u|^2 dx + \varepsilon_R + \varepsilon_{\lambda,R} + \varepsilon_{\mu,R} ,\end{aligned} \tag{4.28}$$

where ε_R, $\varepsilon_{\lambda,R}$ and $\varepsilon_{\mu,R}$ are as above.

Plugging (4.10) and (4.28) into (4.3), we get that

$$\lambda \int_{\mathbb{R}^n} \eta_R^2 |\nabla u|^2 dx + 2\mu \int_{\mathbb{R}^n} \eta_R^2 u^2 dx + \varepsilon_{\lambda,R} + \varepsilon_{\mu,R} + \varepsilon_R = 0 , \tag{4.29}$$

where $\varepsilon_{\lambda,R} = 0$ if $\lambda = 0$ and $\varepsilon_{\lambda,R} \to 0$ as $R \to +\infty$ if $\lambda \neq 0$, where $\varepsilon_{\mu,R} = 0$ if $\mu = 0$ and $\varepsilon_{\mu,R} \to 0$ as $R \to +\infty$ if $\mu \neq 0$, and where $\varepsilon_R \to 0$ as $R \to +\infty$. Letting $R \to +\infty$, it is easily seen that if $\Phi_{\lambda,\mu}(u) < +\infty$ and $\lambda \neq 0$ or $\mu \neq 0$, then (4.29) implies that $u \equiv 0$. This proves the claim we made at the beginning of this section.

5. Global L^2 and ∇L^2-concentration

With the notations of section 3, we let $\mathcal{S} = \{x_1, \ldots, x_p\}$. We let also $\delta > 0$ be such that $B_{x_i}(2\delta) \cap B_{x_j}(2\delta) = \emptyset$ for all $i \neq j$ in $\{1, \ldots, p\}$, and set

$$\begin{aligned}\mathcal{R}_{L^2}(\alpha, \delta) &= \frac{\int_{M\setminus\mathcal{B}_\delta} \tilde{u}_\alpha^2 dv_g}{\int_M \tilde{u}_\alpha^2 dv_g} \\ \mathcal{R}_{\nabla L^2}(\alpha, \delta) &= \frac{\int_{M\setminus\mathcal{B}_\delta} |\nabla \tilde{u}_\alpha|^2 dv_g}{\int_M \tilde{u}_\alpha^2 dv_g} ,\end{aligned}$$

where $\mathcal{B}_\delta$ is the union of the $B_{x_i}(\delta)$'s, $i = 1, \ldots, p$. We claim that the two following propositions hold: for any $\delta > 0$,

(P3) $\mathcal{R}_{L^2}(\alpha, \delta) \to 0$ as $\alpha \to +\infty$, and

(P4) $\mathcal{R}_{\nabla L^2}(\alpha, \delta) \to 0$ as $\alpha \to +\infty$.

Proposition (P3) is what we refer to as global L^2-concentration. Proposition (P4) is what we refer to as global weak ∇L^2-concentration. The notion of global strong ∇L^2-concentration is discussed below. Global L^2-concentration was introduced in Druet-Robert [8] (for $p = 1$) and Druet-Hebey-Vaugon [9] (for p arbitrary) when discussing second order equations. Weak ∇L^2-concentration (in the special case $p = 1$) was introduced in Hebey [16]. The rest of this section is devoted to the proof of (P3) and (P4).

We start with the proof of (P3) and (P4). We use the decomposition (2.2), and let c_α, d_α be as in (2.3). All the constants C below are positive and independent of α. Let $\tilde{v}_\alpha$ be given by

$$\tilde{v}_\alpha = \Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha \ .$$

Noting that $\Delta_g \tilde{v}_\alpha + c_\alpha \tilde{v}_\alpha \geq 0$, we get that $\tilde{v}_\alpha$ is nonnegative. We have that

$$\Delta_g \tilde{v}_\alpha \leq \lambda_\alpha \tilde{u}_\alpha^{2^\sharp - 1} \ .$$

Let $\delta > 0$ be given. The De Giorgi-Nash-Moser iterative scheme and proposition (P2) give that

$$\begin{aligned} &\sup_{M\backslash \mathcal{B}_\delta} \left(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha\right) \\ &\leq C \int_{M\backslash \mathcal{B}_{\delta/2}} \left(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha\right) dv_g + C \int_{M\backslash \mathcal{B}_{\delta/2}} \tilde{u}_\alpha dv_g \ . \end{aligned}$$

Let η be a smooth function such that $0 \leq \eta \leq 1$, $\eta = 0$ in $\mathcal{B}_{\delta/4}$, and $\eta = 1$ in $M\backslash \mathcal{B}_{\delta/2}$. Since $\tilde{v}_\alpha \geq 0$,

$$\begin{aligned} &\int_{M\backslash \mathcal{B}_{\delta/2}} \left(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha\right) dv_g \leq \int_M \eta \left(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha\right) dv_g \\ &\leq C \int_{M\backslash \mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g + d_\alpha \int_{M\backslash \mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g \ , \end{aligned}$$

where $C > 0$ is such that $|\Delta_g \eta| \leq C$. It follows that for any $\delta > 0$,

$$\sup_{M\backslash \mathcal{B}_\delta} \left(\Delta_g \tilde{u}_\alpha + d_\alpha \tilde{u}_\alpha\right) \leq C d_\alpha \int_{M\backslash \mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g \ . \tag{5.1}$$

Now we let η be a smooth function such that $0 \leq \eta \leq 1$, $\eta = 0$ in $\mathcal{B}_\delta$, and $\eta = 1$ in $M\backslash \mathcal{B}_{2\delta}$. Thanks to (5.1), and the Cauchy-Schwarz inequality,

$$\int_M \eta \tilde{u}_\alpha \tilde{v}_\alpha dv_g \leq C d_\alpha \left(\int_{M\backslash \mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g\right)^2 \leq C d_\alpha \int_{M\backslash \mathcal{B}_{\delta/4}} u_\alpha^2 dv_g \ .$$

Noting that

$$\int_M (\Delta_g \tilde{u}_\alpha) \eta \tilde{u}_\alpha dv_g = \int_M \eta |\nabla \tilde{u}_\alpha|^2 dv_g + \frac{1}{2} \int_M (\Delta_g \eta) \tilde{u}_\alpha^2 dv_g \tag{5.2}$$

and writing that $\Delta_g \tilde{u}_\alpha = \tilde{v}_\alpha - d_\alpha \tilde{u}_\alpha$, we get that

$$\int_M \eta |\nabla \tilde{u}_\alpha|^2 dv_g + d_\alpha \int_M \eta \tilde{u}_\alpha^2 dv_g \leq C d_\alpha \int_{M\backslash \mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g + \frac{1}{2} \int_M |\Delta_g \eta| \tilde{u}_\alpha^2 dv_g \ .$$

In particular, for any $\delta > 0$,

$$\int_{M\backslash \mathcal{B}_\delta} |\nabla \tilde{u}_\alpha|^2 dv_g \leq C d_\alpha \int_{M\backslash \mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g \ . \tag{5.3}$$

For η as above, we multiply $(\tilde{E}_\alpha)$ by $\eta \tilde{u}_\alpha$ and integrate over M. Then

$$\begin{aligned} &\int_M (\Delta_g^2 \tilde{u}_\alpha) \eta \tilde{u}_\alpha dv_g + \alpha \int_M (\Delta_g \tilde{u}_\alpha) \eta \tilde{u}_\alpha dv_g \\ &+ a_\alpha \int_M \eta \tilde{u}_\alpha^2 dv_g = \lambda_\alpha \int_M \eta \tilde{u}_\alpha^{2^\sharp} dv_g \ . \end{aligned} \tag{5.4}$$

Thanks to proposition (P2) we can write that

$$\int_M \eta \tilde{u}_\alpha^{2^\sharp} dv_g \le C \int_{M\backslash \mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g \ . \tag{5.5}$$

Integrating by parts,

$$\begin{aligned} &\int_M (\Delta_g^2 \tilde{u}_\alpha)\eta \tilde{u}_\alpha dv_g = \int_M \eta (\Delta_g \tilde{u}_\alpha)^2 dv_g \\ &+ \int_M \tilde{u}_\alpha (\Delta_g \eta)(\Delta_g \tilde{u}_\alpha) dv_g - 2\int_M (\nabla\eta\nabla\tilde{u}_\alpha)(\Delta_g \tilde{u}_\alpha) dv_g \ , \end{aligned} \tag{5.6}$$

where $(\nabla\eta\nabla\tilde{u}_\alpha)$ is the pointwise scalar product of $\nabla\eta$ and $\nabla\tilde{u}_\alpha$ with respect to g. As in (5.2),

$$\int_M \tilde{u}_\alpha (\Delta_g \eta)(\Delta_g \tilde{u}_\alpha) dv_g = \int_M (\Delta_g \eta)|\nabla \tilde{u}_\alpha|^2 dv_g + \frac{1}{2}\int_M (\Delta_g^2 \eta)\tilde{u}_\alpha^2 dv_g \ . \tag{5.7}$$

Independently,

$$\int_M (\nabla\eta\nabla\tilde{u}_\alpha)(\Delta_g \tilde{u}_\alpha) dv_g = \int_M \nabla^2\eta(\nabla\tilde{u}_\alpha,\nabla\tilde{u}_\alpha) dv_g + \int_M \nabla^2\tilde{u}_\alpha(\nabla\eta,\nabla\tilde{u}_\alpha) dv_g \tag{5.8}$$

and it is easily seen that

$$\int_M \nabla^2\tilde{u}_\alpha(\nabla\eta,\nabla\tilde{u}_\alpha) dv_g = \frac{1}{2}\int_M (\Delta_g\eta)|\nabla\tilde{u}_\alpha|^2 dv_g \ . \tag{5.9}$$

Combining (5.2) and (5.5)-(5.9) with (5.4), noting that $\int_M \eta(\Delta_g\tilde{u}_\alpha)^2 dv_g \ge 0$, we get that

$$\begin{aligned} &\frac{1}{2}\int_M (\Delta_g^2\eta)\tilde{u}_\alpha^2 dv_g - 2\int_M \nabla^2\eta(\nabla\tilde{u}_\alpha,\nabla\tilde{u}_\alpha) dv_g + \alpha\int_M \eta|\nabla\tilde{u}_\alpha|^2 dv_g \\ &+ \frac{\alpha}{2}\int_M (\Delta_g\eta)\tilde{u}_\alpha^2 dv_g + a_\alpha\int_M \eta\tilde{u}_\alpha^2 dv_g \le C\int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g \ . \end{aligned} \tag{5.10}$$

Clearly,

$$\left|\int_M \nabla^2\eta(\nabla\tilde{u}_\alpha,\nabla\tilde{u}_\alpha) dv_g\right| \le C\int_{M\backslash\mathcal{B}_\delta} |\nabla\tilde{u}_\alpha|^2 dv_g \ .$$

Then, (5.10) gives that

$$\begin{aligned} &\alpha\int_M \eta|\nabla\tilde{u}_\alpha|^2 dv_g + a_\alpha\int_M \eta\tilde{u}_\alpha^2 dv_g \\ &\le C\int_{M\backslash\mathcal{B}_\delta} |\nabla\tilde{u}_\alpha|^2 dv_g + C\alpha\int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g \\ &+ C\int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g \end{aligned} \tag{5.11}$$

By (5.3) we then get that

$$\int_M \eta|\nabla\tilde{u}_\alpha|^2 dv_g + \frac{a_\alpha}{\alpha}\int_M \eta\tilde{u}_\alpha^2 dv_g \le C\int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g \ . \tag{5.12}$$

It follows from (5.12) that

$$\frac{a_\alpha}{\alpha}\int_{M\backslash\mathcal{B}_{2\delta}} \tilde{u}_\alpha^2 dv_g \le C\int_M \tilde{u}_\alpha^2 dv_g \ .$$

Since $\delta > 0$ is arbitrary, and thanks to (A2), we get that (P3) holds. It also follows from (5.12) that

$$\int_{M\setminus\mathcal{B}_{2\delta}} |\nabla\tilde{u}_\alpha|^2 dv_g \le C \int_{M\setminus\mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g$$

so that (P4) holds also.

As a complement to the notion of global weak ∇L^2-concentration, we can define the notion of global strong ∇L^2-concentration. Given $\delta > 0$, we let

$$\mathcal{R}^s_{\nabla L^2}(\alpha,\delta) = \frac{\int_{M\setminus\mathcal{B}_\delta} |\nabla\tilde{u}_\alpha|^2 dv_g}{\int_M |\nabla\tilde{u}_\alpha|^2 dv_g}$$

and say that global strong ∇L^2-concentration holds for the $\tilde{u}_\alpha$'s if for any $\delta > 0$, $\mathcal{R}^s_{\nabla L^2}(\alpha,\delta) \to 0$ as $\alpha \to +\infty$. We claim that global strong ∇L^2-concentration follows from global weak ∇L^2-concentration when $n \ge 8$. Though we do not need global strong ∇L^2-concentration, we discuss this claim in what follows. Let us suppose first that $n \ge 12$. Then $2^\sharp - 1 \le 2$. Integrating $(\tilde{E}_\alpha)$,

$$a_\alpha \|\tilde{u}_\alpha\|_1 = \lambda_\alpha \|\tilde{u}_\alpha\|_{2^\sharp-1}^{2^\sharp-1} .$$

Since $2^\sharp - 1 \le 2$, we can write that

$$\|\tilde{u}_\alpha\|_{2^\sharp-1}^{2^\sharp-1} \le C\|\tilde{u}_\alpha\|_2^{2^\sharp-1} .$$

Thanks to the Sobolev-Poincaré inequality (see for instance Hebey [14]), there exists positive constants A and B such that for any α,

$$\|\tilde{u}_\alpha\|_2^2 \le A\|\nabla\tilde{u}_\alpha\|_2^2 + B\|\tilde{u}_\alpha\|_1^2 .$$

Noting that λ_α is bounded, we then get that

$$\|\tilde{u}_\alpha\|_2^2 \le A\|\nabla\tilde{u}_\alpha\|_2^2 + \frac{C}{a_\alpha^2}\|\tilde{u}_\alpha\|_2^{2(2^\sharp-1)} .$$

Since $2^\sharp - 1 \ge 1$ and $\|\tilde{u}_\alpha\|_2 \to 0$ as $\alpha \to +\infty$, this gives that

$$\int_M \tilde{u}_\alpha^2 dv_g \le C \int_M |\nabla\tilde{u}_\alpha|^2 dv_g .$$

Writing that

$$\begin{aligned}\frac{\int_{M\setminus\mathcal{B}_\delta} |\nabla\tilde{u}_\alpha|^2 dv_g}{\int_M |\nabla\tilde{u}_\alpha|^2 dv_g} &= \frac{\int_{M\setminus\mathcal{B}_\delta} |\nabla\tilde{u}_\alpha|^2 dv_g}{\int_M \tilde{u}_\alpha^2 dv_g}\frac{\int_M \tilde{u}_\alpha^2 dv_g}{\int_M |\nabla\tilde{u}_\alpha|^2 dv_g}\\ &\le C\frac{\int_{M\setminus\mathcal{B}_\delta} |\nabla\tilde{u}_\alpha|^2 dv_g}{\int_M \tilde{u}_\alpha^2 dv_g}\end{aligned}$$

it easily follows from global weak ∇L^2-concentration (proposition (P4) above) that $\mathcal{R}^s_{\nabla L^2}(\alpha,\delta) \to 0$ as $\alpha \to +\infty$. Let us now suppose that $8 \le n \le 12$. Then $2 \le 2^\sharp - 1 \le 2^\sharp$. Thanks to Hölder's inequality, and since $\|\tilde{u}_\alpha\|_{2^\sharp} = 1$, we can write that

$$\|\tilde{u}_\alpha\|_{2^\sharp-1}^{2^\sharp-1} \le \|\tilde{u}_\alpha\|_2^{2/(2^\sharp-2)} .$$

The above procedure, using the Sobolev-Poincaré inequality, then gives that

$$\|\tilde{u}_\alpha\|_2^2 \le A\|\nabla\tilde{u}_\alpha\|_2^2 + \frac{C}{a_\alpha^2}\|\tilde{u}_\alpha\|_2^{4/(2^\sharp-2)} .$$

Noting that $\frac{2}{2^\sharp-2}\geq 1$ when $n\geq 8$, it follows from this inequality that

$$\int_M \tilde{u}_\alpha^2 dv_g \leq C\int_M |\nabla\tilde{u}_\alpha|^2 dv_g$$

and we get as above that $\mathcal{R}^s_{\nabla L^2}(\alpha,\delta)\to 0$ as $\alpha\to+\infty$. This proves our claim.

6. Control of the Hessian

We use the notations of the preceding section, and thus of section 3. We claim that for $\delta>0$ sufficiently small,

$$\frac{\int_{M\backslash\mathcal{B}_\delta}|\nabla^2\tilde{u}_\alpha|^2 dv_g}{\int_M \tilde{u}_\alpha^2 dv_g}=o\left(a_\alpha\right)\ . \tag{6.1}$$

We let η be a smooth function such that $0\leq\eta\leq 1$, $\eta=0$ in $\mathcal{B}_{\delta/2}$ and $\eta=1$ in $M\backslash\mathcal{B}_\delta$. Multiplying $(\tilde{E}_\alpha)$ by $\eta^2\tilde{u}_\alpha$ and integrating over M, we get that

$$\begin{aligned}&\int_M \Delta_g\tilde{u}_\alpha\Delta_g(\eta^2\tilde{u}_\alpha)dv_g+\alpha\int_M\left(\nabla\tilde{u}_\alpha\nabla(\eta^2\tilde{u}_\alpha)\right)dv_g\\&+a_\alpha\int_M\eta^2\tilde{u}_\alpha^2dv_g=\lambda_\alpha\int_M\eta^2\tilde{u}_\alpha^{2^\sharp}dv_g\ .\end{aligned} \tag{6.2}$$

It is easily checked that

$$\int_M \Delta_g\tilde{u}_\alpha\Delta_g(\eta^2\tilde{u}_\alpha)dv_g=\int_M\left(\Delta_g(\eta\tilde{u}_\alpha)\right)^2dv_g+O\left(\int_{\mathcal{B}_\delta\backslash\mathcal{B}_{\delta/2}}\left(|\nabla\tilde{u}_\alpha|^2+\tilde{u}_\alpha^2\right)dv_g\right)$$

and that

$$\begin{aligned}\int_M\left(\nabla\tilde{u}_\alpha\nabla(\eta^2\tilde{u}_\alpha)\right)dv_g&=\int_M|\nabla(\eta\tilde{u}_\alpha)|^2dv_g-\int_M|\nabla\eta|^2\tilde{u}_\alpha^2dv_g\\&=\int_M|\nabla(\eta\tilde{u}_\alpha)|^2dv_g+O\left(\int_{\mathcal{B}_\delta\backslash\mathcal{B}_{\delta/2}}\tilde{u}_\alpha^2dv_g\right)\ .\end{aligned}$$

Independently, we can write with proposition (P2) of section 3 that

$$\int_M\eta^2\tilde{u}_\alpha^{2^\sharp}dv_g=o\left(\int_M\eta^2\tilde{u}_\alpha^2dv_g\right)\ .$$

Coming back to (6.2), it follows that

$$\begin{aligned}&\int_M\left(\Delta_g(\eta\tilde{u}_\alpha)\right)^2dv_g+\alpha\int_M|\nabla(\eta\tilde{u}_\alpha)|^2dv_g+(a_\alpha+o(1))\int_M\eta^2\tilde{u}_\alpha^2dv_g\\&=O\left(\alpha\int_{\mathcal{B}_\delta\backslash\mathcal{B}_{\delta/2}}\tilde{u}_\alpha^2dv_g\right)+O\left(\int_{\mathcal{B}_\delta\backslash\mathcal{B}_{\delta/2}}|\nabla\tilde{u}_\alpha|^2dv_g\right)\ ,\end{aligned} \tag{6.3}$$

where $o(1)\to 0$ as $\alpha\to+\infty$. Thanks to the Bochner-Lichnerowicz-Weitzenböck formula,

$$\int_M\left(\Delta_g(\eta\tilde{u}_\alpha)\right)^2dv_g=\int_M|\nabla^2(\eta\tilde{u}_\alpha)|^2dv_g+\int_M Rc_g\left(\nabla(\eta\tilde{u}_\alpha),\nabla(\eta\tilde{u}_\alpha)\right)dv_g\ ,$$

where Rc_g is the Ricci curvature of g. Writing that

$$\int_M Rc_g\left(\nabla(\eta\tilde{u}_\alpha),\nabla(\eta\tilde{u}_\alpha)\right)dv_g=O\left(\int_M|\nabla(\eta\tilde{u}_\alpha)|^2dv_g\right)$$

we get with (6.3) that

$$\begin{aligned}&\int_M |\nabla^2(\eta\tilde{u}_\alpha)|^2 dv_g + (\alpha + O(1))\int_M |\nabla(\eta\tilde{u}_\alpha)|^2 dv_g + (a_\alpha + o(1))\int_M \eta^2\tilde{u}_\alpha^2 dv_g\\ &= O\left(\alpha\int_{\mathcal{B}_\delta\backslash\mathcal{B}_{\delta/2}} \tilde{u}_\alpha^2 dv_g\right) + O\left(\int_{\mathcal{B}_\delta\backslash\mathcal{B}_{\delta/2}} |\nabla\tilde{u}_\alpha|^2 dv_g\right)\ ,\end{aligned}$$

where $o(1) \to 0$ as $\alpha \to +\infty$, and $O(1)$ is bounded. Since $\eta = 1$ in $M\backslash\mathcal{B}_\delta$, this implies in turn that

$$\begin{aligned}&\int_{M\backslash\mathcal{B}_\delta} |\nabla^2\tilde{u}_\alpha|^2 dv_g + (\alpha + O(1))\int_{M\backslash\mathcal{B}_\delta} |\nabla\tilde{u}_\alpha|^2 dv_g\\ &\quad + (a_\alpha + o(1))\int_{M\backslash\mathcal{B}_\delta} \tilde{u}_\alpha^2 dv_g\\ &= O\left(\alpha\int_{\mathcal{B}_\delta\backslash\mathcal{B}_{\delta/2}} \tilde{u}_\alpha^2 dv_g\right) + O\left(\int_{\mathcal{B}_\delta\backslash\mathcal{B}_{\delta/2}} |\nabla\tilde{u}_\alpha|^2 dv_g\right)\ ,\end{aligned} \tag{6.4}$$

where $o(1) \to 0$ as $\alpha \to +\infty$, and $O(1)$ is bounded. Thanks to global L^2-concentration, and global weak ∇L^2-concentration, and since $\alpha^{-1}a_\alpha \to +\infty$ as $\alpha \to +\infty$, (6.1) follows from (6.4). This proves our claim. As a remark, it easily follows from the above proof that $o(a_\alpha)$ in (6.1) can be replaced by $o(\alpha)$.

7. Conformal changes of the metric

The Paneitz operator, as discovered by Paneitz [23] and extended by Branson [1] to dimensions $n \geq 5$, reads as

$$P_g^n(u) = \Delta_g^2 u - div_g\left(\frac{(n-2)^2+4}{2(n-1)(n-2)}S_g g - \frac{4}{n-2}Rc_g\right)du + \frac{n-4}{2}Q_g^n u\ ,$$

where Rc_g and S_g are respectively the Ricci curvature and scalar curvature of g, and where

$$Q_g^n = \frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}S_g^2 - \frac{2}{(n-2)^2}|Rc_g|^2\ .$$

Let $\hat{g}$ be a conformal metric to g. We write that $g = \varphi^{4/(n-4)}\hat{g}$. Then, we refer to Branson [1],

$$P_{\hat{g}}^n(u\varphi) = \varphi^{2^\sharp-1}P_g^n(u) \tag{7.1}$$

for any smooth function u. Similarly, if

$$L_g^n(u) = \Delta_g u + \frac{n-2}{4(n-1)}S_g u$$

is the conformal Laplacian with respect to g, and if $g = \phi^{4/(n-2)}\hat{g}$, then, for any smooth function u,

$$L_{\hat{g}}^n(u\phi) = \phi^{2^\star-1}L_g^n(u)\ , \tag{7.2}$$

where $2^\star = 2n/(n-2)$. We let $\hat{u}_\alpha = \tilde{u}_\alpha\varphi$, where $\tilde{u}_\alpha$ is as in section 3. It is easily seen that (7.1) and (7.2) imply that

$$\begin{aligned}&\Delta_{\hat{g}}^2\hat{u}_\alpha + \alpha\varphi^{\frac{4}{n-4}}\Delta_{\hat{g}}\hat{u}_\alpha - B_\alpha(\nabla\varphi,\nabla\hat{u}_\alpha) + h_\alpha\hat{u}_\alpha + \varphi^{\frac{n+4}{n-4}}div_g(\varphi^{-1}A_g d\hat{u}_\alpha)\\ &= div_{\hat{g}}(A_{\hat{g}}d\hat{u}_\alpha) - \frac{n-4}{2}Q_{\hat{g}}^n\hat{u}_\alpha - \frac{n-2}{4(n-1)}\alpha\varphi^{\frac{4}{n-4}}S_{\hat{g}}\hat{u} + \lambda_\alpha\hat{u}_\alpha^{2^\sharp-1}\ ,\end{aligned} \tag{$\hat{E}_\alpha^{\hat{g}}$}$$

where A_g and B_α are given by the expressions

$$\begin{aligned} A_g &= \frac{(n-2)^2+4}{2(n-1)(n-2)} S_g g - \frac{4}{n-2} Rc_g \\ B_\alpha &= \frac{4\alpha}{n-4} \varphi^{\frac{8-n}{n-4}} \hat{g} + \varphi^{\frac{12-n}{n-4}} A_g \end{aligned}$$

and where

$$\begin{aligned} h_\alpha = \alpha\varphi^{\frac{2}{n-4}} \Delta_{\hat{g}} \varphi^{\frac{2}{n-4}} - \frac{n-2}{4(n-1)} \alpha\varphi^{\frac{8}{n-4}} S_g + a_\alpha \varphi^{\frac{8}{n-4}} \\ - \frac{n-4}{2} Q_g^n \varphi^{\frac{8}{n-4}} + \varphi^{\frac{n+4}{n-4}} div_g (A_g d\varphi^{-1}) \end{aligned}$$

Assuming that $\hat{g}$ is the Euclidean metric in Ω, where Ω is an open subset of M, we get that

$$\begin{aligned} \Delta^2 \hat{u}_\alpha + \alpha\varphi^{\frac{4}{n-4}} \Delta\hat{u}_\alpha - B_\alpha(\nabla\varphi, \nabla\hat{u}_\alpha) + h_\alpha \hat{u}_\alpha \\ + \varphi^{\frac{n+4}{n-4}} div_g (\varphi^{-1} A_g d\hat{u}_\alpha) = \lambda_\alpha \hat{u}_\alpha^{2^\sharp - 1} \end{aligned} \tag{$\hat{E}_\alpha$}$$

in Ω, where A_g, B_α, and h_α are as above, and $\hat{g} = \xi$ is the Euclidean metric.

8. Proof of the result

We prove (2.1) by contradiction. We assume that there is a sequence (u_α) of solutions to equation (E_α) such that $E(u_\alpha) \le \Lambda$ for some $\Lambda > 0$. Then the results of the preceding sections apply. For $x_i \in \mathcal{S}$, where $\mathcal{S}$ is as in section 3, we let $\delta > 0$ small, and $\varphi \in C^\infty(M)$, $\varphi > 0$, be such that $\varphi^{-4/(n-4)} g$ is flat in $B_{x_i}(4\delta)$ and $\mathcal{S} \bigcap B_{x_i}(4\delta) = \{x_i\}$. Up to the assimilation through the exponential map at x_i, and according to what we said in section 7, we get a smooth positive function $\hat{u}_\alpha$ in $B_0(3\delta)$, solution of equation $(\hat{E}_\alpha)$ in $B_0(3\delta)$, where $B_0(3\delta)$ is the Euclidean ball of center 0 and radius 3δ. We let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta = 1$ in $B_0(\delta)$, and $\eta = 0$ in $\mathbb{R}^n \backslash B_0(2\delta)$. Thanks to the Pohozaev identity used in section 4,

$$\int_{\mathbb{R}^n} \Delta^2 (\eta\hat{u}_\alpha) x^k \partial_k (\eta\hat{u}_\alpha) dx + \frac{n-4}{2} \int_{\mathbb{R}^n} (\Delta (\eta\hat{u}_\alpha))^2 dx = 0 , \tag{8.1}$$

where x^k is the kth coordinate of x in $\mathbb{R}^n$, and the Einstein summation convention is used so that there is a sum over k in the first term of this equation. Similar computations to the ones that were developed in section 4 easily give that

$$\begin{aligned} &\int_{\mathbb{R}^n} \Delta^2 (\eta\hat{u}_\alpha) x^k \partial_k (\eta\hat{u}_\alpha) dx + \frac{n-4}{2} \int_{\mathbb{R}^n} (\Delta (\eta\hat{u}_\alpha))^2 dx \\ &= \int_{\mathbb{R}^n} \eta^2 \left(\Delta^2 \hat{u}_\alpha\right) x^k \partial_k \hat{u}_\alpha dx + \frac{n-4}{2} \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha \Delta^2 \hat{u}_\alpha dx \\ &\quad + O\left(\int_{B_0(2\delta)\backslash B_0(\delta)} \left(|\nabla^2 \hat{u}_\alpha|^2 + |\nabla\hat{u}_\alpha|^2 + \hat{u}_\alpha^2\right) dx \right) . \end{aligned} \tag{8.2}$$

Multiplying equation $(\hat{E}_\alpha)$ by $\eta^2\hat{u}_\alpha$, and integrating over $\mathbb{R}^n$, it comes that

$$\begin{aligned}&\int_{\mathbb{R}^n}\eta^2\hat{u}_\alpha\Delta^2\hat{u}_\alpha dx+\alpha\int_{\mathbb{R}^n}\varphi^{\frac{4}{n-4}}\eta^2\hat{u}_\alpha\Delta\hat{u}_\alpha dx\\&-\int_{\mathbb{R}^n}\eta^2\hat{u}_\alpha B_\alpha(\nabla\varphi,\nabla\hat{u}_\alpha)dx+\int_{\mathbb{R}^n}\eta^2h_\alpha\hat{u}_\alpha^2dx\\&+\int_{\mathbb{R}^n}\eta^2\varphi^{\frac{n+4}{n-4}}\hat{u}_\alpha div_g(\varphi^{-1}A_gd\hat{u}_\alpha)dx=\lambda_\alpha\int_{\mathbb{R}^n}\eta^2\hat{u}_\alpha^{2^\sharp}dx\ .\end{aligned}\tag{8.3}$$

Integrating by parts,

$$\int_{\mathbb{R}^n}\varphi^{\frac{4}{n-4}}\eta^2\hat{u}_\alpha\Delta\hat{u}_\alpha dx=\int_{\mathbb{R}^n}\eta^2\varphi^{\frac{4}{n-4}}|\nabla\hat{u}_\alpha|^2dx+\frac{1}{2}\int_{\mathbb{R}^n}\Delta(\eta^2\varphi^{\frac{4}{n-4}})\hat{u}_\alpha^2dx\ .$$

Independently,

$$\begin{aligned}&\int_{\mathbb{R}^n}\eta^2\hat{u}_\alpha B_\alpha(\nabla\varphi,\nabla\hat{u}_\alpha)dx\\&\quad=\frac{2\alpha}{n-4}\int_{\mathbb{R}^n}\varphi^{\frac{8-n}{n-4}}\eta^2\left(\nabla\varphi\nabla\hat{u}_\alpha^2\right)dx+\int_{\mathbb{R}^n}\varphi^{\frac{12-n}{n-4}}\eta^2\hat{u}_\alpha A_g\left(\nabla\varphi,\nabla\hat{u}_\alpha\right)dx\ .\end{aligned}$$

Integrating by parts,

$$\int_{\mathbb{R}^n}\varphi^{\frac{8-n}{n-4}}\eta^2\left(\nabla\varphi\nabla\hat{u}_\alpha^2\right)dx=\int_{\mathbb{R}^n}\left(\varphi^{\frac{8-n}{n-4}}\eta^2\Delta\varphi-\left(\nabla(\varphi^{\frac{8-n}{n-4}}\eta^2)\nabla\varphi\right)\right)\hat{u}_\alpha^2dx$$

while

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\varphi^{\frac{12-n}{n-4}}\eta^2\hat{u}_\alpha A_g\left(\nabla\varphi,\nabla\hat{u}_\alpha\right)dx\right|&\leq C\int_{B_0(2\delta)}\hat{u}_\alpha|\nabla\hat{u}_\alpha|dx\\&\leq\frac{C}{2}\int_{B_0(2\delta)}\left(|\nabla\hat{u}_\alpha|^2+\hat{u}_\alpha^2\right)dx\ ,\end{aligned}$$

where $C>0$ is independent of α. At last, writing that

$$\eta^2\varphi^{\frac{n+4}{n-4}}div_g\left(\varphi^{-1}A_gd\hat{u}_\alpha\right)=a^{ij}\partial_{ij}\hat{u}_\alpha+b^k\partial_k\hat{u}_\alpha\ ,\tag{8.4}$$

where a^{ij}, b^k are smooth functions with compact support in $B_0(2\delta)$, we easily get that

$$\left|\int_{\mathbb{R}^n}\eta^2\varphi^{\frac{n+4}{n-4}}\hat{u}_\alpha div_g(\varphi^{-1}A_gd\hat{u}_\alpha)dx\right|\leq C\int_{B_0(2\delta)}\left(|\nabla\hat{u}_\alpha|^2+\hat{u}_\alpha^2\right)dx\ ,$$

where $C>0$ is independent of α. Coming back to (8.3), and thanks to the definition of h_α in section 7, it follows from the above developments that

$$\begin{aligned}&\int_{\mathbb{R}^n}\eta^2\hat{u}_\alpha\Delta^2\hat{u}_\alpha dx+\alpha\int_{\mathbb{R}^n}\eta^2\varphi^{\frac{4}{n-4}}|\nabla\hat{u}_\alpha|^2dx\\&+a_\alpha\int_{\mathbb{R}^n}\eta^2\varphi^{\frac{8}{n-4}}\hat{u}_\alpha^2dx=\lambda_\alpha\int_{\mathbb{R}^n}\eta^2\hat{u}_\alpha^{2^\sharp}dx\\&+O\left(\int_{B_0(2\delta)}|\nabla\hat{u}_\alpha|^2dx\right)+O\left(\alpha\int_{B_0(2\delta)}\hat{u}_\alpha^2dx\right)\ .\end{aligned}\tag{8.5}$$

In a similar way, multiplying equation $(\hat{E}_\alpha)$ by $\eta^2 x^k \partial_k u_\alpha$, and integrating over $\mathbb{R}^n$, it comes that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \eta^2 (x^k \partial_k u_\alpha) \Delta^2 \hat{u}_\alpha dx + \alpha \int_{\mathbb{R}^n} \varphi^{\frac{4}{n-4}} \eta^2 (x^k \partial_k u_\alpha) \Delta \hat{u}_\alpha dx \\
&- \int_{\mathbb{R}^n} \eta^2 (x^k \partial_k u_\alpha) B_\alpha(\nabla\varphi, \nabla\hat{u}_\alpha) dx + \int_{\mathbb{R}^n} \eta^2 h_\alpha (x^k \partial_k u_\alpha) \hat{u}_\alpha dx \\
&+ \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{n+4}{n-4}} (x^k \partial_k u_\alpha) div_g(\varphi^{-1} A_g d\hat{u}_\alpha) dx = \lambda_\alpha \int_{\mathbb{R}^n} \eta^2 (x^k \partial_k u_\alpha) \hat{u}_\alpha^{2^\sharp - 1} dx \ .
\end{aligned} \tag{8.6}$$

Integrating by parts,

$$\int_{\mathbb{R}^n} \eta^2 (x^k \partial_k u_\alpha) \hat{u}_\alpha^{2^\sharp - 1} dx = -\frac{n}{2^\sharp} \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha^{2^\sharp} dx - \frac{1}{2^\sharp} \int_{\mathbb{R}^n} (x^k \partial_k \eta^2) \hat{u}_\alpha^{2^\sharp} dx$$

and we can write that

$$\int_{\mathbb{R}^n} (x^k \partial_k \eta^2) \hat{u}_\alpha^{2^\sharp} dx = O\left(\int_{B_0(2\delta)\backslash B_0(\delta)} \hat{u}_\alpha^{2^\sharp} dx \right) \ .$$

Independently, coming back to the expression of h_α in section 7, and integrating by parts, it is easily seen that

$$\begin{aligned}
\int_{\mathbb{R}^n} \eta^2 h_\alpha (x^k \partial_k u_\alpha) \hat{u}_\alpha dx = &-\frac{n a_\alpha}{2} \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{8}{n-4}} \hat{u}_\alpha^2 dx \\
&+ O\left(\alpha \int_{B_0(2\delta)} \hat{u}_\alpha^2 dx \right) + O\left(a_\alpha \int_{B_0(2\delta)} |x| \hat{u}_\alpha^2 dx \right) \ .
\end{aligned}$$

Similarly, thanks to (8.4), integrating by parts, and noting that $a^{ij} = a^{ji}$, we can also write that

$$\int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{n+4}{n-4}} (x^k \partial_k u_\alpha) div_g(\varphi^{-1} A_g d\hat{u}_\alpha) dx = O\left(\int_{B_0(2\delta)} |\nabla \hat{u}_\alpha|^2 dx \right) \ .$$

Independently, thanks to the expression of B_α in section 7, we can write that

$$\int_{\mathbb{R}^n} \eta^2 (x^k \partial_k u_\alpha) B_\alpha(\nabla\varphi, \nabla\hat{u}_\alpha) dx = O\left(\alpha \int_{B_0(2\delta)} |x| |\nabla \hat{u}_\alpha|^2 dx \right) \ .$$

At last, integrating by parts, we get that

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi^{\frac{4}{n-4}} \eta^2 (x^k \partial_k u_\alpha) \Delta \hat{u}_\alpha dx \quad = \quad & -\frac{n-2}{2} \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|^2 dx \\
& + O\left(\int_{B_0(2\delta)} |x| |\nabla \hat{u}_\alpha|^2 dx \right) \ .
\end{aligned}$$

Coming back to (8.6), it follows from the above developments that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \eta^2 (\Delta \hat{u}_\alpha)^2 x^k \partial_k \hat{u}_\alpha dx - \frac{(n-2)\alpha}{2} \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|^2 dx \\
&- \frac{n a_\alpha}{2} \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{8}{n-4}} \hat{u}_\alpha^2 dx + \frac{(n-4)\lambda_\alpha}{2} \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha^{2^\sharp} dx \\
&= O\left(\alpha \int_{B_0(2\delta)} \hat{u}_\alpha^2 dx\right) + O\left(a_\alpha \int_{B_0(2\delta)} |x| \hat{u}_\alpha^2 dx\right) \\
&\quad + O\left(\int_{B_0(2\delta)\backslash B_0(\delta)} \hat{u}_\alpha^{2^\sharp} dx\right) + O\left(\int_{B_0(2\delta)} |\nabla \hat{u}_\alpha|^2 dx\right) \\
&\quad + O\left(\alpha \int_{B_0(2\delta)} |x| |\nabla \hat{u}_\alpha|^2 dx\right) .
\end{aligned}
\tag{8.7}
$$

Plugging (8.5) and (8.7) into (8.2), and thanks to the Pohozaev identity (8.1) of the beginning of this section, we get that

$$
\alpha \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|^2 dx + 2a_\alpha \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{8}{n-4}} \hat{u}_\alpha^2 dx = A_\alpha , \tag{8.8}
$$

where

$$
\begin{aligned}
A_\alpha = &\, O\left(\alpha \int_{B_0(2\delta)} \hat{u}_\alpha^2 dx\right) + O\left(a_\alpha \int_{B_0(2\delta)} |x| \hat{u}_\alpha^2 dx\right) \\
&+ O\left(\int_{B_0(2\delta)\backslash B_0(\delta)} \hat{u}_\alpha^{2^\sharp} dx\right) + O\left(\int_{B_0(2\delta)} |\nabla \hat{u}_\alpha|^2 dx\right) \\
&+ O\left(\alpha \int_{B_0(2\delta)} |x| |\nabla \hat{u}_\alpha|^2 dx\right) + O\left(\int_{B_0(2\delta)\backslash B_0(\delta)} |\nabla^2 \hat{u}_\alpha|^2 dx\right) .
\end{aligned}
$$

Writing that for $s > 0$, $\varphi^s(x) = \varphi^s(0) + O(|x|)$, and that $\hat{u}_\alpha^{2^\sharp} = \hat{u}_\alpha^{2^\sharp - 2} \hat{u}_\alpha^2$, it follows from (8.8) and proposition (P2) of section 3, that

$$
\alpha \int_{\mathbb{R}^n} \eta^2 |\nabla \hat{u}_\alpha|^2 dx + 2a_\alpha \varphi(0)^{\frac{4}{n-4}} \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha^2 dx = \hat{A}_\alpha , \tag{8.9}
$$

where

$$
\begin{aligned}
\hat{A}_\alpha = &\, O\left(\alpha \int_{B_0(2\delta)} \hat{u}_\alpha^2 dx\right) + O\left(a_\alpha \int_{B_0(2\delta)} |x| \hat{u}_\alpha^2 dx\right) \\
&+ O\left(\int_{B_0(2\delta)} |\nabla \hat{u}_\alpha|^2 dx\right) + O\left(\alpha \int_{B_0(2\delta)} |x| |\nabla \hat{u}_\alpha|^2 dx\right) \\
&+ O\left(\int_{B_0(2\delta)\backslash B_0(\delta)} |\nabla^2 \hat{u}_\alpha|^2 dx\right) .
\end{aligned}
$$

Coming back to our Riemannian metric g, it is easily seen that (8.9) gives the existence of positive constants C_1 and C_2, and of positive constants $t_1 < t_2$, independent of α and δ, such that for $\delta > 0$ small,

$$\begin{aligned}&C_1\alpha\int_{B_{x_i}(t_1\delta)}|\nabla\tilde{u}_\alpha|^2dv_g + C_2a_\alpha\int_{B_{x_i}(t_1\delta)}\tilde{u}_\alpha^2dv_g\\&\le \alpha\int_{B_{x_i}(t_2\delta)}\tilde{u}_\alpha^2dv_g + a_\alpha\delta\int_{B_{x_i}(t_2\delta)}\tilde{u}_\alpha^2dv_g + \int_{B_{x_i}(t_2\delta)}|\nabla\tilde{u}_\alpha|^2dv_g\\&\quad+\alpha\delta\int_{B_{x_i}(t_2\delta)}|\nabla\tilde{u}_\alpha|^2dv_g + \int_{B_{x_i}(t_2\delta)\backslash B_{x_i}(t_1\delta)}|\nabla^2\tilde{u}_\alpha|^2dv_g\ .\end{aligned}$$

Summing over the x_i's in $\mathcal{S}$, it follows that for $\delta > 0$ small,

$$\begin{aligned}&C_1\alpha\int_{\mathcal{B}_{t_1\delta}}|\nabla\tilde{u}_\alpha|^2dv_g + C_2a_\alpha\int_{\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g\\&\le \alpha\int_{\mathcal{B}_{t_2\delta}}\tilde{u}_\alpha^2dv_g + a_\alpha\delta\int_{\mathcal{B}_{t_2\delta}}\tilde{u}_\alpha^2dv_g + \int_{\mathcal{B}_{t_2\delta}}|\nabla\tilde{u}_\alpha|^2dv_g\\&\quad+\alpha\delta\int_{\mathcal{B}_{t_2\delta}}|\nabla\tilde{u}_\alpha|^2dv_g + \int_{\mathcal{B}_{t_2\delta}\backslash\mathcal{B}_{t_1\delta}}|\nabla^2\tilde{u}_\alpha|^2dv_g\ .\end{aligned}\tag{8.10}$$

Thanks to global weak ∇L^2-concentration, see proposition (P4) of section 5,

$$\int_{\mathcal{B}_{t_2\delta}\backslash\mathcal{B}_{t_1\delta}}|\nabla\tilde{u}_\alpha|^2dv_g = o\left(\int_{\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g\right)\ .$$

Writing that

$$\int_{\mathcal{B}_{t_2\delta}}|\nabla\tilde{u}_\alpha|^2dv_g = \int_{\mathcal{B}_{t_1\delta}}|\nabla\tilde{u}_\alpha|^2dv_g + \int_{\mathcal{B}_{t_2\delta}\backslash\mathcal{B}_{t_1\delta}}|\nabla\tilde{u}_\alpha|^2dv_g$$

and choosing $\delta > 0$ sufficiently small such that $\delta < C_1$, it follows from (8.10) that for α sufficiently large,

$$\begin{aligned}C_2a_\alpha\int_{\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g &\le \alpha\int_{\mathcal{B}_{t_2\delta}}\tilde{u}_\alpha^2dv_g + a_\alpha\delta\int_{\mathcal{B}_{t_2\delta}}\tilde{u}_\alpha^2dv_g\\&\quad+ o\left(\int_{\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g\right) + \int_{\mathcal{B}_{t_2\delta}\backslash\mathcal{B}_{t_1\delta}}|\nabla^2\tilde{u}_\alpha|^2dv_g\ .\end{aligned}\tag{8.11}$$

Thanks to global L^2-concentration,

$$\int_{\mathcal{B}_{t_2\delta}\backslash\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g = o\left(\int_{\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g\right)$$

while, thanks to (6.1) and global L^2-concentration,

$$\int_{\mathcal{B}_{t_2\delta}\backslash\mathcal{B}_{t_1\delta}}|\nabla^2\tilde{u}_\alpha|^2dv_g = o\left(a_\alpha\int_{\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g\right)\ .$$

Then, writing that

$$\int_{\mathcal{B}_{t_2\delta}}\tilde{u}_\alpha^2dv_g = \int_{\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g + \int_{\mathcal{B}_{t_2\delta}\backslash\mathcal{B}_{t_1\delta}}\tilde{u}_\alpha^2dv_g$$

and choosing $\delta > 0$ sufficiently small such that $2\delta \le C_2$, it follows from (8.11) that for α sufficiently large,

$$\begin{aligned} \frac{C_2}{2} a_\alpha \int_{\mathcal{B}_{t_1\delta}} \tilde{u}_\alpha^2 dv_g \le \alpha \int_{\mathcal{B}_{t_1\delta}} \tilde{u}_\alpha^2 dv_g + o\left(\int_{\mathcal{B}_{t_1\delta}} \tilde{u}_\alpha^2 dv_g\right) \\ + o\left(\alpha \int_{\mathcal{B}_{t_1\delta}} \tilde{u}_\alpha^2 dv_g\right) + o\left(a_\alpha \int_{\mathcal{B}_{t_1\delta}} \tilde{u}_\alpha^2 dv_g\right) . \end{aligned} \tag{8.12}$$

Dividing (8.12) by $a_\alpha \int_{\mathcal{B}_{t_1\delta}} \tilde{u}_\alpha^2 dv_g$, it follows that

$$C_2 \le C_3 \frac{\alpha}{a_\alpha} + o(1) ,$$

where $C_2, C_3 > 0$ are independent of α, and $o(1) \to 0$ as $\alpha \to +\infty$. Letting $\alpha \to +\infty$, thanks to (A2) of section 2, we get a contradiction. This ends the proof of (2.1). As already mentionned, this ends also the proof of Theorem 0.1.

ACKNOWLEDGEMENTS: Part of this work was carried out while the first author was visiting the university of Cergy-Pontoise. The first author wishes to express his gratitude to the university of Cergy-Pontoise for its warm hospitality. The first author is supported by M.U.R.S.T. under the national project "Variational Methods and Nonlinear Differential Equations". The second and third authors are partially supported by a CNRS/NSF grant, "Equations en grand ordre issues de la géométrie", n^o 12932. The authors are indebted to Antonio Ambrosetti, Olivier Druet and Vladimir Georgescu for stimulating discussions and valuable comments on this work.

References

[1] Branson, T.P., Group representations arising from Lorentz conformal geometry, *J. Funct. Anal.*, 74, 1987, 199-291.

[2] S.Y.A. Chang, On Paneitz operator - a fourth order differential operator in conformal geometry, Harmonic Analysis and Partial Differential Equations, Essays in honor of Alberto P. Calderon, Eds: M. Christ, C. Kenig and C. Sadorsky, Chicago Lectures in Mathematics, 1999, 127-150.

[3] Chang, S.Y.A., Yang, P.C., On a fourth order curvature invariant, Comp. Math. 237, Spectral Problems in Geometry and Arithmetic, Ed: T. Branson, AMS, 1999, 9-28.

[4] Djadli, Z., Hebey, E., and Ledoux, M., Paneitz-type operators and applications, *Duke Math. J.*, 104, 2000, 129-169.

[5] Djadli, Z., Malchiodi, A., and Ould Ahmedou, M., Prescribed fourth order conformal invariant on the standard sphere - Part I: a perturbation result, *Communications in Contemporary Mathematics*, To appear.

[6] Djadli, Z., Malchiodi, A., and Ould Ahmedou, M., Prescribed fourth order conformal invariant on the standard sphere - Part II: blow-up analysis and applications, *Ann. Scuola Norm. Sup. Pisa*, To appear.

[7] Druet, O., Asymptotiques géométriques, phénomènes de concentration et inégalités optimales, *PHD thesis*, 468 pages, 2001.

[8] Druet, O., and Hebey, E., *The AB program in geometric analysis. Sharp Sobolev inequalities and related problems*, Memoirs of the American Mathematical Society, To appear.

[9] Druet, O., Hebey, E., and Vaugon, M., Pohozaev type obstructions and solutions of bounded energy for quasilinear elliptic equations with critical Sobolev growth - The conformally flat case, *Nonlinear Analysis, TMA*, To appear.

[10] Edmunds, D.E., Fortunato, F., Janelli, E., Critical exponents, critical dimensions, and the biharmonic operator, *Arch. Rational Mech. Anal.*, 112, 1990, 269-289.

[11] Esposito, P., and Robert, F., Mountain pass critical points for Paneitz-Branson operators, *Calc. Var. Partial Differential Equations*, To appear.

[12] Felli, V., Existence of conformal metrics on S^n with prescribed fourth order invariant, *Adv. Differential Equations*, 7, 2002, 47-76.

[13] M.J. Gursky, The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE, *Comm. Math. Phys.*, 207, 1999, 131-143.

[14] Hebey, E., *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Institute of Mathematical Sciences, Lecture Notes in Mathematics, 5, 1999. Second Edition published jointly with the American Mathematical Society.

[15] Hebey, E., Nonlinear elliptic equations of critical Sobolev growth from a dynamical viewpoint. Conference in honor of Haïm Brézis and Felix Browder, Rutgers university, *Preprint*, 2001.

[16] Hebey, E., Sharp inequalities of second order, *J. Geom. Anal.*, To appear.

[17] Hebey, E., and Robert, F., Coercivity and Struwe's compactness for Paneitz type operators with constant coefficients, *Calc. Var. Partial Differential Equations*, 13, 2001, 491-517.

[18] Lieb, E.H., Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.*, 118, 1983, 349-374.

[19] Lin, C.S., A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, *Comment. Math. Helv.*, 73, 1998, 206-231.

[20] Lions, P.L., The concentration-compactness principle in the calculus of variations I, II, *Rev. Mat. Iberoamericana*, 1, 145-201 and 45-121, 1985.

[21] Maheux, P., and Saloff-Coste, L., Analyse sur les boules d'un opérateur sous-elliptique, *Math. Ann.*, 303, 1995, 713-740.

[22] Motron, M., On some nonlinear equation of the fourth order and critical Sobolev exponent, *Preprint*, 2001.

[23] Paneitz, S., A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds, *Preprint*, 1983.

[24] Robert, F., Positive solutions for a fourth order equation invariant under isometries, *Proc. Amer. Math. Soc.*, To appear.

[25] Van der Vorst, R.C.A.M., Best constants for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{2N/(N-4)}(\Omega)$, *Differential Integral Equations*, 6, 1993, 259-276.

[26] Van der Vorst, R.C.A.M., Fourth order elliptic equations with critical Sobolev growth, *C. R. Acad. Sci. Paris Sér. I Math.*, 320, 1995, 295-299.

[27] Xu, X., and Yang, P., Positivity of Paneitz operators, *Discrete and Continuous Dynamical Systems* 7, 2001, 329-342.

[28] Xu, X., and Yang, P., On a fourth order equation in $3D$, *Preprint*, 2001.

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