Strategyproof and efficient preference aggregation with Kemeny-based criteria

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Abstract

Suppose a group of agents submit strict linear orderings over a set of alternatives. An aggregation rule is a function mapping this information into a unique social ordering. In a recent paper, Bossert and Sprumont [5] introduced betweeness-based notions of efficiency and strategyproofness for aggregation rules and identified three broad classes of rules which satisfy them. The current paper suggests that such betweeness-based requirements may at times be too weak and introduces stronger concepts based on Kemeny distances, namely K-efficiency and K-strategyproofness. When there are three alternatives, all Condorcet-Kemeny rules are both K-efficient and K-strategyproof for a large subdomain of profiles. Moreover, all status-quo rules are K-strategyproof, though not K-efficient. When the number of alternatives exceeds three none of the rules discussed by Bossert and Sprumont satisfy K-strategyproofness, while just Condorcet-Kemeny rules satisfy K-efficiency. The existence of a nondictatorial and onto K-strategyproof rule is an open question.

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1 Introduction

Suppose a group of agents submit strict linear orderings (i.e., complete, transitive, and anti-symmetric binary relations) over a set of alternatives. An *aggregation rule* (also known as an *Arrovian social welfare function*) is a function mapping this information into a single "social" ordering, that is meant to represent the group's aggregate preferences.[1]

In contrast to other settings of social choice (like, say, that of selecting a single winning alternative on the basis of a set of orderings), strategic issues have not been the object of extensive study in the context of aggregation rules. This is primarily because it has not been clear how to model individual preferences over orderings of alternatives. For instance, if there are four alternatives $\{a, b, c, d\}$ and an agent has the ordering *abcd*,[2] it is not immediately clear whether, on the basis of her ordering, she prefers the outcome *acdb* to *bcda*. As a result, we cannot assess whether this agent would wish to somehow misreport her preferences in order to change an aggregation rule's outcome from *acdb* to *bcda*.

One way of dealing with this issue is through a notion of betweeness discussed in Grandmont [8]. An ordering R is said to be *between* two orderings R_1 and R_2, if and only if it agrees with both R_1 and R_2 whenever the latter two agree. For instance, *abcd* is between *adcb* and *bcad*: the latter two orderings only agree on ordered pair (a, d) and this binary comparison is respected in *abcd*. In recent work, Sato [15] and Bossert and Sprumont [5] used this notion of betweeness to address strategyproofness in preference aggregation. In their work, a rule is deemed strategyproof if misreporting one's ordering cannot lead to a new social ordering that is between that under truthful reporting and the agent's own preferences. This property amounts to requiring that the "truthful" social ordering not be unambiguously dominated by that produced under misreporting. This is a rather weak measure of non-manipulability and I will henceforth refer to it as *weak strategypoofness*.[3] Bossert and Sprumont [5] employed similar betweeness-based reasoning to define an efficiency criterion for orderings, which I refer to as *weak efficiency*. In their model, a rule satisfies weak efficiency if it always gives rise to an ordering such that there exists no other that unambiguously dominates it for all agents.

Sato [15] demonstrated that weak strategyproofness combined with an axiom of so-called *bounded response* leads to a number of impossibility results. In his framework, a rule is said to

[1] Since this paper exclusively deals with strict linear orderings, from now on I use the simpler term "orderings" to denote "strict linear orderings".

[2] Here, as in the rest of the paper, orderings are denoted by strings of alternatives, where an alternative's position in the string corresponds to its rank.

[3] Indeed, Bossert and Sprumont state that theirs *"...is the weakest meaningful definition* [of strategyproofness] *applicable to an aggregation rule."*.

satisfy bounded response if two preference profiles differing along a single binary comparison for a single agent yield a pair of social orderings differing in at most one binary comparison. While intuitive as a technical continuity check, bounded response seems to lack a strong normative justification and is not satisfied by many compelling rules. In contrast, Bossert and Sprumont [5] showed that, absent Sato's strong axiom of bounded response, a rich set of possibility results emerge. They rigorously analyzed three weakly efficient classes of rules which are consistent with weak strategyproofness: monotonic majority alteration rules, status-quo rules, and rules generalizing the Condorcet-Kemeny rule (Kemeny [10]).

However, while weak strategyproofness allows for novel theoretical insights, I argue that it may set too high a bar for manipulability. To demonstrate how it may need to be strengthened, I examine the Condorcet-Kemeny rule and show how it can lead to situations in which misreporting one's preferences seems to represent a compelling course of action. Motivated by this "failure" of the Condorcet-Kemeny rule, I suggest that it may be of interest to examine a stronger notion of strategyproofness that is based on Kemeny (or Kendall-τ) distances, a commonly-used metric of the space of linear orderings [10, 11, 4]. A related examination of the efficiency properties of status-quo rules indicates that weak efficiency may also need to be strengthened in an analogous Kemeny-like fashion.

Thus, in my framework, preferences over the space of orderings are modeled via Kemeny distances. For an agent with ordering R, an ordering R' is strictly preferred to R'' if and only its Kemeny distance from R is strictly smaller. Using this notion of preferences over orderings, a rule is said to be *K-strategyproof* if by misreporting her preferred ordering an agent cannot obtain an outcome that is closer –in the Kemeny sense– to her true ordering. Using the same logic, Bossert and Sprumont's weak efficiency requirement can be strengthened via the concept of *K-efficiency*: an ordering is K-efficient if there does not exist another ordering implying weakly smaller Kemeny distances for all agents, and strictly smaller for some. An ancillary property of K-efficiency is that it implies *local unanimity*, where the latter ensures that if there exists an alternative a that all agents prefer to b, then the social ordering should also rank a above b. By contrast, local unanimity is logically unrelated to Bossert and Sprumont's weaker notion of efficiency.

This Kemeny-inspired way of modeling agent preferences over orderings, as well as its effect on strategic behavior, was first studied by Bossert and Storcken [6]. They established impossibility results for rules satisfying an independence condition (extrema independence) and the much stronger property of *K-coalitional-strategyproofness*, which requires that no coalition of agents can profitably jointly misrepresent its preferences. More recently, others have also used the Kemeny distance as a way to model preferences over orderings (Baldiga [1], Laffond and Laine [12], Laine et al. [13], Baldiga and Green [2]). Finally, as the proposed Kemeny-based model admits a graph-theoretic formulation, prior relevant work can also be found in the literature on distance-based preferences and strategyproof location (Moulin [14], Demange [7], Barbera et al. [3], Schummer and Vohra [16]).

Having introduced these Kemeny-based notions of efficiency and strategyproofness, the natural next question to ask is whether we can find any nontrivial rules that satisfy them. In the case of three alternatives, the answer is by and large affirmative: all Condorcet-Kemeny rules are both K-strategyproof and K-efficient on a large subdomain of preference profiles. Moreover, status-quo rules, though not K-efficient, are K-strategyproof on the entire profile domain. In marked contrast, when there are four or more alternatives these positive results vanish. Indeed, all three classes of rules studied by Bossert and Sprumont violate K-strategyproofness and just Condorcet-Kemeny rules satisfy K-efficiency. The existence of a nondictatorial and onto (and thus nontrivial) K-strategyproof rule is an open question worthy of further study.

2 Model description

In what follows, I adopt the notation of Bossert and Sprumont [5]. Let A be a finite set containing $m \geq 3$ alternatives. Let $\mathbb{N}$ denote the set of natural numbers, and let $\mathcal{N}$ denote the set of all finite nonempty subsets of $\mathbb{N}$. Each $N \in \mathcal{N}$ represents a group of agents.

Agents submit strict linear orderings[4] over alternatives in A (i.e., complete, transitive, and antisymmetric binary relations) and the set of such preferences is denoted by $\mathcal{R}$. Given $N \in \mathcal{N}$, the set of possible preference profiles for that group is given by $\mathcal{R}^N$. An *aggregation rule* is a function that assigns to each preference profile an ordering, i.e., it is a function $f : \bigcup_{N \in \mathcal{N}} \mathcal{R}^N \mapsto \mathcal{R}$.[5]

Let us now introduce a notion of betweeness for orderings due to Grandmont [8]. For any $R, R', R'' \in \mathcal{R}$, we say that R'' is *between* R and R', and write $R'' \in [R, R']$, if and only if $R \cap R' \subseteq R''$. That is, ordering R'' agrees with both R and R' whenever the latter two agree.

Bossert and Sprumont [5] define the *prudent extension* of an ordering $R \in \mathcal{R}$ as the binary relation $\mathbf{R}$ over orderings given by

$$R'' \,\mathbf{R}\, R' \;\Leftrightarrow R'' \in [R, R'], \quad \text{for all } R'', R' \in \mathcal{R}.$$

Hence, for an agent holding the ordering R, R'' is at least as good as R' if and only if R'' is

[4]For brevity, from now on I use the simpler term “orderings”.

[5]Unlike Bossert and Sprumont [5] who allow for rules producing weak orderings (complete, reflexive, transitive binary relations) and then consider strictness as an additional property, I am imposing strictness from the outset.

between R and R'. The relation $\mathbf{R}$ is a strict quasi-ordering (i.e., a reflexive, transitive, and antisymmetric binary relation) that is not complete.

Thus, given an agent i's stated ordering $R_i \in \mathcal{R}$ and two orderings $R' \neq R''$, the expression $R'' \ \mathbf{R}_i \ R'$ implies that R' is unambiguously dominated by R'' for agent i. To be sure, such unambiguous dominance will hold only for a restricted set of pairs of orderings, which is responsible for $\mathbf{R}_i$'s generic incompleteness. It naturally leads to the concepts of efficiency and strategyproofness employed by Bossert and Sprumont [5], to which I add the qualifier "weak".

Weak efficiency. There do not exist $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, and $R' \in \mathcal{R}$ such that $R' \in [R_i, f(R_N)]$ for all $i \in N$ and $R' \neq f(R_N)$.

Weak strategyproofness. There do not exist $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N, i \in N$ and $R'_i \in \mathcal{R}$ such that $f(R'_i, R_{N\setminus\{i\}}) \in [R_i, f(R_N)]$ and $f(R'_i, R_{N\setminus\{i\}}) \neq f(R_N)$.

Weak efficiency sets forth a minimal standard of efficiency, imposing that there not exist an ordering that unambiguously dominates the selected one for all agents. Similarly, weak strategyproofness requires that, by misreporting one's ordering, it should not be possible for an agent to obtain a social ordering that unambiguously dominates (with respect to that agent's true ordering) that under truthful reporting.

An additional property that Bossert and Sprumont discuss is that of *local unanimity*, which is relevant for preference profiles in which there is unanimous agreement over certain binary comparisons.

Local unanimity. For all $N \in \mathcal{N}, R_N \in \mathcal{R}^N$ we have $\bigcap_{i\in N} R_i \subseteq f(R_N)$.

Bossert and Sprumont identified three broad classes of rules that satisfy weak efficiency and weak strategyproofness: (i) Condorcet-Kemeny rules and their suitable generalizations; (ii) monotonic majority alteration rules; and (iii) status-quo rules. They also provided novel characterization of rules (ii) and (iii). Finally, Bossert and Sprumont demonstrated that the well-known Borda and Copeland rules fail to satisfy weak strategyproofness, and are thus extremely vulnerable to strategic manipulation.

To fix ideas and aid the reader in understanding the contribution of Bossert and Sprumont [5], we provide definitions of the aforementioned rules (i)-(ii)-(iii).

(i) Condorcet-Kemeny rules. Originating in the writings of the Marquis de Condorcet, these rules were formalized by Kemeny [10] and axiomatized by Young and Levenglick [18] and Young [17]. Given two orderings $R, R' \in \mathcal{R}$, define their *disagreement set*, denoted by $D(R, R')$,

$$D(R, R') = (R \setminus R') \cup (R' \setminus R),$$

which includes all binary comparisons on which R and R' disagree. The *Kemeny distance* (or, alternatively, *Kendall-τ* distance) between R and R', denoted by $\delta(R, R')$, is defined as

$$\delta(R, R') = |D(R, R')|.$$

Let $\succeq$ be a strict ordering on $\mathcal{R}$. For all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$, let

$$K(R_N) = \arg\min_{R \in \mathcal{R}} \sum_{i \in N} \delta(R, R_i). \tag{1}$$

The $\succeq$-*Condorcet-Kemeny rule* is defined as the aggregation rule which assigns to each $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$ the strict ordering belonging to $K(R_N)$ ranked first according to $\succeq$. Bossert and Sprumont also considered generalized versions of Condorcet-Kemeny rules, in which disagreements over binary comparisons are given arbitrary positive weights that may vary across agents, and showed that they too satisfy weak strategyproofness.

(ii) Monotonic majority alteration rules. Given $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$, the *majority relation* $M(R_N)$ on A is a complete and antisymmetric binary relation defined by

$$a \; M(R_N) \; b \quad \Leftrightarrow |\{i \in N : \; aR_ib\}| \geq |\{i \in N : \; bR_ia\}| \,,$$

for all $(a, b) \in A \times A$. Clearly, the majority relation can fail to be transitive and thus may not always lead to an ordering. A *monotonic majority alteration rule* alters the majority relation to obtain a transitive relation (and thus a unique ordering) in a way that is *agreement-monotonic* (for detailed definitions see Section 4 in [5]). Two such agreement-monotonic alterations are (a) *lexicographic alterations* in which intransitivities are addressed in a step-by-step manner according to an exogenous strict ordering over sets of pairs of alternatives (see Example 3 in [5]) and (b) *Slater alterations* in which intransitivities are addressed by choosing the ordering that has the smallest Kemeny distance from the majority relation, where ties are broken according to an exogenous strict ordering over $\mathcal{R}$ (see Example 4 in [5]).

(iii) Status-quo rules. Status-quo rules are designed to improve upon an exogenously given ordering, which in turn is meant to represent a status-quo solution. Before providing a formal definition, a few additional concepts need to be introduced. Given $R^0 \in \mathcal{R}$ and its prudent extension $\mathbf{R}^0$, Guilbaud and Rosenstiehl [9] proved that $(\mathcal{R}, \mathbf{R}^0)$ is a *lattice* so that every collection $\{R^1, R^2, ..., R^T\} \subseteq 2^{\mathcal{R}}$ has a unique *minimal common upper bound*, i.e., a unique ordering $R \in \mathcal{R}$ such that

$$R \, \mathbf{R}^0 \, R^t, \quad \text{for all } t \in \{1, 2, ..., T\}, \tag{2}$$

and

$$\left[R' \, \mathbf{R}^0 \, R^t, \ \text{ for all } t \in \{1, 2, ..., T\}\right] \ \Rightarrow R' \, \mathbf{R}^0 \, R. \tag{3}$$

The rule f is a *status-quo* rule associated with ordering $R^0 \in \mathcal{R}$ if, for all $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$, $f(R_N)$ equals the unique ordering satisfying Eqs. (2)-(3) for all the orderings in R_N. We denote such a rule f by SQ^{R^0}.

Status-quo rules admit concise reformulations via Kemeny distances. For any two orderings $R', R'' \in \mathcal{R}$ we have (see [6]):

$$R' \, \mathbf{R}^0 \, R'' \ \Leftrightarrow R' \in [R^0, R''] \ \ \Leftrightarrow \ \ R' \in \{R \in \mathcal{R} : \ \delta(R^0, R) + \delta(R, R'') = \delta(R^0, R'')\}. \tag{4}$$

Now, Eqs. (2)-(3)-(4) imply that the status-quo rule associated with R^0 can be rewritten as:

$$SQ^{R^0}(R_N) = \underset{R \in \bigcap_{i \in N}[R_i, R^0]}{\arg\max} \ \delta(R^0, R). \tag{5}$$

Finally, using Eqs. (4)-(5) we further obtain:

$$\begin{aligned} SQ^{R^0}(R_N) &= \underset{R \in \bigcap_{i \in N}[R_i, R^0]}{\arg\max} \ \delta(R^0, R) = \underset{R \in \bigcap_{i \in N}[R_i, R^0]}{\arg\max} \ \delta(R_i, R^0) - \delta(R, R_i), \ \ \forall \, i \in N \\ &= \underset{R \in \bigcap_{i \in N}[R_i, R^0]}{\arg\min} \ \delta(R, R_i), \ \ \forall \, i \in N. \end{aligned} \tag{6}$$

2.1 The limitations of betweeness-based concepts

While the introduction of weak strategyproofness is a conceptual breakthrough that leads to novel insights, it is intuitively clear that it may sometimes place too high a bar for manipulability. As it is based on the prudent extension of agents' orderings (an incomplete relation) it will frequently not take a stand on the desirability of misreporting in order to obtain one ordering over another. Similar problems persist with regard to weak efficiency and the desirability of one ordering versus another.

The following two examples illustrate these potential limitations of weak strategyproofness and efficiency.

Example 1 (the limitations of weak strategyproofness). Suppose $A = \{a, b, c, d, e, f\}$ and $N = \{1, 2, 3, 4, 5, 6, 7\}$ and we have the following preference profile R_N appearing in Table 1:

Consider now the Condorcet-Kemeny rule with a randomly assigned ordering $\succeq$ on $\mathcal{R}$. Let us denote this rule by g. Doing some algebra, we have $K(R_N) = cfbdae$ so that $g(R_N) =$

i	R_i
1	$bfdeca$
2	$bdaefc$
3	$fbdaec$
4	$cbdaef$
5	$dcfeab$
6	$cefbda$
7	$aecfbd$

Table 1: Limitations of weak strategyproofness.

$cfbdae$.[6] Let us identify the pairs of alternatives on which R_1 and $g(R_N)$ disagree:

$$D(R_1, g(R_N)) = \{(b, c), (d, c), (e, c), (f, c), \boldsymbol{(b, f), (e, a)}\}.$$

Hence, we see that the social ordering places alternative c first, which however is agent 1's second-to-last ranked alternative. Moreover, it reverses the order of pairs (b, f) and (e, a). In total, the ordering $g(R_N)$ disagrees with R_1 on six binary comparisons.

Suppose now that voter 1 misreports her preferences by stating $R'_1 = afbedc$. Then, algebraic calculations yield $K(R'_1, R^N_{N\setminus\{1\}}) = fbdaec = g(R'_1, R^N_{N\setminus\{1\}})$. As a result, the pairs on which R_1 and $g(R'_1, R^N_{\setminus\{1\}})$ disagree are:

$$D(R_1, g(R'_1, R_{N\setminus\{1\}})) = \{(c, a), \boldsymbol{(b, f), (e, a)}\}.$$

I have highlighted in bold the common elements of the two disagreement sets. Compared to truthful reporting, the social ordering under misreporting still clashes with 1's preferences regarding pairs (b, f) and (e, a). However, by ranking c last instead of first, it has replaced the previous disagreement over pairs $(b, c), (d, c), (e, c), (f, c)$, with a single disagreement over (c, a), i.e., the order of her two least-preferred alternatives.

Clearly, $g(R'_1, R_{N\setminus\{1\}}) \notin [R_1, g(R_N)]$, so that misreporting does not unambiguously dominate truthfulness for agent 1. However, it is plausible that agent 1 will prefer a social ordering which results in three, as opposed to six, disagreeing pairs of alternatives.

Example 2 (the limitations of weak efficiency). Suppose $A = \{a_1, a_2, ..., a_m\}$ for $m \geq 3$, $N = \{1, 2, ..., m-1\}$, and we have the preference profile listed in Table 2:[7]

Let $R^0 = a_m a_{m-1} ... a_2 a_1$ and consider the associated status-quo rule SQ^{R^0}. For all $i \in N$ we have $[R_i, R^0] = \{R \in \mathcal{R} : (a_{i+1}, a_i) \in R\}$. Thus, it is easy to see that $\bigcap_{i \in N} [R_i, R^0] = R^0$, so

[6] This and all similar calculations to follow were performed in MATLAB. Programs available upon request.

[7] The following is a generalization of a simpler example found in Bossert and Sprumont [5].

i	R_i
1	$\boxed{a_2 a_1} a_3 \ldots a_{m-1} a_m$
2	$a_1 \boxed{a_3 a_2} a_4 \ldots a_{m-1} a_m$
$\vdots$	$\vdots$
$m-2$	$a_1 a_2 a_3 \ldots a_{m-3} \boxed{a_{m-1} a_{m-2}} a_m$
$m-1$	$a_1 a_2 a_3 \ldots a_{m-3} a_{m-2} \boxed{a_m a_{m-1}}$

Table 2: Limitations of weak efficiency (boxes draw attention to key pairs of alternatives).

that $SQ^{R^0}(R_N) = R^0$. As the rule satisfies weak efficiency, we are not surprised to find that $\bigcap_{i \in N} \left[R_i, SQ^{R^0}(R_N)\right] = SQ^{R^0}(R_N)$ implying that the ordering $SQ^{R^0}(R_N)$ is undominated. However, note that we have $\delta\left(R_i, SQ^{R^0}(R_N)\right) = \binom{m}{2} - 1$ for all $i \in N$, so that with regard to every agent's preferences R_i the chosen ordering disagrees on all binary comparisons but one. Moreover, observe that we have $\bigcap_{i \in N} R_i = \{(a_k, a_l) : k, l \in N,\ l > k+1\ \} \not\subseteq R^0 = SQ^{R^0}(R_N)$. Indeed, $\bigcap_{i \in N} R_i \cap R^0 = \emptyset$. Evidently, the rule $SQ^{R^0}(R_N)$ violates local unanimity in a significant way.

Now, consider the ordering $R' = a_1 a_2 ... a_{m-1} a_m$, i.e., the exact opposite of R^0. Here we have $\delta(R_i, R') = 1$ for all $i \in N$, so that R' differs along a single binary comparison with regard to all R_i. This ordering respects local unanimity and seems like a significantly more appealing outcome for all agents in N, especially for medium or relatively high values of m.

3 Alternative concepts of efficiency and strategyproofness

Examples 1 and 2 highlight the weakness of betweeness-based efficiency and non-manipulability criteria. They further suggest that a problematic feature of such criteria lies in their insensitivity to the consideration of *all* binary comparisons when deciding between orderings, not just those related to unambiguous dominance. Using Kemeny distances, one may strengthen Bossert and Sprumont's concepts of strategyproofness and efficiency in a manner that addresses these concerns.

K-strategyproofness. There do not exist $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N, i \in N$ and $R'_i \in \mathcal{R}$ such that $\delta(f(R'_i, R_{N\setminus\{i\}}), R_i) < \delta(f(R_N), R_i)$.

K-efficiency. There do not exist $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, and $R' \in \mathcal{R}$ such that $\delta(R', R_i) \leq \delta(f(R_N), R_i)$ for all $i \in N$ and there exists at least one $j \in N$ such that $\delta(R', R_j) < \delta(f(R_N), R_j)$.

Example 1 shows that the Condorcet-Kemeny rule fails K-strategyproofness in a significant way since $\delta(f(R'_1, R_{N\setminus\{1\}}), R_i) = 3 < 6 = \delta(f(R_N), R_1)$. Similarly, Example 2 demonstrates how status-quo rules can result in outcomes that are extremely K-inefficient.

The following propositions collect a few straightforward implications of K-efficiency and K-strategyproofness.

Proposition 1 *If a rule satisfies K-efficiency then it satisfies weak efficiency.*

Proof. Suppose rule f does not satisfy weak efficiency. Then there exists $R_N \in \mathcal{R}^N$ and $R' \in \mathcal{R}$ such that $R' \neq f(R_N)$ and $R' \in [R_i, f(R_N)]$ for all $i \in N$. Note however, that, for any $i \in N$, if $R' \neq f(R_N)$ and $R' \in [R_i, f(R_N)]$ then $D(R_i, R') \subset D(R_i, f(R_N))$. Hence we will have $\delta(R_i, R') < \delta(R_i, f(R_N))$ for all $i \in N$, thus violating K-efficiency. ■

Bossert and Sprumont showed that weak efficiency is logically unrelated to local unanimity. This does not hold for K-efficiency.

Proposition 2 *If a rule satisfies K-efficiency then it satisfies local unanimity.*

Proof. Follows by the proof of a stronger version of its contrapositive outlined in Remark 5 in Bossert and Sprumont [5]. ■

It should be noted that the opposite direction of Proposition 2 does not hold. For instance, consider $N = \{1, 2, 3\}$ and $R_1 = bcad, R_2 = abcd, R_3 = badc$. Since we have $\delta(R_1, R_2) = \delta(R_1, R_3) = \delta(R_2, R_3) = 2$, the set of K-efficient orderings is easily seen to be $\{R_1, R_2, R_3, bacd\}$. Meanwhile, we have $\bigcap_{i \in N} R_i = \{(b, c), (b, d), (a, d)\}$ so that the set of locally unanimous rankings is $\{R_1, R_2, R_3, bacd, abdc\}$. Thus, if a rule f sets $f(R_N) = abdc$ it will violate K-efficiency without violating local unanimity.

Proposition 3 *If a rule satisfies K-strategyproofness then it satisfies weak strategyproofness.*

Proof. Identical to Proposition 1. ■

4 Main results

Having laid out the concepts of K-efficiency and K-strategyproofness in the previous section, can we find any nontrivial (i.e., non-dictatorial and onto) rule that satisfies them? A natural way to start this inquiry is by drawing on the work of Bossert and Sprumont [5] and examining the three broad classes of rules that they proved satisfy weak strategyproofness.

We distinguish between two cases regarding the number of alternatives: $m = 3$ and $m > 3$. As we will see shortly, this distinction turns out to be important.

4.1 The case m=3

The following Theorem establishes that all Condorcet-Kemeny rules are K-strategyproof when the number of alternatives is three and we are dealing with a restricted domain of profiles that precludes the existence of electorates that are perfectly split with respect to *all* binary comparisons of alternatives. Formally, this profile subdomain is denoted by $\mathcal{K}$ and satisfies

$$\mathcal{K} = \left\{R_N \in \mathcal{R}^N : \ N \in \mathcal{N} \text{ and } \ \exists (a,b) \in A \times A \ \text{ s.t. } \ |\{i \in N : \ aR_i b\}| > |\{i \in N : \ bR_i a\}|\right\}.$$

Theorem 1 *Consider the restricted domain of profiles $\mathcal{K}$. On this domain, all Condorcet-Kemeny rules satisfy K-strategyproofness when $m = 3$. In particular, this implies that all Condorcet-Kemeny rules are K-strategyproof when $m = 3$ and $|N|$ is odd.*

Proof. See Appendix A1. ■

The intuition behind the proof of Theorem 1 is straightforward. Recall that all Condorcet-Kemeny rules are weakly strategyproof [5]. When there are just three alternatives this implies that K-strategyproofness can be violated only if there exists an agent for which (i) the social ordering under truthfulness results in exactly two disagreeing binary comparisons and; (ii) the ordering obtained when this agent misreports his preferences is its exact opposite. By first principles this is shown to be impossible, unless the truthful or misreporting preference profile does not belong in $\mathcal{K}$ [meaning that all orderings have identical Kemeny scores as per Eq. (1)].[8]

Since Condorcet-Kemeny rules are by definition K-efficient, a corollary to Theorem 1 is that, in the restricted domain $\mathcal{K}$, there exists a K-efficient and K-strategyproof rule when $m = 3$.

On the other hand, Theorem 2 establishes that all status-quo rules will satisfy K-strategyproofness when $m = 3$ without the need for any domain restrictions. However, this gain in non-manipulability comes at a significant cost to efficiency.

Theorem 2 *When $m = 3$ all status-quo rules satisfy K-strategyproofness. Conversely, none satisfy local unanimity and thus also K-efficiency.*

Proof. See Appendix A1. ■

[8]Note that Slater and lexicographic alteration rules, being based on the majority relation, will encounter similar problems for profiles $R_N \notin \mathcal{K}$ as delineated in the proof of Theorem 1.

The proof of Theorem 2 is simple. The K-strategyproofness of status-quo rules is established by recalling their weak strategyproofness [5] and enumerating all six possible choices for the status-quo ordering. Conversely, the violation of local unanimity can be easily seen by referring to Example 2 for $m = 3$.

4.2 The case $m > 3$

In contrast to the case of three alternatives, when $m > 3$ all positive results on K-strategyproofness quickly vanish.

Proposition 4 *No Condorcet-Kemeny, Slater majority-alteration, or status-quo rule satisfies K-strategyproofness for $m > 3$.*

Proof. Let g be a Condorcet-Kemeny rule with any ordering $\succeq$. Suppose $A = \{a, b, c, d\}$ and $N = \{1, 2, 3, 4, 5\}$ and we have the preference profile R_N shown in Table 3.

i	R_i
1	*dcba*
2	*dacb*
3	*bdac*
4	*cbda*
5	*abcd*

Table 3: No Condorcet-Kemeny or Slater rule is K-strategyproof when $m > 3$.

Here, it is easy to verify that $K(R_N) = bdac$, so $g(R_N) = bdac$. Consider now voter 1. We have $\delta(R_1, g(R_N)) = 3$. Suppose now that voter 1 changes her preferences to $R'_1 = cdba$, by simply flipping the positions of adjacent alternatives c and d. Then, we may verify that $K(R'_1, R_{N\setminus\{1\}}) = cbda$, so $g(R'_1, R_{N\setminus\{1\}}) = cbda$, leading to $\delta(R_1, g(R'_1, R_{N\setminus\{1\}})) = 2$. Thus, all $\succeq$-Condorcet-Kemeny rules will fail K-strategyproofness.

Let us now turn to Slater rules. Computing the majority relations corresponding to profiles R_N and $(R'_1, R_{N\setminus\{1\}})$, we obtain: $M(R_N) = \{(a, c), (b, a), (b, d), (c, b), (d, a), (d, c)\}$ and $M(R'_1, R_{N\setminus\{1\}}) = (M(R_N) \setminus \{(d, c)\}) \cup \{(c, d)\}$. Using them it is easy to see that every Slater majority-alteration rule f, regardless of its ordering $\succeq$, will also yield $f(R_N) = bdac$ and $f(R'_1, R_{N\setminus\{1\}}) = cbda$ and thus also fail K-strategyproofness.[9]

Finally, we address status-quo rules. Consider the profile of Example 2 and the corresponding status-quo rule SQ^{R^0} for $R^0 = a_m a_{m-1} ... a_2 a_1$. Recall that $SQ^{R^0}(R_N) = R^0$. Focus

[9]Note how this discussion shows that all Condorcet-Kemeny and Slater rules fail Sato's [15] axiom of bounded response.

on agent 1 and suppose she submits ordering $R'_1 = a_1 a_m a_{m-1} \dots a_3 a_2$ instead of her truthful preferences $R_1 = a_2 a_1 a_3 ... a_{m-1} a_m$. Then we have

$$\left(\bigcap_{i \in N \setminus \{1\}} [R_i, R^0] \right) \cap [R'_1, R^0] = R^0 \cup \{R^*_j : \quad j = 1, 2, ..., m-1\},$$

where (bold fonts placed for emphasis):

$$R^*_1 = \boldsymbol{a_1} a_m a_{m-1} \dots a_2 = R'_1$$

$$R^*_j = a_m a_{m-1} \dots a_{m-j+2} \boldsymbol{a_1} a_{m-j+1} \dots a_4 a_3 a_2, \quad j = 2, 3, ..., m-1.$$

Then, applying Eq. (5) to profile $(R'_1, R_{N \setminus \{1\}})$ obtains

$$SQ^{R^0}(R'_1, R_{N \setminus \{1\}}) = R'_1.$$

When $m > 3$ this violates K-strategyproofness since, $\delta(R_1, SQ^{R^0}(R_N)) = \binom{m}{2} - 1 > \delta(R_1, SQ^{R^0}(R'_1, R_{N \setminus \{1\}})) = \binom{m}{2} - (m-2)$, for $m > 3$.

Now take an arbitrary ordering $\tilde{R}^0$ and construct a profile $\tilde{R}$ by taking the one in Table 2 and relabeling $a_{m-k+1} \leftarrow \tilde{a}_k$ for $k = 1, 2, ..., m$. An identical argument to the above shows that the misreport $\tilde{R}'_1 = \tilde{a}_m \tilde{a}_1 \tilde{a}_2 \dots \tilde{a}_{m-2} \tilde{a}_{m-1}$ is profitable for agent 1 when $m > 3$. This establishes that no status-quo rule can be K-strategyproof when $m > 3$. ■

Bossert and Sprumont [5] showed that Slater majority-alteration rules are weakly efficient as well as locally unanimous. However, the following proposition demonstrates that they fail to be K-efficient when $m > 3$.

Proposition 5 *No Slater majority-alteration rule is K-efficient for $m > 3$.*

Proof. Let h be a Slater majority-alteration rule with an ordering $\succeq$ to be specified shortly. Suppose $A = \{a, b, c, d\}$ and $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and we have the preference profile R_N shown in Table 4.

i	R_i	i	R_i
1	*dcab*	5	*cdab*
2	*abcd*	6	*cdab*
3	*abcd*	7	*dbca*
4	*abcd*	8	*bdac*

Table 4: Slater rules violate K-efficiency.

There are 4 orderings that are minimal with respect to total Kemeny distance from the majority relation $M(R_N) = \{(a, b), (a, c), (b, c), (b, d), (c, a), (c, d), (d, a), (d, b)\}$, namely $R =$

$abcd$, $R' = bcda$, $R'' = cdab$ and $R''' = dabc$. Let us suppose $\succeq$ is such that R' is ranked first. Then, $h(R_N) = R' = bcda$. However, it is easy to see that $\delta(R_i, R''') = \delta(R_i, R''')$ for all $i \in N \setminus \{1\}$ and $\delta(R_1, R''') = 2 < 4 = \delta(R_1, R')$, implying that all such $\succeq$-Slater rules are not K-efficient.

Now take a $\succeq$-Slater rule, call it again h, such that an arbitrary ordering on A, $a_1a_2a_3a_4$, is ranked first in $\succeq$. Construct a profile $\tilde{R}_N$ by taking the one appearing in Table 4 and relabeling $b \leftarrow a_1$, $c \leftarrow a_2$, $d \leftarrow a_3$ and $a \leftarrow a_4$. Repeating the above argument we see that $h(\tilde{R}_N) = a_1a_2a_3a_4$ which is K-dominated by ordering $a_3a_4a_1a_2$.

Since the ordering $a_1a_2a_3a_4$ is arbitrary, the above establishes that all $\succeq$-Slater rules fail K-efficiency for $m > 3$. ■

Remark 1. Let us now address the properties of lexicographic majority-alteration rules. The results we obtain are qualitatively similar to those above. With regard to strategyproofness, consider the profile of Table 3 and the same misreport as in the proof of Proposition 4. Define rule f as a lexicographic majority-alteration rule with an ordering $\succeq$ over pairs of alternatives such that $\{c, d\} \succeq \{a, d\} \succeq \{a, c\} \succeq \{b, d\} \succeq \{b, c\} \succeq \{a, b\}$. A few straightforward calculations show that $f(R_N) = bdac$ and $f(R'_1, R_{N\setminus\{1\}}) = cbda$, and thus f will also fail K-strategyproofness. Meanwhile, as regards efficiency, Bossert and Sprumont [5] show that when $m > 3$ there exist lexicographic majority monotonic rules that fail local unanimity (see Remark 3 in [5]) and thus by Proposition 2 also K-efficiency. I suspect that both of the above negative results can be extended to *all* lexicographic majority-alteration rules by a judicious relabeling of alternatives and subsequent consideration of the transformed profiles, in much the same way as the proofs of Propositions 4 and 5. The argument would likely be more elaborate though, since there are $\binom{m}{2}!$ orderings of pairs of alternatives. In any event, the above remarks show that lexicographic alteration rules fall prey to violations of both strengthened notions of efficiency and strategyproofness.

Remark 2. An interesting candidate for a K-efficient and K-strategyproof rule is the following family of rules, which I denote as *Rawlsian*. Given $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$ and $R \in \mathcal{R}$, consider the $|N|$-dimensional vector

$$\delta^*(R, R_N) = \left(\delta_1(R, R_N), \delta_2(R, R_N), ..., \delta_{|N|}(R, R_N)\right),$$

whose elements are equal to the elements of set $\{\delta(R, R_i) : i \in N\}$ listed in decreasing order. That is, $\delta_1(R, R_N)$ is the maximum value of the $\delta(R, R_i)$'s and it corresponds to some agent $i_1 \in N$ (if there are two or more agents attaining the maximum value of $\delta(R, R_i)$ pick one at random). Then, $\delta_2(R, R_N)$ is the maximum of the remaining $\delta(R, R_i)$'s excluding that of agent i_1, and it corresponds to some agent $i_2 \in N \setminus \{i_1\}$ (similarly, if there are two or more agents belonging to $N \setminus \{i_1\}$ that attain the maximum value of the remaining $\delta(R, R_i)$'s, pick one at random). Using similar recursive logic we can define all other $\delta_k(R, R_N)$ and i_k up until $k = |N|$.

Let $\succeq$ denote a strict ordering on the elements of $\mathcal{R}$. For $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$, define

$$Rw(R_N) = \underset{R \in \mathcal{R}}{\text{argmin}^{\text{lex}}} \; \delta^*(R, R_N). \tag{7}$$

That is, $Rw(R_N)$ denotes the set of orderings that are the lexicographic-minimizers of the vector-valued function $\delta^*(\cdot, R_N) : \mathcal{R} \mapsto \Re^{|N|}$. The $\succeq$ *Rawlsian rule* is the aggregation rule which assigns to each $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$ the strict ordering belonging to $Rw(R_N)$ ranked first according to $\succeq$.

Consistent to Rawls' principles of justice, Rawlsian rules search for an ordering that minimizes the discontent of the worst-off agents in N. If there are several such orderings, then they focus on minimizing the discontent of the (weakly) second worst-off, and so on. This attention on the least fortunate agents is indicative of a certain sort of fairness.

Clearly, all $\succeq$-Rawlsian rules are K-efficient. However, their desirable fairness and efficiency properties come at a high price, as the following example shows that no Rawlsian rule can ever be weakly strategyproof, even when $m = 3$. To wit, let g be a Rawlsian rule with any ordering $\succeq$. Suppose $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$ and consider the profile R_N, where $R_1 = R_2 = bca$ and $R_3 = abc$. It is clear that there is just one ordering that attains the argmin^{lex} of Eq. (7) applied to profile R_N, namely bac, so $Rw(R_N) = g(R_N) = bac$. Suppose now that agent 1 changes her preferences to $R'_1 = cba$. Then, a few brief calculations establish that $Rw((R_{N\setminus\{1\}}, R'_1)) = bca = g((R_{N\setminus\{1\}}, R'_1))$. Thus, weak strategyproofness is violated no matter how the ordering $\succeq$ is chosen. ■

4.3 Relation to Bossert and Storcken [6]

As mentioned in the introduction, Bossert and Storcken [6] were the first to consider Kemeny-based concepts of strategyproofness for aggregation rules. Their analysis employed a stronger version of non-manipulability, *K-coalitional strategyproofness*, which extends K-strategyproofness to strategic behavior involving coalitions of agents. Moreover, Bossert and Storcken introduced two versions (one weak, one strong) of an independence condition known as *extrema independence*. Extrema independence and its weak counterpart are technical requirements that ensure robustness to special kinds of changes in extreme preferences. (It is straightforward to see that status-quo rules satisfy extrema independence while monotonic-majority-alteration and Condorcet-Kemeny rules violate the strong version while satisfying the weak.)

When $m > 3$, Bossert and Storcken showed that extrema independence is incompatible with K-coalitional strategyproofness, unless we are willing to entertain trivial rules. Theorem 3 summarizes this result.

Theorem 3 *(Bossert and Storcken [6]) Suppose $m > 3$. There exists no onto rule satisfying K-coalitional strategyproofness and extrema independence. If $|N|$ is even, there exists no onto rule satisfying K-coalitional strategyproofness and weak extrema independence.*

Bossert and Storcken's impossibility result is clearly relevant to the inquiry of this paper. For example, it implies that all the rules examined by Bossert and Sprumont [5] do not satisfy K-coalitional strategyproofness. Yet, the relevance of Theorem 3 to our context is tempered by the fact that K-coalitional strategyproofness is a very significant strengthening of K-strategyproofness. Insisting on it would nullify even the few possibility results this work has been able to establish. In particular, when $m = 3$ all status-quo rules and all Condorcet-Kemeny rules (even when, in the case of the latter, $|N|$ is odd and therefore the domain restriction $\mathcal{K}$ of Theorem 1 is automatically satisfied) will fail K-coalitional strategyproofness (see section A2 in the Appendix).

5 Conclusion

This paper has been concerned with Arrovian preference aggregation. In this setting strategic behavior had not, until the recent work of Bossert and Sprumont [5] and Sato [15], been the object of much systematic study. But while the introduction of betweeness-based notions of efficiency and strategyproofness by these authors was a conceptual breakthrough leading to interesting theory, we have demonstrated that they may at times lead to unsatisfying conclusions. This in turn prompted the introduction of stronger requirements based on Kemeny distances, namely K-efficiency and K-strategyproofness, and the search for rules that may satisfy them.

Let us briefly recap the paper's main results. When there are three alternatives, all Condorcet-Kemeny rules (which are generically K-efficient) are K-strategyproof in a restricted, but still quite broad, profile domain. Conversely, all status-quo rules are K-strategyproof, but fail local unanimity and therefore also K-efficiency.

These positive results regarding K-strategyproofness vanish when $m > 3$, as all three classes of rules considered by Bossert and Sprumont [5] fail to be K-strategyproof. Mean-

	Cond.-Kem.	Status quo	Slater alteration	Lex. alteration	Rawlsian
Weak efficiency	yes	yes	yes	yes	yes
Weak strategyproofness	yes	yes	yes	yes	no
Local unanimity	yes	no	yes	no*	yes
K-Efficiency	yes	no	no	no*	yes
K-strategyproofness	no	no	no	no*	no

Table 5: Rules and their properties for $m > 3$. Simple "yes" and "no" entries mean that the result holds for all members of the respective class. An asterisk indicates that the result has been established for some, but not necessarily all, members of the respective class.

while, Rawlsian rules are K-efficient but fail even weak strategyproofness for $m = 3$. Table 5 summarizes what we currently know about the case of more than three alternatives.

When $m = 3$, the existence of a non-dictatorial, K-efficient (or even locally unanimous), and K-strategyproof rule on the unrestricted domain of profiles is an open question. Similarly, when $m > 3$ the existence of a non-dictatorial and onto K-strategyproof rule remains unsettled. Addressing these questions in a definitive manner is a topic worthy of further research.

While we do not know the answer to the above questions, the fact that all weakly strategyproof classes of rules examined by Bossert and Sprumont have not been successful suggests that milder strengthenings of weak strategyproofness may be needed to achieve general possibility results. What these adjusted requirements may look like is not clear.

Appendix

A1: Proofs not in Main Text

Theorem 1. Suppose g is a Condorcet-Kemeny rule with ordering $\succeq$. Let $A = \{a, b, c\}$, $N \in \mathcal{N}$ and $R_N \in \mathcal{R}^N$. Suppose, without loss of generality, that voter i's preferences are given by $R_i = abc$ and that there exists $R'_i \in \mathcal{R}$ such that $\delta(R_i, g(R_N)) > \delta(R_i, g(R'_i, R_{N\setminus\{i\}}))$. We distinguish between 4 cases:

(i) $\delta(R_i, g(R_N)) = 0$. But since $0 \leq \delta(R_i, R)$ for all $R \in \mathcal{R}$, we immediately reach a contradiction.

(ii) $\delta(R_i, g(R_N)) = 1$. Then, we must have $\delta(R_i, g(R'_i, R_{N\setminus\{i\}})) = 0$. Hence, $R_i = g(R'_i, R_{N\setminus\{i\}})$. This implies that rule g is not weakly strategyproof which contradicts Proposition 5 in [5].

(iii) $\delta(R_i, g(R_N)) = 3$. Then, we must have $\delta(R_i, g(R'_i, R_{N\setminus\{i\}})) < 3$. Let $\tilde{R}_i$ denote the ordering which is exactly the opposite of R_i (which reverses the direction of all binary comparisons). Then, it must be the case that $g(R_N) = \tilde{R}_i$ and $g(R_i', R_{N\setminus\{i\}}) \neq \tilde{R}_i$. This again contradicts the weak strategyproofness of g.

(iv) $\delta(R_i, g(R_N)) = 2$. This is the only nontrivial case and we address it in what follows.

To violate K-strategyproofness we must have $\delta(R_i, g(R_i', R_{N\setminus\{i\}})) < 2$. Suppose, first, that $\delta(R_i, g(R_i', R_{N\setminus\{i\}})) = 0$. Repeating the argument of case (ii), we arrive at a contradiction.

Thus, we must have $\delta(R_i, g(R_i', R_{N\setminus\{i\}})) = 1$. Now, $\delta(R_i, g(R_N)) = 2$ implies that we must have either $g(R_N) = cab$ or $g(R_N) = bca$. Suppose that $g(R_N) = cab$ (the proof for case $g(R_N) = bca$ is similar). Then, to avoid violating weak strategyproofness we must have $g(R_i', R_{N\setminus\{i\}}) = bac$. I will argue how this cannot happen unless $R_N \notin \mathcal{K}$ or $(R_i', R_{N\setminus\{i\}}) \notin \mathcal{K}$.

Given profile R_N, define the 3×3 matrix E, where E_{xy} denotes the number of agents ranking alternative x over y. For all pairs $(x, y) \in A \times A$ such that $x \neq y$ we must have $E_{xy} + E_{yx} = |N|$ (the diagonal elements of E are defined to equal 0). Hence, matrix E tabulates the results of all head-to-head contests between alternatives under truthful preferences. Now, denote by E' the altered matrix w.r.t. to E, in which agent i misreports her true preferences $R_i = abc$ by submitting $R_i' \neq R_i$. We have the following five possibilities:

(I) $R_i' = bac$, implying $E'_{ab} = E_{ab} - 1$, $E'_{ca} = E_{ca}$, $E'_{cb} = E_{cb}$;

(II) $R_i' = bca$, implying $E'_{ab} = E_{ab} - 1$, $E'_{ca} = E_{ca} + 1$, $E'_{cb} = E_{cb}$;

(III) $R_i' = acb$, implying $E'_{ab} = E_{ab}$, $E'_{ca} = E_{ca}$, $E'_{cb} = E_{cb} + 1$.

(IV) $R_i' = cba$, implying $E'_{ab} = E_{ab} - 1$, $E'_{ca} = E_{ca} + 1$, $E'_{cb} = E_{cb} + 1$.

(V) $R_i' = cab$, implying $E'_{ab} = E_{ab}$, $E'_{ca} = E_{ca} + 1$, $E'_{cb} = E_{cb} + 1$.

Now, since $g(R_N) = cab$ and $g(R_i', R_{N\setminus\{i\}}) = bac$, it must be the case that:

$$E_{ca} + E_{cb} + E_{ab} \geq E_{ac} + E_{bc} + E_{ba} \tag{8}$$

$$E'_{ca} + E'_{cb} + E'_{ab} \leq E'_{ac} + E'_{bc} + E'_{ba}. \tag{9}$$

Given agent i's five possible modifications to matrix E listed above, the only way that Eqs. (8)-(9) do not lead to a contradiction is if either case (I) or (II) applies.[10] If case (II) applies then we must have $E_{ca} + E_{cb} + E_{ab} = E'_{ca} + E'_{cb} + E'_{ab}$ in turn implying that both Eqs. (8)-(9) are equalities. But then we cannot have $g(R_N) = cab$ and $g(R_i', R_{N\setminus\{i\}}) = bac$ (this would imply that $cab \succeq bac \succeq cab$, a contradiction).

[10] Recall that pairs of elements symmetric to the main diagonals of E and E' must sum to $|N|$.

Thus it must be that case (I) applies. Since $g(R_N) = cab$ we must have $E_{ab} \geq E_{ba}$ (otherwise, $cab \notin K(R_N)$ because ordering cba would have better Kemeny performance for profile R_N). For similar reasons, we must also have $E_{ca} + E_{cb} \geq E_{ac} + E_{bc}$, and $E_{ca} \geq E_{ac}$.

We now distinguish between two cases:

1. $E_{ca} + E_{cb} > E_{ac} + E_{bc}$. In this case we cannot have $bac \in K(R'_i, R_{N\setminus\{i\}})$, since ordering cba would have better Kemeny performance for profile $(R'_i, R_{N\setminus\{i\}})$.

2. $E_{ca} + E_{cb} = E_{ac} + E_{bc}$. Here, suppose first that $E_{ca} > E_{ac}$. Then we cannot have $bac \in K(R'_i, R_{N\setminus\{i\}})$ since bca would have better Kemeny performance for profile $(R'_i, R_{N\setminus\{i\}})$. Hence, it must be that $E_{ac} = E_{ca}$ implying $E_{cb} = E_{bc}$. Thus, $|N|$ must be even. If $E_{ab} = E_{ba}$, then $\succeq$ must rank cab first, and bac before cba or bca. If $E_{ab} = E_{ba} + 2$, then $\succeq$ must rank bac first, and cab before abc or acb. In the former case, we have $R_N \notin \mathcal{K}$, while in the latter $(R'_i, R_{N\setminus\{i\}}) \notin \mathcal{K}$. ■

Theorem 2. Let $A = \{a, b, c\}, N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $R^0 \in \mathcal{R}$ and consider the status-quo rule $g = SQ^{R^0}$. We first address K-strategyproofness. Suppose, without loss of generality, that voter i's preferences are given by $R_i = abc$ and that there exists $R'_i \in \mathcal{R}$ such that $\delta(R_i, g(R_N)) > \delta(R_i, g(R'_i, R_{N\setminus\{i\}}))$.

Similar to the proof of Theorem 1, when $m = 3$ the only way that K-strategyproofness can be violated without also contradicting the weak strategyproofness established by [5] is if $\delta(R_i, g(R'_i, R_{N\setminus\{i\}})) = 1$ and $\delta(R_i, g(R_N)) = 2$ implying that $g(R_N) \in \{cab, bca\}$. Suppose that $g(R_N) = cab$ (the proof for case $g(R_N) = bca$ is similar). Then, once again to avoid violating weak strategyproofness we must have $g(R'_i, R_{N\setminus\{i\}}) = bac$. I will argue how this cannot happen by considering all 6 possible choices of R^0 and showing how each one leads to a contradiction. Recall that status-quo rules satisfy Eqs. (5)-(6).

(i) $R^0 = abc$. But then $g(R_N) \notin [R_i, R^0] = \{abc\}$, a contradiction.

(ii) $R^0 = acb$. But then $g(R_N) \notin [R_i, R^0] = \{abc, acb\}$, a contradiction.

(iii) $R^0 = bac$. But then $g(R_N) \notin [R_i, R^0] = \{abc, bac\}$, a contradiction.

(iv) $R^0 = bca$. But then $g(R_N) \notin [R_i, R^0] = \{abc, bac, bca\}$, a contradiction.

(v) $R^0 = cab$. Then, there must exist a $j \in N \setminus \{i\}$ such that $R_j \in \{bca, cba, cab\}$; otherwise $g(R_N) \in \{abc, acb\}$, a contradiction. Hence, this implies that $[R_j, R^0] \subseteq \{bca, cba, cab\} \Rightarrow \bigcap_{k \in N\setminus\{i\}} [R_k, R^0] \subseteq \{bca, cba, cab\}$. Thus, we must have $g(R, R_{N\setminus\{i\}}) \in \{bca, cba, cab\}$ for all $R \in \mathcal{R}$ contradicting $g(R'_i, R_{N\setminus\{i\}}) = bac$.

(vi) $R^0 = cba$. Then, all $j \in N$ must satisfy $R_j \in \{abc, acb, cab\}$; otherwise $g(R_N) \in \{bac, bca, cba\}$, a contradiction. Moreover, there must exist at least one $j \in N \setminus \{i\}$ such that $R_j = cab$; otherwise $g(R_N) \in \{abc, acb\}$, a contradiction. Putting these two pieces of information together, we see that $\bigcap_{k \in N\setminus\{i\}} [R_k, R^0] = \{cab, cba\}$ so that $g(R, R_{N\setminus\{i\}}) \in \{cab, cba\}$ for all $R \in \mathcal{R}$. This contradicts $g(R_i', R_{N\setminus\{i\}}) = bac$.

Thus, g must be K-strategyproof. Let us now turn to efficiency. Consider a status-quo rule $SQ^{\tilde{R}^0}$ where $\tilde{R}^0 = \tilde{a}_1\tilde{a}_2\tilde{a}_3$ is an arbitrary ordering on A. Now, refer to Example 2 for $m = 3$ and construct the profile $\tilde{R}_N = \{\tilde{R}_1, \tilde{R}_2\}$ where $\tilde{R}_1 = \tilde{a}_2\tilde{a}_3\tilde{a}_1$ and $\tilde{R}_2 = \tilde{a}_3\tilde{a}_1\tilde{a}_2$. In other words, the profile $\tilde{R}$ is built by taking the one appearing in Table 2 for $m = 3$ and relabeling $a_{m-k+1} \leftarrow \tilde{a}_k$ for $k = 1, 2, 3$. Then, following the logic of Example 2, we will have $SQ(\tilde{R}_N) = \tilde{R}^0$, which fails local unanimity, and is K-dominated by the locally unanimous ordering $\tilde{R}' = \tilde{a}_3\tilde{a}_2\tilde{a}_1$. This argument demonstrates how, for any choice of R^0, there will always exist a profile such that SQ^{R^0} will fail local unanimity when applied to this profile. ■

A2: The possibility results of section 4.1 do not extend to K-coalitional strategyproofness

Here we show that the possibility results of Section 4.1 vanish when we consider the K-coalitional-strategyproofness property of Bossert and Storcken [6]. Suppose $A = \{a, b, c\}$.

First, focus on Condorcet-Kemeny rules. Let $N = \{1, 2, 3, 4, 5\}$ (so that the domain restriction $\mathcal{K}$ is automatically satisfied) and consider the profile R_N where: $R_1 = abc$, $R_2 = cba$, $R_3 = cab$, $R_4 = abc$, and $R_5 = bca$. Let g denote the $\succeq$-Condorcet-Kemeny rule such that bca is ranked first in $\succeq$. We have $K(R_N) = \{bca, cab, abc\}$, implying $g(R_N) = bca$. Now suppose agents 3 and 4 submit $R_3' = R_4' = acb$. Then $K(R_{N\setminus\{3,4\}}, R_3', R_4') = acb$, so that $g(R_{N\setminus\{3,4\}}, R_3', R_4') = acb$. Since $\delta(R_3, bca) = \delta(R_4, bca) = 2 < 1 = \delta(R_3, acb) = \delta(R_4, acb)$, K-coalitional strategyproofness is violated.

Now consider status-quo rules. Let $N = \{1, 2\}$ and suppose $R_1 = abc$ and $R_2 = cab$. Suppose $R^0 = bca$ and let g denote the status-quo rule SQ^{R^0}. We will have $g(R_N) = bca$. Suppose now that agents 1 and 2 misreport $R_1' = R_2' = acb$. Then $g(R_N') = acb$ violating K-coalitional-strategyproofness, since $\delta(R_1, g(R_N)) = \delta(R_2, g(R_N) = 2 < 1 = \delta(R_1, g(R_1', R_2')) = \delta(R_2, g(R_1', R_2')$.

Analogous relabeling arguments to those in the proof of Proposition 4 establish that the above violations of K-coalitional-strategyproofness hold for all Condorcet-Kemeny and status-quo rules.

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