# Critical Wavenumbers in the Classification of Fractal Radiation Patterns

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## Abstract—

**Motivation:** Fractal models apply to radiation patterns from the so-called fractal antennas and to the description of wave propagation through a complicated environment. In antenna *design* the prototype antenna array factor has been provided by Weierstraß functions and their band-limited approximations. In antenna *characterization* one may want to assign a given (synthesized or measured) radiation pattern to a fractal class.

**Fractal Pattern Synthesis:** Array antenna factors are synthesized by means of Weierstraß functions [1]. In one spatial dimension ($x_1$) one has

$$f[x_1] := \sum_{m=0}^{\infty} b^{(D-2)m} \cos[2\pi b^m x_1]$$

where $1 < D < 2$, and $b$ is the wavenumber such that $b > 1$. It can be shown that the box-counting ($B$) fractal dimension ($\dim_B$) of the graph of $f[.]$ (graph$[f]$) satisfies $\dim_B[\text{graph}[f]] = D$. Two-dimensional fractal radiation patterns can be synthesized by separation of variables.

**Classification of Synthesized Fractal Patterns:** The spectrum enhancement ($\sigma\eta$) algorithm, introduced a few years ago [2 and references quoted therein] operates on the Fourier transform of a function of two variables. It evaluates derivatives of integer [3] or of fractional order [4], followed by some non-linear operations. By applying $\sigma\eta$ to a synthesized fractal pattern one obtains a vector of morphological descriptors, which are submitted to a trainable classifier. A typical classification result is displayed by Figure 1 where the center panel shows the fractal pattern class centroids on the plane of the first two principal components $z_1$ and $z_2$. Each of the five classes (labelled $1, 3, 5, 6, 9$) is composed of patterns such that $b = 14$ and $16$. Moreover, in class 1, $D = 1.1$; in class 3, $D = 1.3$ (sample pattern on the right panel); in class 5, $D = 1.5$; in class 6, $D = 1.6$ and in class 9, $D = 1.9$ (sample on the left panel).

**Problem to be Addressed:** There are critical values of the wavenumber [5] at which the numerical estimation of $\dim_B[\text{graph}[f]]$ is affected by relevant errors. The goal of this investigation is to determine whether or not the same wavenumber values are critical for the $\sigma\eta$ algorithm as well and, in case they are, provide an explanation.

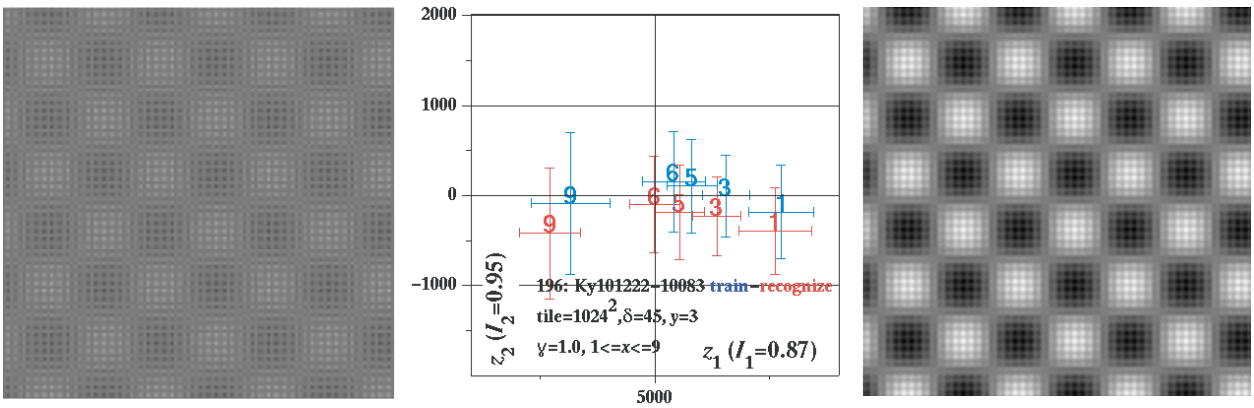


Figure 1.

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